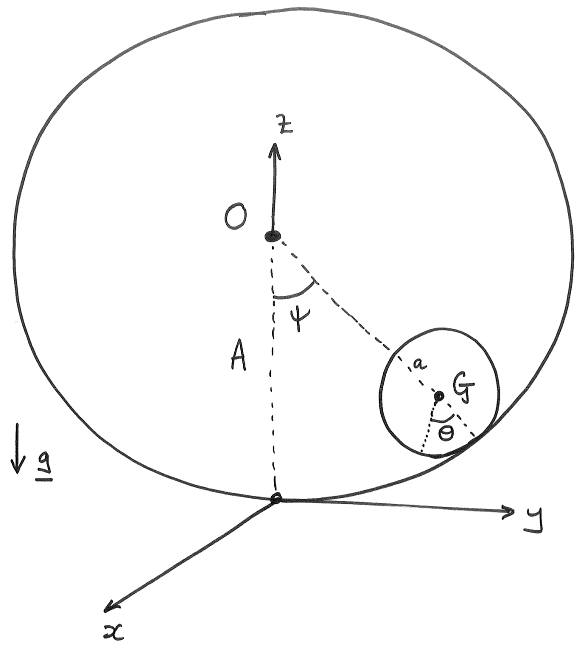
Dynamics: Problem Sheet 8 (of 8)



1. Gravitational acceleration is given by $-g\boldsymbol{k}$. A uniform cylinder of mass M, radius a and length l rolls on the inside of a stationary larger cylinder given by

$$y^2 + z^2 = A^2 > a^2,$$

where x, y, z are Cartesian coordinates of an inertial reference frame (see above image). The symmetry axes of the two cylinders are parallel for all time and the centre of mass of the smaller cylinder, G, always remains in the y-z plane.

Let ψ denote the angle between the downward vertical and the radial vector from the origin to the centre of mass of the smaller cylinder. Let θ denote the angle the smaller cylinder rotates around its symmetry axis. Assuming the smaller cylinder rolls without slipping and contact between the cylinders is always maintained, you are given that for a suitable choice of direction corresponding to $\theta = 0$, one has

$$a\theta = (A - a)\psi. \qquad (*)$$

Determine the equation of motion for the small cylinder, in the form

$$\ddot{\psi} = -\beta \sin\psi,$$

where the constant β is to be determined. How could you deduce that β is proportional to g and that β is positive, without explicit calculation.

[*You may quote the inertia tensor about the centre of mass for the smaller cylinder rather than derive it.*]

Optional. Show that (*) holds. You may wish to look up the coin rotation paradox.

2. (a) A heavy pendulum consists of a uniform rigid rod of mass M and length l, pivoted freely at one end, without loss the origin O, and swings freely in a vertical plane under gravity. Show that its kinetic energy is given by

$$T = \frac{1}{6} M l^2 \dot{\theta}^2 ,$$

where θ is the angle the pendulum makes with the downward vertical.

(b) Given that the potential energy is $V = MgZ_G$, where Z_G is the height of the centre of mass of the pendulum, hence write down the total energy $E = T + V$. Show that conservation of E is implied by the equation of motion (8.47) derived in the lecture notes.

3. A smooth straight wire rotates with constant angular speed ω about the vertical axis through a fixed point O on the wire, and the angle between the wire and the upward vertical is constant and equal to α, where $0 < \alpha < \pi/2$. A bead of mass m is free to slide on the wire.

(a) Starting from the general form of Newton's second law in a rotating frame, show that

$$m \left(\frac{\mathrm{d}^2 \mathbf{r}}{\mathrm{d}t^2} \right)_{\mathcal{S}} = \mathbf{N} - mg\,\mathbf{k} - 2m\omega\,\mathbf{k} \wedge \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right)_{\mathcal{S}} - m\omega^2\,\mathbf{k} \wedge (\mathbf{k} \wedge \mathbf{r}) ,$$

where $\mathbf{r}(t)$ is the position of the bead in the frame $\mathcal{S}$ that rotates with the wire, $\mathbf{k}$ is a unit vector pointing vertically, and $\mathbf{N}$ is the normal reaction of the wire.

(b) Hence show that $z(t)$, the height of the bead above O, satisfies the equation

$$\ddot{z} - (\omega^2 \sin^2 \alpha) z = -g \cos^2 \alpha .$$

Show that an equilibrium point for the bead exists, and determine its stability.

4. A bead P of mass m slides on a smooth circular wire of radius a and centre C. The wire lies in a horizontal plane and is forced to rotate at a constant angular speed ω about the vertical axis through a fixed point O in the plane of the wire. The distance from O to C is constant and equal to b.

   Let θ be the angle that the line joining the centre C to the bead makes with the diameter through O. (See the figure below.)

   (a) Show that the position of the bead may be written as

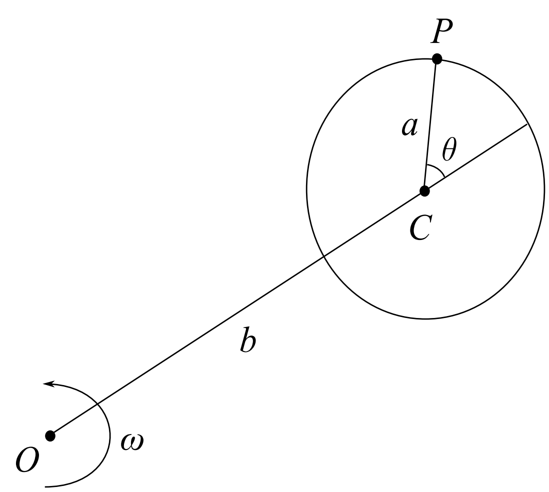
   $$\mathbf{r} \quad = \quad (b + a\,\cos\theta)\,\mathbf{e}_1 + a\,\sin\theta\,\mathbf{e}_2\ ,$$

   where $\mathbf{e}_i$, $i = 1, 2, 3$, are an orthonormal basis for a frame that rotates with the wire.

   (b) Using Newton's second law in this rotating frame, hence show that

   $$\ddot{\theta} + \frac{b}{a}\omega^2 \sin\theta \quad = \quad 0\ .$$

   (c) Show that the bead can remain in equilibrium relative to the wire at two points. Decide whether these positions of equilibrium are stable or unstable.



Please send comments and corrections to `gaffney@maths.ox.ac.uk`.