Dynamics: Problem Sheet 7 (of 8)

1. Consider a system of particles with masses m_I and position vectors $\mathbf{r}_I(t)$, where $I = 1, \ldots, N$ labels the particles. Particle I is acted on by internal forces $\mathbf{F}_{IJ}$, due to particles $J \neq I$, together with an external force $\mathbf{F}_I^{\mathrm{ext}}$.

   (a) Define the *centre of mass* G of the system of particles, and the *total angular momentum* $\mathbf{L}$ about G. Assuming that the internal forces satisfy the strong form of Newton's third law, using Newton's laws of motion show that

   $$\dot{\mathbf{L}} \;=\; \boldsymbol{\tau}^{\mathrm{ext}} \;,$$

   where $\boldsymbol{\tau}^{\mathrm{ext}}$ is the *total external torque about* G, which you should define.

   (b) Suppose now that all the masses are equal, $m_I = m$ for all $I = 1, \ldots, N$, and that the external force on particle I is $\mathbf{F}_I^{\mathrm{ext}} = -b\,\dot{\mathbf{r}}_I$, for each $I = 1, \ldots, N$, where $b > 0$ is a constant. Show that

   $$\mathbf{L}(t) \;=\; \mathrm{e}^{-bt/m}\,\mathbf{L}(0) \;.$$

2. Consider a binary star system, with stars of mass m_1, m_2. Recall from section 7.3 of the lectures that the position of each star in the *centre of mass frame* is

   $$\mathbf{r}_1 \;=\; \frac{m_2}{m_1+m_2}\mathbf{r} \;, \qquad \mathbf{r}_2 \;=\; -\frac{m_1}{m_1+m_2}\mathbf{r} \;,$$

   where $\mathbf{r}(t)$ satisfies the equation of motion

   $$\frac{m_1 m_2}{m_1+m_2}\ddot{\mathbf{r}} \;=\; -\frac{\kappa}{r^2}\frac{\mathbf{r}}{r} \;, \qquad \text{where } \kappa \;=\; G_N m_1 m_2 \;.$$

   (a) Suppose that the orbit of each star in the centre of mass frame is *circular*, with angular velocity Ω. Show that

   $$r_1 \;=\; \frac{G_N^{1/3} m_2}{(m_1+m_2)^{2/3}\Omega^{2/3}} \;, \qquad r_2 \;=\; \frac{G_N^{1/3} m_1}{(m_1+m_2)^{2/3}\Omega^{2/3}} \;,$$

   where r_1 and r_2 are the radii of the stars' orbits.

   (b) Now introduce a small planet into the system, with mass small enough that it does not affect the orbits of the stars. Suppose also that the planet orbits the centre of mass of the stars in a circle of radius R with the same angular velocity Ω as the stars, so that the whole configuration is stationary in a rotating frame, with the stars and planet colinear. By writing down the equation of motion of the planet, show that

   $$\text{either} \qquad -R\,\Omega^2 \;=\; \mathrm{sign}(r_1 - R)\frac{G_N m_1}{(r_1-R)^2} - \frac{G_N m_2}{(r_2+R)^2} \;,$$

   $$\text{or} \qquad -R\,\Omega^2 \;=\; -\frac{G_N m_1}{(r_1+R)^2} + \mathrm{sign}(r_2 - R)\frac{G_N m_2}{(r_2-R)^2} \;.$$

[There are three positive solutions to these equations for R, known as *Lagrange points*. There are two other Lagrange points in which the planet is not colinear with the stars.]

3. Consider a rigid body that is rotating about a general point O that is fixed both in the body *and* fixed in an inertial frame. Starting from the point particle model of a rigid body, show that its kinetic energy is

$$T \equiv \sum_{I=1}^{N} \frac{1}{2} m_I |\dot{\mathbf{r}}_I|^2 = \frac{1}{2} \sum_{i,j=1}^{3} \mathcal{I}_{ij}^{(O)} \omega_i \omega_j ,$$

where $\mathcal{I}^{(O)}$ is the inertia tensor of the body about O, and $\boldsymbol{\omega}$ is its angular velocity. [*Hint*: You might find the vector identity $|\mathbf{a} \wedge \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$ helpful.]

4. (a) Consider a continuum distribution of mass of bounded extent, with density $\rho(\boldsymbol{x})$, centre of mass $\boldsymbol{R}_G$ and total mass M within a uniform gravitational field, $-g\boldsymbol{k}$. Given there is no other external force, show that the total external torque about a point P is given by

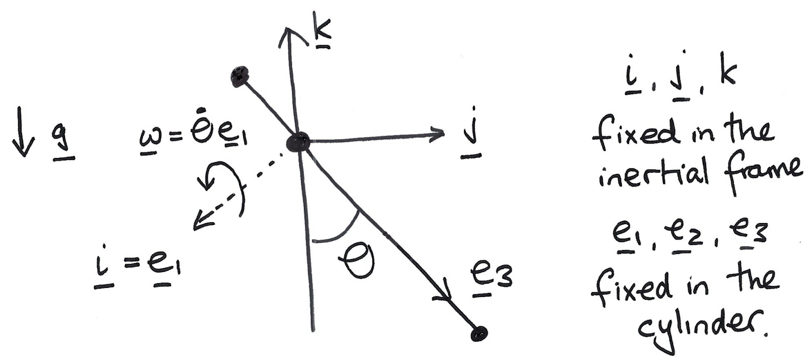
$$\boldsymbol{\tau} = -Mg(\boldsymbol{R}_G - \boldsymbol{p}) \wedge \boldsymbol{k}$$

where $\boldsymbol{p}$ is the position vector of P.

(b) A uniform circular cylinder of length l, radius a and mass M, is pivoting about a point, P_α, which is on the cylinder axis of symmetry and a distance αl from one of the cylinder ends. Show that the inertia tensor $\mathcal{I}$ is given by

$$\mathcal{I} = \begin{pmatrix} \frac{1}{3}Ml^2(\alpha^3 + (1-\alpha)^3) + \frac{1}{4}Ma^2 & 0 & 0 \\ 0 & \frac{1}{3}Ml^2(\alpha^3 + (1-\alpha)^3) + \frac{1}{4}Ma^2 & 0 \\ 0 & 0 & \frac{1}{2}Ma^2 \end{pmatrix}.$$

Here we have taken the z direction along the axis of the cylinder. [*Hint*: Use cylindrical polar coordinates.]



(c) The above cylinder swings in a vertical plane, hinged at a point half way between an end of the cylinder and its centre of mass. Let $\theta(t)$ denote the angle from the downward vertical to the longer segment of the cylinder, of length $3l/4$. Show that the equation of motion for the cylinder is given by

$$\left(\frac{7}{12} + \frac{a^2}{l^2}\right)\ddot{\theta} = -\frac{g}{l}\sin\theta.$$

Please send comments and corrections to `gaffney@maths.ox.ac.uk`.