Analysis III

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[1]The original version of these notes was created by Ben Green.

Contents

Preface

These are the notes for Analysis III at Oxford. The objective of this course is to present a rigorous theory of what it means to integrate a function $f : [a, b] \to \mathbb{R}$. For which functions f can we do this, and what properties does the integral have? Can we give rigorous and general versions of facts you learned in school, such as integration by parts, integration by substitution, and the fact that the integral of f' is just f?

We will present the theory of the *Riemann integral*, although the way we will develop it is much closer to what is known as the *Darboux integral*. The end product is the same (the Riemann integral and the Darboux integral are equivalent) but the Darboux development tends to be easier to understand and handle.

This is not the only way to define the integral. In fact, it has certain deficiencies when it comes to the interplay between integration and limits or the integrability of functions with singularities, for example. To handle these situations one needs the *Lebesgue integral*, which is discussed in the part A course A4 Integration.

Students should be aware that every time we write "integrable" we mean "Riemann integrable". For example, later on we will exhibit a non-integrable function, but it turns out that this function *is* integrable in the sense of Lebesgue.

These lecture notes are based on the lecture notes from previous years by Ben Green and Marc Lackenby, but I have made some minor changes this year and have likely added a non-empty set of typos to the notes. Hence I would be grateful if you could report any corrections to

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CHAPTER 1

Step functions and the Riemann integral

1.1. Step functions

We are going to define the (Riemann) integral of a function by approximating it using simple functions called step functions.

DEFINITION 1.1. Let $[a, b]$ be an interval. A function $\phi : [a, b] \to \mathbb{R}$ is called a *step function* if there is a finite sequence $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ such that ϕ is constant on each open interval (x_{i-1}, x_i).

Remarks. We do not care about the values of f at the endpoints $x_0, x_1, \ldots, x_n$.

We call a sequence $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ a *partition* $\mathcal{P}$, and we say that ϕ is a step function *adapted* to $\mathcal{P}$.

DEFINITION 1.2. A partition $\mathcal{P}'$ given by $a = x'_0 \leq \cdots \leq x'_{n'} \leq b$ is a *refinement* of $\mathcal{P}$ if every x_i is an x'_j for some j.

LEMMA 1.3. *We have the following facts about partitions:*

(i) *If ϕ is a step function adapted to $\mathcal{P}$, and if $\mathcal{P}'$ is a refinement of $\mathcal{P}$, then ϕ is also a step function adapted to $\mathcal{P}'$.*

(ii) *If $\mathcal{P}_1, \mathcal{P}_2$ are two partitions then there is a common refinement of both of them.*

(iii) *If ϕ_1, ϕ_2 are step functions then so are* $\max(\phi_1, \phi_2)$, $\phi_1 + \phi_2$ *and $\lambda\phi_i$ for any scalar λ.*

PROOF. All completely straightforward; for (iii), suppose that ϕ_1 is adapted to $\mathcal{P}_1$ and that ϕ_2 is adapted to $\mathcal{P}_2$, and pass to a common refinement of $\mathcal{P}_1, \mathcal{P}_2$. □

If $X \subset \mathbb{R}$ is a set, the *indicator function* of X is the function $\mathbb{1}_X$ taking the value 1 for $x \in X$ and 0 elsewhere. In the literature this function is also called the characteristic function of X and an alternative notation that is frequently used is χ_X.

LEMMA 1.4. *A function $\phi : [a, b] \to \mathbb{R}$ is a step function if and only if it is a finite linear combination of indicator functions of intervals* (*open and closed*).

PROOF. Suppose first that ϕ is a step function adapted to some partition $\mathcal{P}$, $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$. Then ϕ can be written as a weighted sum of the functions $\mathbb{1}_{(x_{i-1},x_i)}$ (each an indicator function of an open interval) and the functions $\mathbb{1}_{\{x_i\}}$ (each an indicator function of a closed interval containing a single point).

Conversely, the indicator function of any interval is a step function, and hence so is any finite linear combination of these by Lemma 1.3. □

In particular, the step functions on $[a, b]$ form a vector space, which we occasionally denote by $\mathscr{L}_{\text{step}}[a, b]$.

1.2. I of a step function

It is obvious what the integral of a step function "should" be.

DEFINITION 1.5. Let ϕ be a step function adapted to some partition $\mathcal{P}$, and suppose that $\phi(x) = c_i$ on the interval (x_{i-1}, x_i). Then we define

$$I(\phi) = \sum_{i=1}^{n} c_i(x_i - x_{i-1}).$$

We call this $I(\phi)$ rather than $\int_a^b \phi$, because we are going to define $\int_a^b f$ for a class of functions f much more general than step functions. It will then be a theorem that $I(\phi) = \int_a^b \phi$, rather than simply a definition.

Actually, there is a small subtlety to the definition. Our notation suggests that $I(\phi)$ depends only on ϕ, but its definition depended also on the partition $\mathcal{P}$. In fact, it does not matter which partition one chooses. If one is pedantic and writes

$$I(\phi; \mathcal{P}) = \sum_{i=1}^{n} c_i(x_i - x_{i-1})$$

then one may easily check that

$$I(\phi; \mathcal{P}) = I(\phi; \mathcal{P}')$$

for any refinement $\mathcal{P}'$ of $\mathcal{P}$. Now if ϕ is a step function adapted to both $\mathcal{P}_1$ and $\mathcal{P}_2$ then one may locate a common refinement $\mathcal{P}'$ and conclude that

$$I(\phi; \mathcal{P}_1) = I(\phi; \mathcal{P}') = I(\phi; \mathcal{P}_2).$$

LEMMA 1.6. *The map* $I : \mathscr{L}_{\text{step}}[a, b] \to \mathbb{R}$ *is linear, i.e.*

$$I(\lambda\phi_1 + \mu\phi_2) = \lambda I(\phi_1) + \mu I(\phi_2)$$

and order-preserving in the sense that if $\phi_1 \leq \phi_2$ *(pointwise) then*

$$I(\phi_1) \leq I(\phi_2).$$

PROOF. This is obvious on passing to a common refinement of the partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ to which ϕ_1, ϕ_2 are adapted. □

1.3. Definition of the integral

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. We say that a function ϕ_- is a *minorant* for f if ϕ_- is a step function and $\phi_- \leq f$ pointwise. We say that ϕ_+ is a *majorant* for f if ϕ_+ is a step function and $f \leq \phi_+$ pointwise.

We will sometimes use the shorthand $\mathcal{M}_+(f) := \{\phi_+ \text{ majorant of } f\}$, $\mathcal{M}_-(f) := \{\phi_- \text{ minorant of } f\}$, or just $\mathcal{M}_\pm$ if is is clear from the context what f.

Definition 1.7. A function f is *integrable* if

$$\sup_{\phi_- \in \mathcal{M}_-(f)} I(\phi_-) = \inf_{\phi_+ \in \mathcal{M}_+(f)} I(\phi_+). \tag{1.1}$$

If f is integrable then we define the integral $\int_a^b f$ to be the common value of the two quantities in (1.1).

We sometimes write for short

$$I_+(f) := \inf_{\phi_+ \in \mathcal{M}_+(f)} I(\phi_+) \text{ and } I_-(f) := \inf_{\phi_- \in \mathcal{M}_-(f)} I(\phi_-)$$

and note that this sup and inf exist for any bounded function f. Indeed if $|f| \leq M$ then the constant function $\phi_- = -M$ is a minorant for f (so there is at least one) and evidently $I(\phi_-) \leq (b - a)M$ for all minorants. A similar proof applies to majorants.

We note moreover that $I(\phi_-) \leq I(\phi_+)$ for all $\phi_\pm \in \mathcal{M}_\pm(f)$ since I is order-preserving and since $\phi_- \leq f \leq \phi_+$. In particular, for any bounded function f we always have

$$\sup_{\phi_- \in \mathcal{M}_-} I(\phi_-) \leq \inf_{\phi_+ \in \mathcal{M}_+} I(\phi_+), \tag{1.2}$$

so to show that a bounded function is integrable all we need to check is whether the reverse inequality $I_+(f) \leq I_-(f)$ holds.

It follows from (1.2) that if f is integrable then

$$I(\phi_-) \leq \int_a^b f \leq I(\phi_+) \tag{1.3}$$

whenever $\phi_- \leq f \leq \phi_+$ are minorant and majorants.

Remark. If a function f is only defined on an open interval (a, b), then we say that it is integrable if an arbitrary extension of it to $[a, b]$ is. It follows immediately from the definition of step functions and their integral(which does not care about the endpoints) that it does not matter which extension we choose.

Remark on dx. Integrals are often written using the dx notation. For example, $\int_0^1 x^2 dx$. This means the same as $\int_0^1 f$, where $f(x) = x^2$. We emphasise that in this course this is nothing more than a piece of notation. The dx tells us which variable f is a function of. This can sometimes be very useful to avoid confusion.

1.4. A useful lemma and some examples

The definition of the integral given in the previous section seems, at first sight, to be hard to verify in practice. It is defined in terms of *all* majorants ϕ_+ and minorants ϕ_- for the function f. How might we compute $\sup_{\phi_-\in\mathcal{M}_-} I(\phi_-)$ and $\inf_{\phi_+\in\mathcal{M}_+} I(\phi_+)$? The following very useful lemma provides a necessary and sufficient condition for a function f to be integrable. We will see that it can also be used to compute the integral in specific examples.

LEMMA 1.8. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then the following are equivalent:*

(i) *f is integrable;*
(ii) *For every $\varepsilon > 0$, there is a majorant ϕ_+ and a minorant ϕ_- for f such that $I(\phi_+) - I(\phi_-) < \varepsilon$.*
(iii) *There exists a sequence of majorants ϕ_+^n and minorants ϕ_-^n for f so that*

$$I(\phi_+^n - \phi_-^n) \to 0 \text{ as } n \to \infty.$$

We note that for sequences of majorants and minorants as in (iii) the limits $\lim_{n\to\infty} I(\phi_+^n)$ and $\lim_{n\to\infty} I(\phi_+^n)$ must exist and be equal to $\int_a^b f$.

This follows by the sandwich theorem for sequences since we can always bound

$$I(\phi_-^n) \leq \sup_{\mathcal{M}_-} I(\phi_-) = I_-(f) \leq I_+(f)$$

while using that $\varepsilon_n := I(\phi_+^n - \phi_-^n) = I(\phi_+^n) - (\phi_-^n) \to 0$ also gives that the right hand side of

$$I(\phi_-^n) = I(\phi_+^n) - \varepsilon_n \geq \inf_{\mathcal{M}_+} I(\phi_+) - \varepsilon_n = I_+(f) - \varepsilon_n$$

converges to $I_+(f)$.

PROOF. Suppose first that f is integrable. Let $\varepsilon > 0$. Then by the approximation property for sup and inf, there is a minorant ϕ_- such that

$$I(\phi_-) > \sup_{\phi_-} I(\phi_-) - (\varepsilon/2)$$

and a majorant ϕ_+ such that

$$I(\phi_+) < \inf_{\phi_+} I(\phi_+) + (\varepsilon/2).$$

Since the sup and inf are assumed to be equal, we deduce that

$$I(\phi_+) - I(\phi_-) < \varepsilon.$$

Now suppose that (ii) holds. Let $\varepsilon > 0$ be arbitrary, and let ϕ_+ and ϕ_- be the majorant and minorant provided by (ii). Then

$$I(\phi_+) < I(\phi_-) + \varepsilon \leq \sup_{\phi_-} I(\phi_-) + \varepsilon.$$

So, taking the infimum over all majorants, we deduce that

$$\inf_{\phi_+} I(\phi_+) < \sup_{\phi_-} I(\phi_-) + \varepsilon.$$

Therefore, $\inf_{\phi_+} I(\phi_+)$ is squeezed between $\sup_{\phi_-} I(\phi_-)$ and $\sup_{\phi_-} I(\phi_-)+\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we deduce that the inf and sup must be equal. In other words, f is integrable.

Finally if (ii) holds then we can apply this with $\varepsilon = \frac{1}{n}$ to get a sequences of minorants/majorants $\phi_\pm^n$ with $0 \leq I(\phi_+^n - \phi_-^-) \leq \frac{1}{n} \to 0$, while (ii) follows from (iii) as given $\varepsilon > 0$ and a sequences of minorants/majorants ϕ_+^n, ϕ_-^n as in (iii) there will be some N so that $I(\phi_+^n - \phi_-^n) < \varepsilon$ for all $n \geq N$ and any of those majorants/minorants work for (ii).

□

Once we know that f is integrable, then any majorant ϕ_+ and minorant ϕ_- as in (ii) gives an approximation to the integral, by (1.3). This is because $\int_a^b f$ lies between $I(\phi_-)$ and $I(\phi_+)$ which differ by less than ε.

The following example demonstrates how useful this is in practice.

Example. The function $f(x) = x$ is integrable on $[0, 1]$, and $\int_0^1 f(x)dx = \frac{1}{2}$.

Proof. We define explicit minorants and majorants. Let n be an integer to be specified later, and set $\phi_-(x) = \frac{i}{n}$ for $\frac{i}{n} \leq x < \frac{i+1}{n}$, $i = 0, 1, \ldots, n-1$. Set $\phi_+(x) = \frac{j}{n}$ for $\frac{j-1}{n} \leq x < \frac{j}{n}$, $j = 1, \ldots, n$. Then $\phi_- \leq f \leq \phi_+$ pointwise, so ϕ_-, ϕ_+ (being step functions) are minorant/majorant for f. We have

$$I(\phi_-) = \sum_{i=0}^{n-1} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{2}(1 - \frac{1}{n})$$

and

$$I(\phi_+) = \sum_{j=1}^{n} \frac{j}{n} \cdot \frac{1}{n} = \frac{1}{2}(1 + \frac{1}{n}).$$

So, by Lemma 1.8, f is integrable. Moreover, the integral of f must lie between $\frac{1}{2}(1 - \frac{1}{n})$ and $\frac{1}{2}(1 + \frac{1}{n})$. Since n was arbitrary, the integral must be $\frac{1}{2}$. □

Earlier we defined $I(\phi)$ for a step function ϕ. We can now prove that this actually agrees with the integral of ϕ.

PROPOSITION 1.9. *Suppose that ϕ is a step function on $[a,b]$. Then ϕ is integrable, and $\int_a^b \phi = I(\phi)$.*

PROOF. Take $\phi_- = \phi_+ = \phi$, and the result is immediate from Lemma 1.8 and (1.3). □

COROLLARY 1.10. *There is a non-negative integrable function f on $[a,b]$ which is not identically zero, but for which $\int_a^b f = 0$.*

PROOF. Simply take f to be the zero function, modified at one point. □

But now we come to a "non-example".

EXAMPLE. There is a bounded function $f : [0,1] \to \mathbb{R}$ which is not (Riemann) integrable.

PROOF. Consider the function f such that $f(x) = 1$ if $x \in \mathbb{Q}$ and 0 if $x \notin \mathbb{Q}$. Since any open interval contains both rational points and points which are not rational, any step function majorising f must satisfy $\phi_+(x) \geq 1$ except possibly at the finitely many endpoints x_i, and hence $I(\phi_+) \geq 1$. Similarly any minorant ϕ_- satisfies $\phi_-(x) \leq 0$ except at finitely many points, and so $I(\phi_-) \leq 0$. This function f cannot possibly be integrable. □

Remark. Students will see in next year's course on Lebesgue integration that the Lebesgue integral of this function *does* exist (and equals 0).

1.5. Basic theorems about the integral

In this section we assemble some basic facts about the integral. Their proofs are all essentially routine, but there are some labour-saving tricks to be exploited.

PROPOSITION 1.11. *Suppose that f is integrable on $[a,b]$. Then, for any c with $a < c < b$, f is Riemann integrable on $[a,c]$ and on $[c,b]$. Moreover $\int_a^b f = \int_a^c f + \int_c^b f$.*

Conversely if $f : [a,b] \to \mathbb{R}$ is so that f is integrable on both $[a,c]$ and $[c,b]$ then it is integrable on $[a,b]$.

PROOF. Let M be a bound for f, thus $|f(x)| \leq M$ everywhere. In this proof it is convenient to assume that (i) all partitions of $[a,b]$ include the point c and that (ii) all minorants take the value $-M$ at c, and all majorants the value M. By refining partitions if necessary, this makes no difference to any computations involving $I(\phi_-), I(\phi_+)$.

Now observe that a minorant ϕ_- of f on $[a,b]$ is precisely the same thing as a minorant $\phi_-^{(1)}$ of f on $[a,c]$ juxtaposed with a minorant $\phi_-^{(2)}$ of f on $[c,b]$, and that

$I(\phi_-) = I(\phi_-^{(1)}) + I(\phi_-^{(2)})$. A similar comment applies to majorants. Thus, since f is integrable,

$$\sup_{\phi_-} I(\phi_-) = \sup_{\phi_-^{(1)}} I(\phi_-^{(1)}) + \sup_{\phi_-^{(2)}} I(\phi_-^{(2)}) = \inf_{\phi_+^{(1)}} I(\phi_+^{(1)}) + \inf_{\phi_+^{(2)}} I(\phi_+^{(2)}) = \inf_{\phi_+} I(\phi_+). \tag{1.4}$$

Since $\sup_{\phi_-^{(i)}} I(\phi_-^{(i)}) \leq \inf_{\phi_+^{(i)}} I(\phi_+^{(i)})$ for $i = 1, 2$, we are forced to conclude that equality holds: $\sup_{\phi_-^{(i)}} I(\phi_-^{(i)}) = \inf_{\phi_+^{(i)}} I(\phi_+^{(i)})$ for $i = 1, 2$. (Here, we used the fact that if $x \leq x'$, $y \leq y'$ and $x + y = x' + y'$ then $x = x'$ and $y = y'$.) Thus f is indeed integrable on $[a, c]$ and on $[c, b]$, and it follows from (1.4) that $\int_a^b f = \int_a^c f + \int_c^b f$.

The final part of the lemma follows immediately as given any $\varepsilon > 0$ we can use that if f is integrable on both $I_1 = [a, c]$ and $I_2 = [c, b]$ then there exist majorants $\phi_+^{1,2}$ and minorants $\phi_-^{1,2}$ on $I_{1,2}$ so that $\int_a^c \phi_+^1 - \phi_-^1 < \frac{1}{2}\varepsilon$ and $\displaystyle\int_c^b \phi_+^2 - \phi_-^2 < \frac{1}{2}\varepsilon$. Defining $\phi_\pm$ as $\phi_\pm = \phi_\pm^1$ on $[a, c]$ and as $\phi_\pm = \phi_\pm^2$ on $(c, b]$ then gives a majorant-minorant pair for f on $[a, b]$ for which (ii) of Lemma 1.8 holds.

□

Corollary 1.12. *Suppose that $f : [a, b] \to \mathbb{R}$ is integrable, and that $[c, d] \subseteq [a, b]$. Then f is integrable on $[c, d]$.*

Proof. This is immediate. □

Proposition 1.13. *If f, g are integrable on $[a, b]$ then so is $\lambda f + \mu g$ for any $\lambda, \mu \in \mathbb{R}$. Moreover $\int_a^b (\lambda f + \mu g) = \lambda \int_a^b f + \mu \int_a^b g$. That is, the integrable functions on $[a, b]$ form a vector space and the integral is a linear functional (linear map to $\mathbb{R}$) on it.*

Proof. Suppose that $\lambda > 0$. If $\phi_- \leq f \leq \phi_+$ are minorant/majorant for f, then $\lambda\phi_- \leq \lambda f \leq \lambda\phi_+$ are minorant and majorant for λf. Moreover $I(\lambda\phi_+) - I(\lambda\phi_-) = \lambda(I(\phi_+) - I(\phi_-))$ can be made arbitrarily small. Thus λf is integrable. Moreover $\inf_{\phi_+} I(\lambda\phi_+) = \lambda \inf_{\phi_+} I(\phi_+)$, $\sup_{\phi_-} I(\lambda\phi_-) = \lambda \sup_{\phi_-} I(\phi_-)$, and so $\int_a^b (\lambda f) = \lambda \int_a^b f$. If $\lambda < 0$ then we can proceed in a very similar manner. We leave this to the reader. If $\lambda = 0$, then λf is identically zero and hence is integrable by Proposition 1.9.

Now suppose that $\phi_- \leq f \leq \phi_+$ and $\psi_- \leq g \leq \psi_+$ are minorant/majorants for f, g. Then $\phi_- + \psi_- \leq f + g \leq \phi_+ + \psi_+$ are minorant/majorant for $f + g$ (note these are steps functions) and by Lemma 1.6 (linearity of I)

$$\inf_{\phi_+, \psi_+} I(\phi_+ + \psi_+) = \inf_{\phi_+} I(\phi_+) + \inf_{\psi_+} I(\psi_+) = \int_a^b f + \int_a^b g,$$

whilst

$$\sup_{\phi_-,\psi_-} I(\phi_- + \psi_-) = \sup_{\phi_-} I(\phi_-) + \sup_{\psi_-} I(\psi_-) = \int_a^b f + \int_a^b g.$$

It follows that indeed $f+g$ is integrable and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

That $\int_a^b (\lambda f + \mu g) = \lambda \int_a^b f + \mu \int_a^b g$ follows immediately by combining these two facts. □

Corollary 1.14. *If f is integrable on $[a,b]$, and if $\tilde{f}$ differs from f in finitely many points, then $\tilde{f}$ is also integrable.*

Proof. The function $\tilde{f} - f$ is zero except at finitely many points. Suppose that these points are $x_1, \ldots, x_{n-1}$. Then $\tilde{f} - f$ is a step function adapted to the partition $a = x_0 \le x_1 \le \cdots \le x_{n-1} \le x_n = b$. By Proposition 1.9, $\tilde{f} - f$ is integrable, and hence so is $\tilde{f} = (\tilde{f} - f) + f$, by Proposition 1.13. □

Just like the map $I : \mathcal{L}_{step} \to \mathbb{R}$, the integral is not only linear, but also order-preserving, i.e. we have

Proposition 1.15. *Suppose that f and g are integrable on $[a,b]$ and that $f \le g$ on $[a,b]$. Then*

$$\int_a^b f \le \int_a^b g.$$

Proof. As $f - g \ge 0$, any $\phi_- \in \mathcal{M}_-(f-g)$ must be so that $\phi_- \le 0$ on $[a,b]$ and hence so that $I(\phi_-) \le I(0) = 0$ as I is order-preserving on $\mathcal{L}_{step}$. Thus $I_-(f-g) \ge 0$ and since $f-g$ is integrable this gives

$$0 \le \int_a^b f - g = \int_a^b f - \int_a^b g$$

since the integral is linear. □

A special cases of the above lemma is the inequality

$$(b-a) \inf_{x\in[a,b]} f(x) \le \int_a^b f \le (b-a) \sup_{x\in[a,b]} f(x)$$

which we could also just directly get from (1.2) by using $\phi_- = \inf f$ and $\phi_+ = \sup f$ as minorant and majorant.

We can also use that functions which are constructed as minimum or maximum of two integrable functions, and hence in particular $|f| = \max(f, -f)$ are again integrable.

Proposition 1.16. *Suppose that f and g are integrable on $[a,b]$. Then $\max(f,g)$ and $\min(f,g)$ are both Riemann integrable. In particular $|f|$ is Riemann integrable* *and we have*

$$|\int_a^b f| \leq \int_a^b |f|. \tag{1.5}$$

This inequality is often called the triangle inequality for integrals and is extremely useful in applications.

Proof. We have $\max(f,g) = g + \max(f-g,0)$, $\min(h,0) = -\max(-h,0)$ and $|f| = \max(f,0) - \min(f,0)$. Using these relations and Proposition 1.13, it is enough to prove that if f is integrable on $[a,b]$, then so is $\max(f,0)$.

Now the function $x \mapsto \max(x,0)$ is order-preserving (if $x \leq y$ then $\max(x,0) \leq \max(y,0)$) and non-expanding (we have $|\max(x,0) - \max(y,0)| \leq |x-y|$, as can be established by an easy case-check, according to the signs of x, y). It follows that if $\phi_- \leq f \leq \phi_+$ are minorant and majorant for f then $\max(\phi_-,0) \leq \max(f,0) \leq \max(\phi_+,0)$ are minorant and majorant for $\max(f,0)$ (it is obvious that they are both step functions). Moreover,

$$I(\max(\phi_+,0)) - I(\max(\phi_-,0)) \leq I(\phi_+) - I(\phi_-).$$

Since f is integrable, this can be made arbitrarily small.

Finally, as both $f \leq |f|$ and $-f \leq |f|$ we can apply Proposition 1.15 to see that both $\int_a^b f \leq \int_a^b |f|$ and $-\int_a^b f = \int_a^b -f \leq \int_a^b |f|$, which yields the triangle inequality (1.5). □

Finally, we look at products.

Proposition 1.17. *Suppose that $f, g : [a,b] \to \mathbb{R}$ are two integrable functions. Then their product fg is integrable.*

Proof. Write $f = f_+ - f_-$, where $f_+ = \max(f,0)$ and $f_- = -\min(f,0)$, and similarly for g. Then $fg = f_+g_+ - f_-g_+ - f_+g_- + f_-g_-$, and so it suffices to prove the statement for non-negative functions such as $f_\pm, g_\pm$. Suppose, then, that $f, g \geq 0$. Let $\varepsilon > 0$, and let $\phi_- \leq f \leq \phi_+$, $\psi_- \leq g \leq \psi_+$ be minorants and majorants for f, g with $I(\phi_+) - I(\phi_-), I(\psi_+) - I(\psi_-) \leq \varepsilon$. Replacing ϕ_- with $\max(\phi_-,0)$ if necessary (and similarly for ψ_-), we may assume that $\phi_-, \psi_- \geq 0$ pointwise. Replacing ϕ_+ with $\min(\phi_+, M)$, where $M = \max\{\sup_{[a,b]} f, \sup_{[a,b]} g\}$ (and similarly for ψ_+) we may assume that $\phi_+, \psi_+ \leq M$ pointwise. By refining partitions if necessary, we may assume that all of these step functions are adapted to the same partition $\mathcal{P}$. Now observe that $\phi_-\psi_-, \phi_+\psi_+$ are both step functions and that $\phi_-\psi_- \leq fg \leq \phi_+\psi_+$ pointwise. Moreover, if $0 \leq u, v, u', v' \leq M$ and $u \leq u'$, $v \leq v'$ then we have

$$u'v' - uv = (u'-u)v' + (v'-v)u \leq M(u'-u+v'-v). \tag{1.6}$$

Applying this on each interval of the partition $\mathcal{P}$, with $u = \phi_-$, $u' = \phi_+$, $v = \psi_-$, $v' = \psi_+$, we have

$$I(\phi_+\psi_+) - I(\phi_-\psi_-) \leq M(I(\phi_+) - I(\phi_-) + I(\psi_+) - I(\psi_-)) \leq 2\varepsilon M.$$

Since $\varepsilon > 0$ was arbitrary, the result follows. □

Remark. Here is a sketch of an alternative proof, which is arguably a little slicker, or at least easier notationally. Note the identity $fg = \frac{1}{4}(f+g)^2 - \frac{1}{4}(f-g)^2$. Thus it suffices to show that if f is integrable then so is f^2. Replacing f by $|f|$, we may assume that $f \geq 0$ pointwise. Then proceed as above but with $f = g$, $\phi_- = \psi_-$, $\phi_+ = \psi_+$. In place of (1.6) one may instead use $(u')^2 - u^2 \leq 2M(u' - u)$.

CHAPTER 2

Basic theorems about the integral

In this section we show that the integrable functions are in rich supply.

2.1. Continuous functions are integrable

Let $\mathcal{P}$ be a partition of $[a,b]$, $a = x_0 < x_1 < \cdots < x_n = b$. The *mesh* of $\mathcal{P}$ is defined to be $\max_i(x_i - x_{i-1})$. Thus if $\mathrm{mesh}(\mathcal{P}) \le \delta$ then every interval in the partition $\mathcal{P}$ has length at most δ. To give an example, if $[a,b] = [0,1]$ and if $x_i = \frac{i}{N}$ then the mesh is $1/N$.

THEOREM 2.1. *Continuous functions $f : [a,b] \to \mathbb{R}$ are integrable.*

PROOF. Since f is continuous on a closed and bounded interval, f is also bounded. We will also use the fact that a continuous function f is *uniformly* continuous. Let $\varepsilon > 0$, and let $\delta > 0$ be so small that $|f(x) - f(y)| \le \varepsilon$ whenever $|x-y| \le \delta$. Let $\mathcal{P}$ be a partition with mesh $< \delta$. Let ϕ_+ be the step function whose value on (x_{i-1}, x_i) is $\sup_{x\in[x_{i-1},x_i]} f(x)$ and which takes the value $f(x_i)$ at the points x_i, and let ϕ_- be the step function whose value on (x_{i-1}, x_i) is $\inf_{x\in[x_{i-1},x_i]} f(x)$ and which takes the value $f(x_i)$ at the points x_i.

By construction, ϕ_+ is a majorant for f and ϕ_- is a minorant. Since a continuous function on a closed bounded interval attains its bounds, there are $\xi_-, \xi_+ \in [x_{i-1}, x_i]$ such that $\sup_{x\in[x_{i-1},x_i]} f(x) = f(\xi_+)$ and $\inf_{x\in[x_{i-1},x_i]} f(x) = f(\xi_-)$.

For $x \in (x_{i-1}, x_i)$ we have $\phi_+(x) - \phi_-(x) \le f(\xi_+) - f(\xi_-) \le \varepsilon$. Therefore $\phi_+(x) - \phi_-(x) \le \varepsilon$ for all $x \in [a,b]$.

It follows that $I(\phi_+) - I(\phi_-) \le \varepsilon(b-a)$. Since ε was arbitrary, this concludes the proof. □

We can strengthen this result, and allow the function to be discontinuous at finitely many points.

THEOREM 2.2. *Let $f : [a,b] \to \mathbb{R}$ be bounded. Suppose that that there is a finite set $S \subset [a,b]$ so that f is continuous at every point $x \in [a,b] \setminus S$. Then f is integrable.*

The theorem implies in particular that if $f : (a,b) \to \mathbb{R}$ is continuous and bounded, then it is integrable as any extension to $[a,b]$ will be bounded on $[a,b]$

and continuous on $(a, b) = [a, b] \setminus \{a, b\}$. Indeed, as we shall see below, proving this special case will essentially be all we need to do to get the full theorem. This result would apply, for example, to the function $f(x) = \sin(1/x)$ on $(0, 1)$.

Proof of Theorem 2.2. We first show

Claim 1: Suppose that $f : [c, d] \to \mathbb{R}$ is bounded and continuous on (c, d). Then f is integrable on $[c, d]$.

To prove this we let M be so that $|f| \leq M$ on $[c, a]$ and fix any $\varepsilon > 0$, which we can assume without loss of generatlity to be so that $\varepsilon < \frac{1}{2}(d - c)$. Then f is continuous, and hence uniformly continuous, on $[c + \varepsilon, d - \varepsilon]$. Let $\delta > 0$ be such that if $x, y \in [c + \varepsilon, c - \varepsilon]$ and $|x - y| \leq \delta$ then $|f(x) - f(y)| \leq \varepsilon$, and consider a partition $\mathcal{P}$ with $c = x_0$, $c + \varepsilon = x_1$, $d - \varepsilon = x_{n-1}$, $d = x_n$ and mesh $\leq \delta$.

Let ϕ_+ be the step function whose value on (x_{i-1}, x_i) is $\sup_{x \in [x_{i-1}, x_i]} f(x)$ when $i = 2, \ldots, n - 1$, and whose value on (x_0, x_1) and (x_{n-1}, x_n) is M.

Let ϕ_- be the step function whose value on (x_{i-1}, x_i) is $\inf_{x \in [x_{i-1}, x_i]} f(x)$ when $i = 2, \ldots, n - 1$, and whose value on (x_0, x_1) and (x_{n-1}, x_n) is $-M$.

Then $\phi_- \leq f \leq \phi_+$ on $[c, d]$. As in the proof of the previous theorem, we have $|\phi_+(x) - \phi_-(x)| \leq \varepsilon$ when $x \in (x_{i-1}, x_i)$, $i = 2, \ldots, n - 1$. On (x_0, x_1) and (x_{n-1}, x_n) we have the trivial bound $|\phi_+(x) - \phi_-(x)| \leq 2M$. Thus

$$I(\phi_+) - I(\phi -) \leq (d - c - 2\varepsilon)\varepsilon + 2M \cdot 2\varepsilon,$$

which can be made arbitrarily small by taking ε arbitrarily small.

Having thus proved the auxiliary Claim 1, we can now obtain the full claim of the theorem by writing the elements of $S = \{s_1, \ldots, s_n\}$ in increasing order $a \leq s_1 < \ldots < s_n \leq b$ and using that f is continuous on each of the open intervals $(a, s_1), (s_1, s_2), \ldots, (s_n, b)$, so integrable over all closed intervals $[a, s_1], [s_1, s_2], \ldots, [s_n, b]$. The second part of Proposition 1.11 (applied n times) thus gives that f is integrable over $[a, b]$. □

In the first chapter, we gave a simple example of a nonnegative function f which has zero integral, but is not identically zero. The following simple lemma shows that this cannot happen in the world of continuous functions.

Lemma 2.3. *Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function with $f \geq 0$ pointwise and $\int_a^b f = 0$. Then $f(x) = 0$ for $x \in [a, b]$.*

Proof. Suppose not. Then there is some point $x \in [a, b]$ with $f(x) > 0$. We can thus set $\varepsilon := f(x) > 0$. Since f is continuous, there is some $\delta > 0$ such that if $|x - y| \leq \delta$ then $|f(x) - f(y)| \leq \varepsilon/2$, and hence $f(x) \geq \varepsilon/2$. The set of all $y \in [a, b]$

with $|x-y| \leq \delta$ is a subinterval $I \subset [a,b]$ with length at least $\min(b-a,\delta)$, and so

$$\int f \geq \int_I f \geq \frac{\varepsilon}{2}\min(b-a,\delta) > 0.$$

□

2.2. Mean value theorems

The integrals of continuous functions satisfy various "mean value theorems". Here is a simple instance of such a result.

PROPOSITION 2.4. *Suppose that $f : [a,b] \to \mathbb{R}$ is continuous. Then there is some $\xi \in [a,b]$ such that*

$$\int_a^b f = (b-a)f(\xi).$$

PROOF. Since f is continuous, it attains its maximum M and its minimum m. As $m \leq f \leq M$, and as the integral is order-preserving, see Proposition 1.15, we hence get

$$m(b-a) \leq \int_a^b f \leq M(b-a),$$

which implies that

$$m \leq \frac{1}{b-a}\int_a^b f \leq M.$$

By the intermediate value theorem, f attains every value in $[m,M]$, and in particular there is some c such that

$$f(\xi) = \frac{1}{b-a}\int_a^b f.$$

□

The following slightly more complicated result, which generalises the above, may be established in essentially the same way.

PROPOSITION 2.5. *Suppose that $f : [a,b] \to \mathbb{R}$ is continuous, and that $w : [a,b] \to \mathbb{R}$ is a nonnegative integrable function. Then there is some $\xi \in [a,b]$ such that*

$$\int_a^b fw = f(\xi)\int_a^b w.$$

PROOF. First one should remark that fw is indeed integrable, this being a consequence of Proposition 1.17. As in the proof of Proposition 2.4, write M, m for the maximum and minimum of f respectively. Then $mw \leq fw \leq Mw$ pointwise, and so

$$m\int_a^b w \leq \int_a^b fw \leq M\int_a^b w.$$

If $\int_a^b w = 0$ then the result follows immediately; otherwise, we must have $\int_a^b w$ (since $w \geq 0$ on $[a, b]$) so we may divide through to get

$$m \leq \frac{\int_a^b fw}{\int_a^b w} \leq M.$$

Since both m and M are values attained by f, the result now follows from the intermediate value theorem. □

Remark. Just to be clear, Proposition 2.4 is the case $w = 1$ of Proposition 2.5.

2.3. Monotone functions are integrable

A function $f : [a, b] \to \mathbb{R}$ is said to be *monotone* if it is either increasing (meaning $x \leq y$ implies $f(x) \leq f(y)$) or decreasing (meaning $x \leq y$ implies $f(x) \geq f(y)$).

THEOREM 2.6. *Monotone functions $f : [a, b] \to \mathbb{R}$ are integrable.*

PROOF. By replacing f with $-f$ if necessary we may suppose that f is monotone increasing, i.e. $f(x) \leq f(y)$ whenever $x \leq y$. Since $f(a) \leq f(x) \leq f(b)$, f is automatically bounded.

Let n be a positive integer, and consider the partition of $[a, b]$ into n equal parts. Thus $\mathcal{P}$ is $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$, with $x_i = a + \frac{i}{n}(b-a)$. On (x_{i-1}, x_i), define $\phi_+(x) = f(x_i)$ and $\phi_-(x) = f(x_{i-1})$. Define $\phi_-(x_i) = f(x_i)$ and $\phi_+(x_i) = f(x_i)$. Then ϕ_+ is a majorant for f and ϕ_- is a minorant. We have

$$\begin{aligned} I(\phi_+) - I(\phi_-) &= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(x_i - x_{i-1}) \\ &= \frac{b-a}{n} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \\ &= \frac{1}{n}(b-a)(f(b) - f(a)). \end{aligned}$$

Taking n large, this can be made as small as desired. □

CHAPTER 3

Riemann sums

The way in which we have been developing the integral is closely related to the approach taken by Darboux. In this chapter we discuss what is essentially Riemann's original way of defining the integral, and show that it is equivalent. This is of more than merely historical interest: the equivalence of the definitions has several useful consequences.

If $\mathcal{P}$ is a partition and $f : [a,b] \to \mathbb{R}$ is a function then by a *Riemann sum* adapted to $\mathcal{P}$ we mean an expression of the form

$$\Sigma(f;\mathcal{P},\vec{\xi}) = \sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1}),$$

where $\vec{\xi} = (\xi_1, \dots, \xi_n)$ and $\xi_j \in [x_{j-1}, x_j]$.

PROPOSITION 3.1. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Fix a sequence of partitions $\mathcal{P}^{(i)}$. For each i, let $\Sigma(f,\mathcal{P}^{(i)},\vec{\xi}^{(i)})$ be a Riemann sum adapted to $\mathcal{P}^{(i)}$. Suppose that there is some constant c such that, no matter how $\vec{\xi}^{(i)}$ is chosen, $\Sigma(f;\mathcal{P}^{(i)},\vec{\xi}^{(i)}) \to c$. Then f is integrable and $c = \int_a^b f$.*

PROOF. Let $\varepsilon > 0$. Let i be chosen so that $\Sigma(f;\mathcal{P}^{(i)},\vec{\xi}^{(i)}) \le c + \varepsilon$, no matter which $\vec{\xi}^{(i)}$ is chosen. Write $\mathcal{P} = \mathcal{P}^{(i)}$, and suppose that $\mathcal{P}$ is $a = x_0 \le \cdots \le x_n = b$. For each j, choose some point $\xi_j \in [x_{j-1}, x_j]$ such that $f(\xi_j) \ge \sup_{x \in [x_{j-1},x_j]} f(x) - \varepsilon$. (Note that f does not necessarily attain its supremum on this interval.) Let ϕ_+ be a step function taking the value $f(\xi_j)+\varepsilon$ on (x_{j-1}, x_j), and with $\phi_+(x_j) = f(x_j)$. Then ϕ_+ is a majorant for f. It is easy to see that

$$I(\phi_+) = \varepsilon(b-a) + \Sigma(f;\mathcal{P},\vec{\xi}).$$

We therefore have

$$I(\phi_+) \le \varepsilon(b-a) + c + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$\inf_{\phi_+} I(\phi_+) \le c.$$

By an identical argument,

$$\sup_{\phi_-} I(\phi_-) \ge c.$$

Therefore

$$c \le \sup_{\phi_-} I(\phi_-) \le \inf_{\phi_+} I(\phi_+) \le c,$$

and so all these quantities equal c. □

This suggests that we could use such Riemann sums to *define* the integral, perhaps by taking some natural choice for the sequences of partitions $\mathcal{P}^{(i)}$ such as $x_j^{(i)} = a + \frac{j}{i}(b-a)$ (the partition into i equal parts). However, Proposition 3.1 does not imply that this definition is equivalent to the one we have been using, since we have not shown that the Riemann sums converge if f is integrable. In fact, this requires an extra hypothesis. Recall that the *mesh* $\mathrm{mesh}(\mathcal{P})$ of a partition is the length of the longest subinterval in $\mathcal{P}$.

Proposition 3.2. *Let $\mathcal{P}^{(i)}$, $i = 1, 2, \dots$ be a sequence of partitions satisfying $\mathrm{mesh}(\mathcal{P}^{(i)}) \to 0$. Suppose that f is integrable. Then $\lim_{i\to\infty} \Sigma(f; \mathcal{P}^{(i)}, \vec{\xi}^{(i)}) = \int_a^b f$, no matter what choice of $\vec{\xi}^{(i)}$ we make.*

Proof. Throughout the proof, write $M := \sup_{x\in[a,b]} |f(x)|$. Let $\mathcal{P} : a = x_0 \le x_1 \le \cdots \le x_n = b$ be a partition. In this proof it is convenient to introduce the notion of the *optimal* majorant $\phi_+^{\mathcal{P}}$ for f relative to $\mathcal{P}$ (and similarly minorant). This is the majorant defined by

$$\phi_+^{\mathcal{P}} := \begin{cases} \sup_{x\in(x_{i-1},x_i)} f(x) & \text{on } (x_{i-1}, x_i) \\ f(x_i) & \text{at the points } x_i. \end{cases}$$

It is easy to see that if ϕ_+ is any majorant for f adapted to $\mathcal{P}$, then $I(\phi_+^{\mathcal{P}}) \le I(\phi_+)$. Similarly, $I(\phi_-^{\mathcal{P}}) \ge I(\phi_-)$, and so

$$I(\phi_+^{\mathcal{P}}) - I(\phi_-^{\mathcal{P}}) \le I(\phi_+) - I(\phi_-).$$

Let $\varepsilon > 0$. Since f is integrable it follows from what we just said that there is a partition $\mathcal{P} : a = x_0 \le x_1 \le \cdots \le x_n = b$ such that $I(\phi_+^{\mathcal{P}}) - I(\phi_-^{\mathcal{P}}) < \varepsilon$. In particular, since $I(\phi_-) \le \int_a^b f$ for any minorant ϕ_-,

$$I(\phi_+^{\mathcal{P}}) \le \int_a^b f + \varepsilon. \tag{3.1}$$

Set $\delta := \varepsilon/nM$. Let $\mathcal{P}' : a = x'_0 \le x'_1 \le \cdots \le x'_{n'} = b$ be any partition with $\mathrm{mesh}(\mathcal{P}') \le \delta$, and consider an arbitrary Riemann sum

$$\Sigma(f; \mathcal{P}', \vec{\xi}') = \sum_{j=1}^{n'} f(\xi'_j)(x'_j - x'_{j-1}).$$

This is equal to $I(\psi)$, where the step function ψ is defined to be $f(\xi'_j)$ on (x'_{j-1}, x'_j) and $f(x'_j)$ at the x'_j.

Let us compare ψ and the optimal majorant $\phi_+^{\mathcal{P}}$.

Say that j is *good* if $[x'_{j-1}, x'_j] \subset (x_{i-1}, x_i)$ for some i. If j is good then, for $t \in (x'_{j-1}, x'_j)$,

$$\psi(t) = f(\xi'_j) \leq \sup_{x \in [x'_{j-1}, x'_j]} f(x) \leq \sup_{x \in (x_{i-1}, x_i)} f(x) = \phi_+^{\mathcal{P}}(t). \tag{3.2}$$

If j is bad (i.e. not good) then we cannot assert such a bound, but we do have the trivial bound

$$\psi(t) \leq \phi_+^{\mathcal{P}}(t) + 2M \tag{3.3}$$

for all j.

Now if j is bad then we have $x_i \in [x'_{j-1}, x'_j]$ for some i. No x_i can belong to more than two intervals $[x'_{j-1}, x'_j]$, so there cannot be more than $2n$ bad values of j. Therefore the total length of the corresponding intervals (x'_{j-1}, x'_j) is at most $2\delta n = 2\varepsilon/M$.

It therefore follows, using (3.2) on the good intervals and (3.3) on the bad, that

$$\Sigma(f; \mathcal{P}', \vec{\xi}') = I(\psi) \leq I(\phi_+^{\mathcal{P}}) + 2M \cdot \frac{2\varepsilon}{M} = I(\phi_+^{\mathcal{P}}) + 4\varepsilon. \tag{3.4}$$

Combining this with (3.1) yields

$$\Sigma(f; \mathcal{P}', \vec{\xi}') \leq \int_a^b f + 5\varepsilon.$$

There is a similar lower bound, proven in an analogous manner.

Since ε was arbitrary, this concludes the proof. □

Proposition 3.1 and 3.2 together allow us to give an alternative definition of the integral. This is basically Riemann's original definition.

PROPOSITION 3.3. *Let $f : [a, b] \to \mathbb{R}$ be a function. Let $\mathcal{P}^{(i)}$, $i = 1, 2, \ldots$ be a sequence of partitions with* $\mathrm{mesh}(\mathcal{P}^{(i)}) \to 0$. *Then f is integrable if and only if $\lim_{i\to\infty} \Sigma(f, \mathcal{P}^{(i)}, \vec{\xi^{(i)}})$ is equal to some constant c, independently of the choice of $\vec{\xi^{(i)}}$. If this is so, then $\int_a^b f = c$.*

Finally, we caution that it is important that the limit must exist for *any* choice of $\vec{\xi}^{(i)}$. Suppose, for example, that $[a, b] = [0, 1]$ and that $\mathcal{P}^{(i)}$ is the partition into i equal parts, thus $x_j^{(i)} = \frac{j}{i}$ for $j = 1, \ldots, i$. Take $\xi_j^{(i)} = \frac{j}{i}$; then the Riemann sum $\Sigma(f, \mathcal{P}^{(i)}, \vec{\xi}^{(i)})$ is equal to

$$S_i(f) := \frac{1}{i} \sum_{j=1}^{i} f(\frac{j}{i}).$$

By Proposition 3.2, if f is integrable then

$$S_i(f) \to \int_a^b f.$$

However, the converse is not true. Consider, for example, the function f introduced in the first chapter, with $f(x) = 1$ for $x \in \mathbb{Q}$ and $f(x) = 0$ otherwise. This function is not integrable, as we established in that chapter. However,

$$S_i(f) = 1 \qquad \text{for all } i.$$

CHAPTER 4

The Fundamental Theorem of Calculus

It is a well-known fact, which goes by the name of "the fundamental theorem of calculus" that "integration and differentiation are inverse to one another". Our objective in this chapter is to prove rigorous versions of this fact. We will prove two statements, sometimes known as the first and second fundamental theorems of calculus respectively, though there does not seem to be complete consensus on this matter. The first theorem deals with integration followed by differentiation. In the second theorem, we differentiate, then integrate.

So far, we have considered integrals of the form $\int_a^b f$. But we now want to vary the interval over which we integrate, as follows. We define the function

$$F(x) = \int_a^x f$$

for $x \in [a, b]$. Under suitable assumptions, we will show that F is differentiable with derivative f.

4.1. First fundamental theorem of calculus

The first thing to notice is that it is just not true that integration and differentiation are inverses without some additional assumptions.

Example. If f is not continuous, then F can be differentiable but it need not be the case that $F' = f$. For example, let $f : [0, 1] \to \mathbb{R}$ be the function that takes value 1 at $x = \frac{1}{2}$ but that is 0 elsewhere. Then F is identically zero. Hence, F is differentiable and F' is the zero function. This shows that in general, when you integrate and then differentiate, you might not get the original function back.

Example. We note that F is not necessarily differentiable assuming only that f is Riemann-integrable. For example if we take the function f defined by $f(t) = 0$ for $t \leq \frac{1}{2}$ and $f(t) = 1$ for $t > \frac{1}{2}$ then f is integrable on $[0, 1]$, and the function $F(x) = \int_0^x f(t)dt$ is given by $F(x) = 0$ for $x \leq \frac{1}{2}$ and $F(x) = x - \frac{1}{2}$ for $\frac{1}{2} \leq x \leq 1$. Evidently, F fails to be differentiable at $\frac{1}{2}$.

However, the first fundamental theorem of calculus asserts that the function F *is* differentiable and $F' = f$, as long as f is continuous.

Theorem 4.1 (First fundamental theorem of calculus). *Suppose that f is integrable on (a,b). Define a new function $F : [a,b] \to \mathbb{R}$ by*

$$F(x) := \int_a^x f.$$

Then F is continuous. Moreover, if f is continuous at $c \in (a,b)$ then F is differentiable at c and $F'(c) = f(c)$.

Proof. The fact that F is continuous follows immediately from the fact that f is bounded (which it must be, as it is integrable), say by M. Then

$$|F(c+h) - F(c)| = \left| \int_c^{c+h} f \right| \leq \int_c^{c+h} |f| \leq Mh.$$

In fact, this argument directly establishes that F is uniformly continuous (and in fact Lipschitz).

Now we turn to the second part. Suppose that $c \in (a,b)$ and that $h > 0$ is sufficiently small that $c + h < b$. We have

$$F(c+h) - F(c) = \int_c^{c+h} f.$$

Let $\varepsilon > 0$. Since f is continuous at c, there is a $\delta > 0$ such that for all $t \in [c, c+\delta]$, we have $|f(t) - f(c)| \leq \varepsilon$. Therefore, for any $h \in (0, \delta)$,

$$|F(c+h) - F(c) - hf(c)| = \left| \int_c^{c+h} (f(t) - f(c))dt \right| \leq \varepsilon h.$$

Divide through by h:

$$\left| \frac{F(c+h) - F(c)}{h} - f(c) \right| \leq \varepsilon. \tag{4.1}$$

Essentially the same argument works for $h < 0$ (in fact, exactly the same argument works if we interpret $\int_c^{c+h} f$ in the natural way as $-\int_{c+h}^c f$). Statement (4.1) is exactly the definition of F being differentiable at c with derivative $f(c)$. □

4.2. Second fundamental theorem of calculus

We turn now to the "second form" of the fundamental theorem, which deals with differentiation, followed by integration. Here, we cannot get such a strong result as the first fundamental theorem.

Consider, for instance, the function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(0) = 0$ and $F(x) = x^2 \sin \frac{1}{x^2}$ for $x \neq 0$. Then it is a standard exercise to show that F is differentable everywhere, with $f = F'$ given by $f(0) = 0$ and $f(x) = 2x\sin(1/x^2) - \frac{2}{x}\cos(1/x^2)$. In particular, f is unbounded on any interval containing 0, and so it has no majorants and is not integrable according to our definition.

An even worse example (the Volterra function) can be constructed with f bounded, but still not integrable. This construction is rather elaborate and we will not give it here. These constructions show that a hypothesis of integrability of F' should be built into any statement of the second fundamental theorem of calculus.

THEOREM 4.2 (Second fundamental theorem of calculus). *Suppose that* $F : [a,b] \to \mathbb{R}$ *is continuous on* $[a,b]$ *and differentiable on* (a,b)*. Suppose furthermore that its derivative* F' *is integrable on* (a,b)*. Then*

$$\int_a^b F' = F(b) - F(a).$$

PROOF. Let $\mathcal{P}$ be a partition, $a = x_0 < x_1 < \cdots < x_n = b$. We claim that some Riemann sum $\Sigma(F';\mathcal{P},\xi)$ is equal to $F(b) - F(a)$. By Proposition 3.2 (the harder direction of the equivalence between integrability and limits of Riemann sums), the second fundamental theorem follows immediately from this.

The claim is an almost immediate consequence of the mean value theorem. By that theorem, we may choose $\xi_i \in (x_{i-1}, x_i)$ so that $F'(\xi_i)(x_i - x_{i-1}) = F(x_i) - F(x_{i-1})$. Summing from $i = 1$ to n gives

$$\Sigma(F';\mathcal{P},\xi) = \sum_{i=1}^{n}(F(x_i) - F(x_{i-1})) = F(b) - F(a).$$

□

4.3. Integration by parts

Everyone knows that integration by parts says that

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g.$$

We are now in a position to prove a rigorous version of this.

PROPOSITION 4.3. *Suppose that* $f, g : [a,b] \to \mathbb{R}$ *are continuous functions, differentiable on* (a,b)*. Suppose that the derivatives* f', g' *are integrable on* (a,b)*. Then* fg' *and* $f'g$ *are integrable on* (a,b)*, and*

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g.$$

PROOF. We use the second form of the fundamental theorem of calculus, applied to the function $F = fg$. We know from basic differential calculus that F is differentiable and $F' = f'g + fg'$. By Proposition 1.17 and the assumption that f', g' are integrable, F' is integrable on (a,b). Applying the fundamental theorem gives

$$\int_a^b F' = F(b) - F(a),$$

which is obviously equivalent to the stated claim. □

4.4. Substitution

PROPOSITION 4.4 (Substitution rule). *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and that $\phi : [c, d] \to [a, b]$ is continuous on $[c, d]$, has $\phi(c) = a$ and $\phi(d) = b$, and maps (c, d) to (a, b). Suppose moreover that ϕ is differentiable on (c, d) and that its derivative ϕ' is integrable on this interval. Then*

$$\int_a^b f = \int_c^d (f \circ \phi)\phi'.$$

Remark. It may help to see the statement written out in full:

$$\int_a^b f(x)dx = \int_c^d f(\phi(t))\phi'(t)dt.$$

PROOF. Let us first remark that $f \circ \phi$ is continuous and hence integrable on $[c, d]$. It therefore follows from Proposition 1.17 that $(f \circ \phi)\phi'$ is integrable on $[c, d]$, so the statement does at least make sense.

Since f is continuous on $[a, b]$, it is integrable. The first fundamental theorem of calculus implies that its antiderivative

$$F(x) := \int_a^x f$$

is continuous on $[a, b]$, differentiable on (a, b) and that $F' = f$.

By the chain rule and the fact that $\phi((c, d)) \subset (a, b)$, $F \circ \phi$ is differentiable on (c, d), and

$$(F \circ \phi)' = (F' \circ \phi)\phi' = (f \circ \phi)\phi'.$$

By the remarks at the start of the proof, it follows that $(F \circ \phi)'$ is an integrable function. By the second form of the fundamental theorem,

$$\begin{aligned}
\int_c^d (f \circ \phi)\phi' &= \int_c^d (F \circ \phi)' \\
&= (F \circ \phi)(d) - (F \circ \phi)(c) \\
&= F(b) - F(a) \\
&= F(b) = \int_a^b f.
\end{aligned}$$

□

CHAPTER 5

Limits and the integral

5.1. Interchanging the order of limits and integration

Suppose we have a sequence of functions f_n converging to a limit function f. If this convergence is merely pointwise, integration need not preserve the limit.

EXAMPLE. There is a sequence of integrable functions $f_n : [0,1] \to \mathbb{R}$ (in fact, step functions) such that $f_n(x) \to 0$ pointwise for all $x \in [0,1]$ but $\int f_n = 1$ for all n. Thus $\lim_{n\to\infty} \int_0^1 f_n = 1$, whilst $\int_0^1 \lim_{n\to\infty} f_n = 0$, and so interchange of integration and limit is not valid in this case.

PROOF. Define $f_n(x)$ to be equal to n for $0 < x < \frac{1}{n}$ and 0 elsewhere. □

However, if $f_n \to f$ uniformly then the situation is much better.

THEOREM 5.1. *Suppose that $f_n : [a,b] \to \mathbb{R}$ are integrable, and that $f_n \to f$ uniformly on $[a,b]$. Then f is also integrable, and*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f = \int_a^b \lim_{n\to\infty} f_n.$$

PROOF. Let $\varepsilon > 0$. Since $f_n \to f$ uniformly, there is some choice of n such that we have $|f_n(x) - f(x)| \le \varepsilon$ for all $x \in [a,b]$.

Now f_n is integrable, and so there is a majorant ϕ_+ and a minorant ϕ_- for f_n with $I(\phi_+) - I(\phi_-) \le \varepsilon$.

Define $\tilde{\phi}_+ := \phi_+ + \varepsilon$ and $\tilde{\phi}_- := \phi_- - \varepsilon$. Then $\tilde{\phi}_-, \tilde{\phi}_+$ are minorant/majorant for f. Moreover

$$\begin{aligned} I(\tilde{\phi}_+) - I(\tilde{\phi}_-) &\le 2\varepsilon(b-a) + I(\phi_+) - I(\phi_-) \\ &\le 2\varepsilon(b-a) + \varepsilon. \end{aligned}$$

Since ε was arbitrary, this shows that f is integrable. Now

$$\left|\int_a^b f_n - \int_a^b f\right| \le \int_a^b |f_n - f| \le (b-a) \sup_{x\in[a,b]} |f_n(x) - f(x)|.$$

Since $f_n \to f$ uniformly, it follows that

$$\lim_{n\to\infty} \left|\int_a^b f_n - \int_a^b f\right| = 0,$$

and hence that

$$\lim_{n\to\infty}\int_a^b f_n = \int_a^b f = \int_a^b \lim_{n\to\infty} f_n.$$

This concludes the proof. □

An immediate corollary of this is that we may integrate series term-by-term under suitable conditions.

Corollary 5.2. *Suppose that $\phi_i : [a,b] \to \mathbb{R}$, $i = 1, 2, \ldots$ are integrable functions and that $|\phi_i(x)| \le M_i$ for all $x \in [a,b]$, where $\sum_{i=1}^{\infty} M_i < \infty$. Then the sum $\sum_i \phi_i$ is integrable and*

$$\int_a^b \sum_i \phi_i = \sum_i \int_a^b \phi_i.$$

Proof. This is immediate from the Weierstrass M-test and Theorem 5.1, applied with $f_n = \sum_{i=1}^{n} \phi_i$. □

Now suppose we have a sequence $(a_i)_{i=0}^{\infty}$ of real numbers. Then the expression $\sum_{i=0}^{\infty} a_i x^i$ is called a (formal) power series. The word "formal" means that we are not actually evaluating the sum over i; indeed, this may well not be possible for a given choice of the a_i and x.

Definition 5.3. Given a formal power series $\sum_i a_i x^i$, we define its radius of convergence R to be the supremum of all $|x|$ for which the sum $\sum_{i=0}^{\infty} |a_i x^i|$ converges. If this sum converges for all x, we write $R = \infty$.

As a special case of the above results we obtain

Theorem 5.4. *Suppose a formal power series $\sum_{i=0}^{\infty} a_i x^i$ has radius of convergence R. Then the series converges for $|x| < R$, giving a well-defined function $f(x) = \sum_{i=0}^{\infty} a_i x^i$. Moreover, f is integrable over $[-r, r]$ for every $0 < r < R$ and we have*

$$\int_0^x f = \sum_{j=0}^{\infty} \tfrac{a_j}{j+1} x^{j+1} \text{ for every } x \in (-R, R).$$

Note that here we again use the standard convention of writing $\int_0^x f$ for $-\int_x^0 f$ if $x < 0$.

Proof. You have shown in Analysis 2 that a powerseries converges pointwise on $(-R, R)$, making f well defined on $(-R, R)$ and that it converges uniformly on every interval $[-r, r]$, $r < R$. Thus we can apply Theorem 5.1 to see that f is integrable on every such interval $[-r, r]$ and that

$$\int_c^d f(t)dt = \sum_{j=0}^{\infty} \int_c^d a_j t^j \, dt$$

for all $c \leq d$ with $[c, d] \subset [-r, r]$.

If $x \in (-R, R)$ we can apply this for $[0, x]$ (respectively $[x, 0]$ if $x < 0$) as these intervals are in $[-r, r]$ for $r := |x| < R$. Combined with he second version of the fundamental theorem of calculus (applied to the functions $\phi_j(x) = \frac{a_j}{j+1}x^{j+1}$ whose derivative is $\phi_j'(x) = a_j x^j$) we thus get

$$\int_0^x f(t)dt = \sum_{j=0}^{\infty} \int_0^x a_j t^j dt = \sum_{j=0}^{\infty} \tfrac{a_j}{j+1} x^{j+1} \tag{5.1}$$

as claimed. □

5.2. Interchanging the order of limits and differentiation

The behaviour of limits with respect to differentiation is much worse than the behaviour with respect to integration.

EXAMPLE. There is a sequence of functions $f_n : [0, 1] \to \mathbb{R}$, each continuously differentiable on $(0, 1)$, such that $f_n \to 0$ uniformly but such that f_n' does not converge at every point.

PROOF. Take $f_n(x) = \frac{1}{n}\sin(n^2 x)$. Then $f_n'(x) = -n\cos(n^2 x)$. Taking $x = \frac{\pi}{4}$, we see that if n is a multiple of 4 then $f_n'(x) = -n$, which certainly does not converge. □

If, however, we assume that the derivatives f_n' converge uniformly then we do have a useful result.

PROPOSITION 5.5. *Suppose that $f_n : [a, b] \to \mathbb{R}$, $n = 1, 2, \ldots$ is a sequence of continuous functions with the property that f_n is continuously differentiable on (a, b), that f_n converges pointwise to some function f on $[a, b]$, and that f_n' converges uniformly to some bounded function g on (a, b). Then f is differentiable on (a,b) and $f' = g$. In particular,* $\lim_{n\to\infty} f_n' = (\lim_{n\to\infty} f_n)'$.

PROOF. First note that, since the f_n' are continuous and $f_n' \to g$ uniformly, g is continuous. Since we are also assuming g is bounded, it follows from Theorem 2.2 that g is integrable.

We may therefore apply the first form of the fundamental theorem of calculus to g. Since g is continuous, the theorem says that if we define a function $F : [a, b] \to \mathbb{R}$ by

$$F(x) := \int_a^x g(t)dt$$

then F is differentiable with $F' = g$. By the second form of the fundamental theorem of calculus applied to f_n, we have

$$\int_a^x f_n'(t)dt = f_n(x) - f_n(a).$$

Taking limits as $n \to \infty$ and using the fact that $f_n \to f$ pointwise, we obtain

$$\lim_{n\to\infty} \int_a^x f_n'(t)dt = f(x) - f(a).$$

However, since $f_n' \to g$ uniformly, it follows from Theorem 5.1 that

$$\lim_{n\to\infty} \int_a^x f_n'(t)dt = \int_a^x g(t)dt.$$

Thus

$$F(x) = \int_a^x g(t)dt = f(x) - f(a).$$

It follows immediately that f is differentiable and that its derivative is the same as that of F, namely g. □

Remark. Note that the statement of Proposition 5.5 involves only differentiation. However, the proof involves a considerable amount of the theory of integration. This is a theme that is seen throughout mathematical analysis. For example, the nice behaviour of complex differentiable functions (which you will see in course A2 next year) is a consequence of Cauchy's *integral* formula.

With these results in hand, we can now also give alternative proofs to some results you will have seen in Analysis II. The proofs you saw there were slightly unpleasant. The use of integration is the "correct" way to prove these statements.

Let us begin by recording a "series variant" of Proposition 5.5.

COROLLARY 5.6. *Suppose we have a sequence of continuous functions ϕ_i : $[a, b] \to \mathbb{R}$, continuously differentiable on (a, b), with $\sum_i \phi_i$ converging pointwise. Suppose that $|\phi_i'(x)| \leq M_i$ for all $x \in (a, b)$, where $\sum_i M_i < \infty$. Then $\sum \phi_i$ is differentiable and*

$$(\sum_i \phi_i)' = \sum_i \phi_i'.$$

PROOF. We can apply Proposition 5.5 with $f_n := \sum_{i=1}^n \phi_i$ as the Weierstrass M-test ensures that $f_n' = \sum_{i=1}^n \phi_i'$ converges uniformly to some bounded function g. □

We want to apply this result in particular to provide an alternative proof of the 'differentiation theorem for power series" from Analysis II, i.e. of

THEOREM 5.7. *Suppose a formal power series $\sum_{i=0}^\infty a_i x^i$ has radius of convergence R. Then the series converges for $|x| < R$, giving a well-defined function $f(x) = \sum_{i=0}^\infty a_i x^i$. Moreover, f is differentiable on $(-R, R)$, and its derivative is given by term-by-term differentiation, that is to say $f'(x) = \sum_{i=1}^\infty i a_i x^{i-1}$. Moreover, the radius of convergence for this power series for f' is at least R.*

For the proof of this theorem, we use the following simple lemma

LEMMA 5.8. *Suppose that $0 \le \lambda < 1$. Then $\sum_{i=0}^\infty \lambda^i$ and $\sum_{i=1}^\infty i\lambda^{i-1}$ both converge.*

PROOF OF LEMMA 5.8. By the well-known geometric series formula we have

$$\sum_{i=0}^{n-1} \lambda^i = \frac{1-\lambda^n}{1-\lambda}.$$

Letting $n \to \infty$ gives the first statement immediately, the value of the sum being $\frac{1}{1-\lambda}$.

For the second statement, we differentiate the geometric series formula. This gives

$$\sum_{i=1}^{n-1} i\lambda^{i-1} = \frac{1 + (n-1)\lambda^n - n\lambda^{n-1}}{(1-\lambda)^2},$$

which tends to $\frac{1}{(1-\lambda)^2}$ as $n \to \infty$. □

PROOF OF THEOREM 5.7. If $R = 0$, there is nothing to prove. Suppose that $R > 0$. Let R_1 satisfy $0 < R_1 < R$. We apply Corollary 5.6 with $\phi_i(x) = a_i x^i$ and $[a, b] = [-R_1, R_1]$. We need to check that the hypotheses of that result are satisfied. By definition of the radius of convergence, there is some R_0 satisfying $R_1 < R_0 \le R$ for which $\sum_i |a_i R_0^i|$ converges, and in particular $|a_i R_0^i|$ is bounded, uniformly in i: let us say that $|a_i R_0^i| \le K$. Then if $x \in [a, b]$ we have

$$|\phi_i(x)| \le K(\frac{R_1}{R_0})^i$$

and

$$|\phi_i'(x)| \le \frac{K}{R_0} i (\frac{R_1}{R_0})^{i-1}. \tag{5.2}$$

The first condition of Corollary 5.6, that is to say pointwise convergence of $\sum_i \phi_i(x)$, is now immediate from the first part of Lemma 5.8. Taking $M_i := \frac{K}{R_0} i (\frac{R_1}{R_0})^{i-1}$, we obtain the other condition of Corollary 5.6 from the second part of Lemma 5.8.

It now follows from Corollary 5.6 that f is differentiable on $(-R_1, R_1)$, and that is derivative is given by term-by-term differentiation of the power series for f. Since $R_1 < R$ was arbitrary, we may assert the same on $(-R, R)$.

Finally, it follows from (5.2) and Lemma 5.8 that the radius of convergence of the power series for f' is at least R_1. Since $R_1 < R$ was arbitrary, the radius of convergence of this power series is at least R, as claimed. □

By applying this theorem repeatedly, it follows that under the same assumptions f is infinitely differentiable on $(-R, R)$, with all of its derivatives being given by term-by-term differentiation.

CHAPTER 6

Improper integrals

If one attempts to assign a meaning to the integral of an unbounded function, or to the integral of a function over an unbounded domain, then one is trying to understand an *improper integral.*

Example. Consider the function $f(x) = \log x$. This is continuous on $(0, 1]$ but it is not integrable there since it is not bounded (it tends to $-\infty$ as $x \to 0$). However, it *is* integrable on any interval $[\varepsilon, 1]$, $\varepsilon > 0$.

By the second fundamental theorem of calculus (and the fact that if $F(x) = x \log x - x$ then $F'(x) = \log x$) we have

$$\int_{\varepsilon}^{1} \log x dx = [x \log x - x]_{\varepsilon}^{1} = -1 - \varepsilon \log \varepsilon + \varepsilon. \tag{6.1}$$

We now claim that

$$\lim_{\varepsilon \to 0^+} \varepsilon \log \varepsilon = 0, \tag{6.2}$$

This can either be shown using the fact that log is inverse to exp, see Example Sheet 4, Q2, or using the following proof that is based upon the fundamental theorem of Calculus:

As $\log(\sqrt{\varepsilon}) = \frac{1}{2}\log(\varepsilon)$ and as $\log(x)$ is continuously differentiable on $(0, \infty)$ with $\log'(x) = \frac{1}{x}$, we can use the second fundamental theorem of calculus to write

$$-\tfrac{1}{2}\log(\varepsilon) = \log(\sqrt{\varepsilon}) - \log(\varepsilon) = \int_{\varepsilon}^{\sqrt{\varepsilon}} \frac{1}{x} dx.$$

As $0 < \frac{1}{x} < \frac{1}{\varepsilon}$ on this interval we can hence bound

$$|\log \varepsilon| \leq 2 \int_{\varepsilon}^{\sqrt{\varepsilon}} \frac{1}{x} dx \leq 2 \int_{\varepsilon}^{\sqrt{\varepsilon}} \frac{1}{\varepsilon} dx \leq \frac{2}{\sqrt{\varepsilon}},$$

which implies that $|\varepsilon \log \varepsilon| \leq 2\sqrt{\varepsilon} \to 0$ as $\varepsilon \to 0^+$.

Combined, (6.1) and (6.2) imply that

$$\lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \log x dx = -1.$$

This will often be written as

$$\int_0^1 \log x dx = -1,$$

but strictly speaking, as remarked above, this is not an integral as discussed in this course.

EXAMPLE. Consider the function $f(x) = 1/x^2$. We would like to discuss the integral of this "on $[1,\infty)$", but this is not permitted by the way we have defined the integral, which requires a bounded interval. However, on any bounded interval $[1,K]$ we have

$$\int_1^K \frac{1}{x^2} dx = \big[-\frac{1}{x}\big]_1^K = 1 - \frac{1}{K}$$

by the second fundamental theorem of calculus. Therefore

$$\lim_{K\to\infty} \int_1^K \frac{1}{x^2} dx = 1.$$

This is invariably written

$$\int_1^\infty \frac{1}{x^2} dx = 1.$$

EXAMPLE. Define $f(x)$ to be $\log x$ if $0 < x \leq 1$, and $f(x) = \frac{1}{x^2}$ for $x \geq 1$. Then it makes sense to write

$$\int_0^\infty f(x) dx = 0,$$

by which we mean

$$\lim_{K\to\infty, \varepsilon\to 0} \int_\varepsilon^K f(x) dx = 0.$$

This is a combination of the preceding two examples.

More generally, given $-\infty \leq a < b \leq \infty$ and a function $f : (a,b) \to \mathbb{R}$ so that

- f is integrable over every interval $[c,d] \subset (a,b)$
- the limit
$$\lim_{\substack{c\to a^+ \\ d\to b^-}} \int_c^d f$$
exists, i.e. if for *all* sequences $c_n \to a$ and $d_n \to b$ with $a < c_n$ and $d_n < b$ the sequence $\int_{c_n}^{d_n} f$ converges to the same limit.

we can say that the *improper Riemann integral* of f over (a,b) exists. In such a situation one often writes $\int_a^b f$ also for such improper integrals.

Similarly, if a function is so that its improper Riemann integrals over (a,b) and over (b,c) exist, then one often writes $\int_a^c f$ for $\int_a^b f + \int_b^c f$.

However you should be aware that this "standard abuse of notation" hides the fact that these are *not* Riemann integrals as defined in this course and that one has to be very careful when working with improper integrals.

WARNING. Key results you have seen in this course, including the fundamental theorem of calculus and the fact that products of integrable functions are integrable etc, are NOT valid for improper integrals, though can be used on the closed intervals $[c, d]$ where f is integrable, to check whether the above improper integral exists.

WARNING. While a function $[c, d] \to \mathbb{R}$ which is Riemann integrable will always be integrable in the more general sense defined in A4 Integration (i.e. Lebesgue-integrable), the existence of an improper Riemann integral does NOT imply Lebesgue integrablity.

EXAMPLE. Define $f(x)$ to be $1/x$ for $0 < |x| \leq 1$, and $f(0) = 0$. Then f is unbounded as $x \to 0$, and so not Riemann integrable on $[-1, 1]$. If we excise the problematic region around 0, and look at

$$I_{\varepsilon,\varepsilon'} := \int_{\varepsilon}^{1} f(x)dx + \int_{-1}^{-\varepsilon'} f(x)dx$$

then we can apply the fundamental theorem of calculus on $[-1, -\varepsilon']$ and on $[\varepsilon, 1]$ to get

$$I_{\varepsilon,\varepsilon'} = \log \frac{\varepsilon'}{\varepsilon}.$$

Note that while for $\varepsilon' = \varepsilon$ this integral would be zero, in order for the improper integral to exist and to be zero, we would need to have that this expression tends to zero no matter how we send $\varepsilon, \varepsilon' \to 0$. This is clearly *not* the case as we e.g. have that $I_{\varepsilon_n,\varepsilon_n'} = \log(\varepsilon_n) \to -\infty$ if consider any sequence $\varepsilon_n \to 0^+$ and set $\varepsilon_n' = \varepsilon_n^2$.

While this means that the improper Riemann-integral does not exist, in practice it can be useful to consider the so-called *Cauchy principal value* (PV) of this integral, which is defined as the limit $\lim_{\varepsilon\to 0} I_{\varepsilon,\varepsilon}$, which in this case equals 0. We won't discuss principal values any further in this course, and here simply stress that it is *not* appropriate to write $\int_{-1}^{1} \frac{1}{x} dx$ for such a principal value, but that one could possibly write PV $\int_{-1}^{1} \frac{1}{x} dx = 0$.

EXAMPLE. Consider $f(x) = \frac{1}{\sqrt{|x|}}$ for $x \neq 0$ and set $f(0) = 0$. As above we can apply the fundamental theorem of calculus on intervals which don't contain 0. In this case we get that both $\int_{-1}^{-\varepsilon'} f = 2(1 - \sqrt{\varepsilon'})$ and $\int_{\varepsilon}^{1} f = 2(1 - \sqrt{\varepsilon})$ converge to 2 no matter how we send $\varepsilon', \varepsilon \to 0^+$ so the improper Riemann integral exists and it is hence ok to write $\int_{-1}^{1} f = 4$.