Dynamics: Problem Sheet 2 (of 8)

1. Consider a particle of mass m moving vertically in a fluid with quadratic drag force Dv^2, where v is its velocity and $D > 0$ is a constant. The particle is also acted on by gravity, with acceleration due to gravity g.

   (a) Consider dropping the particle from rest through the fluid, so that its velocity is $v = \dot{z} \leq 0$, with z measured upwards. Show that the equation of motion may be written as

   $$m\dot{v} = -mg + Dv^2 \ .$$

   Show that this may be integrated to

   $$t = -\int_0^v \frac{\mathrm{d}u}{g - \frac{Du^2}{m}} \ .$$

   By evaluating the integral, hence show that the solution is

   $$v(t) = -\sqrt{\frac{mg}{D}} \tanh\left(\sqrt{\frac{Dg}{m}}\, t\right) \ .$$

   What is the terminal velocity?

   (b) Now consider projecting the particle upwards through the fluid, starting at $z = 0$ with speed u. Show that the equation of motion may be written as

   $$\frac{\mathrm{d}(v^2)}{\mathrm{d}z} = 2\dot{v} = -2g - \frac{2Dv^2}{m} \ .$$

   Regarding this as an equation for $v^2(z)$, by integrating it show that the maximum height reached is

   $$z_{\mathrm{max}} = \frac{m}{2D} \log\left(1 + \frac{Du^2}{mg}\right) \ .$$

   What happens as $D \to 0$?

2. A particle of mass m moves along the x axis with one end attached to a spring of spring constant $k > 0$, and is subjected to an additional force $F_0 \cos \Omega t$.

   (a) Show that the equation of motion is

   $$\ddot{x} + \omega^2 x = A \cos \Omega t \ ,$$

   where $x = 0$ corresponds to the unstretched position of the spring, $\omega = \sqrt{k/m}$, and $A = F_0/m$.

   (b) Suppose that $x = \dot{x} = 0$ at time $t = 0$. Derive the solution

   $$x(t) = \frac{A}{\omega^2 - \Omega^2}(\cos \Omega t - \cos \omega t)$$

   for $\Omega \neq \omega$ and the solution

   $$x(t) = \frac{A}{2\omega} t \sin \omega t$$

   for $\Omega = \omega$. What is the qualitative difference between the two solutions?

3. Consider a particle of charge q moving in a constant electromagnetic field. Without loss of generality we take the magnetic field $\mathbf{B} = (0, 0, B) \neq 0$ to point along the z axis, while the electric field $\mathbf{E} = (E_1, E_2, E_3)$ is constant, but arbitrary.

   (a) Assuming the particle has mass m, but ignoring gravity, show that Newton's second law implies the coupled ODEs

$$\begin{aligned} m\ddot{x} &= q\,E_1 + qB\,\dot{y}\ , \\ m\ddot{y} &= q\,E_2 - qB\,\dot{x}\ , \\ m\ddot{z} &= q\,E_3\ , \end{aligned}$$

   for the position $\mathbf{r} = (x, y, z)$ of the particle.

   (b) Verify that

$$\begin{aligned} x(t) &= x_0 + \frac{E_2}{B}\,t + R\,\cos(\omega t + \phi)\ , \\ y(t) &= y_0 - \frac{E_1}{B}\,t - R\,\sin(\omega t + \phi)\ , \\ z(t) &= z_0 + u\,t + \frac{q}{2m}E_3\,t^2\ , \end{aligned}$$

   solves the equations of motion in part (a), where $\omega = qB/m$ is the cyclotron frequency, (x_0, y_0, z_0) is a constant vector, and u, R and ϕ are also constants.

   [*Optional*: For a more challenging version of this question, rather than verifying the solution, instead *derive* it, hence showing it is the general solution.]

4. Consider a particle of mass m moving in a plane with position vector $\mathbf{r} = (x, y)$, subject to a force $\mathbf{F} = -k\,\mathbf{r}$, where $k > 0$ is constant.

   (a) Show that the general solution to the equation of motion is

$$\mathbf{r}(t) = \mathbf{A}\,\sin\omega t + \mathbf{B}\,\cos\omega t\ ,$$

   where $\mathbf{A}$ and $\mathbf{B}$ are constant vectors, and $\omega = \sqrt{k/m}$. (You might find it helpful to write out the vector equation of motion in terms of its components.)

   (b) Show that the solution in part (a) may be rewritten as

$$\mathbf{r}(t) = \mathbf{a}\,\sin(\omega t + \phi) + \mathbf{b}\,\cos(\omega t + \phi)\ ,$$

   where now $\mathbf{a}$ and $\mathbf{b}$ are constant *orthogonal* vectors, and ϕ is a constant phase.

   (c) Hence show that the trajectory of the particle traces out an ellipse, with centre the origin.

Please send comments and corrections to `gaffney@maths.ox.ac.uk`.