ASO Integral Transforms, Hilary Term 2025

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Housekeeping

- • Reading: see Lecture Notes (LN) for a list.
- • Lectures: will follow LN closely but abridged and with different examples.
- • Numbering: labels of theorems, examples etc follow LN. Example 12a is related to, but different from, Example 12 in LN. Read LN as well as these slides!
- • Captioning: is rather inaccurate to put it kindly. You can usually make it out looking at slides but ask if impossible...
- • See LN for a preamble about integration (READ THIS!).
- • There are two problem sheets. I suggest first tutorial covers lectures 1–6 although there is some overlap.

Course outline

There are three main themes.

1. (a) Distributions: how to interpret a point mass/charge/heat source/. . . as a mathematical object; how to differentiate a step function.
2. (b) Integral transforms: representations of functions akin to Fourier Series but valid on an infinite interval, so the output is a function rather than a series.
3. (c) Applications of all the above, to differential equations, probability and much more.

1: The delta function & other distributions

1.1: Motivation.

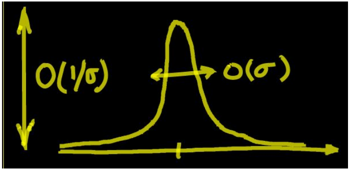
Example 4a: An example from probability. Let $X \sim N(0, \sigma^2)$ be a Normal random variable with density function

$$f_X^\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-x^2/2\sigma^2}$$

which satisfies

$$\int_{-\infty}^{\infty} f_X^\sigma(x)\,\mathrm{d}x = 1 \qquad \text{for all } \sigma^2 > 0.$$

What happens as $\sigma \downarrow 0$?



As $\sigma \to 0$,

$$\text{for } x \neq 0, \qquad f_X^\sigma(x) \to 0,$$

BUT

$$\lim_{\sigma \to 0} \int_{-\infty}^{\infty} f_X^\sigma(x)\,\mathrm{d}x = 1.$$

This suggests that there is a “function”, which we call $\delta(x)$, such that

$$\delta(x) = 0 \qquad x \neq 0,$$
$$\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1,$$

whatever this means rigorously.

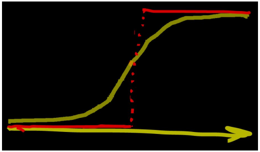
Note that $f_X^0(x)$ is the “PDF” of a random variable for which $\mathbb{P}[X = 0] = 1$.

Now look at the CDF

$$F_X^\sigma(x) = \mathbb{P}[X \le x] = \int_{-\infty}^{x} f_X^\sigma(s)\,\mathrm{d}s$$

so, at least for $\sigma > 0$,

$$\frac{\mathrm{d}F_X^\sigma}{\mathrm{d}x} = f_X^\sigma. \qquad (*)$$



As $\sigma \to 0$,

$$F_X^\sigma(x) \to H(x) = \begin{cases} 0 & x < 0, \\ 1 & x \ge 0. \end{cases}$$

Given (*) above, and that $f_X^0(x) = \delta(x)$, we expect that

$$\frac{\mathrm{d}H}{\mathrm{d}x} = \delta(x)$$

and this holds everywhere. We can differentiate a step function!

Example 4b: Impulses. A particle with position $x(t)$ is at rest with $x(t) = 0$ for $t \leq 0$. For $t > 0$ we apply a force $f(t)$ so

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = m\ddot{x} = f(t) \qquad \text{(Newton 2)}$$

where $f(t) = 0$ for $t \leq 0$.

Now suppose $f(t) > 0$ for $0 < t < \tau$, and then $f(t) = 0$ for $t > \tau$. Also let $\int_0^\tau f(t)\,\mathrm{d}t = I$, a constant. Integrating once,

$$\int_0^\tau m\ddot{x}(t)\,\mathrm{d}t = [m\dot{x}(t)]_0^\tau = m\dot{x}(\tau) = I.$$

Let $\tau \to 0$ with I fixed (eg $f(t) = I/\tau$ for $0 < t < \tau$). This called an *impulse* and the velocity is

$$\dot{x}(t) = \begin{cases} 0 & x \le 0, \\ I/m & x > 0. \end{cases}$$

Using the idea of differentiating a step function as above,[1] the equation of motion should be

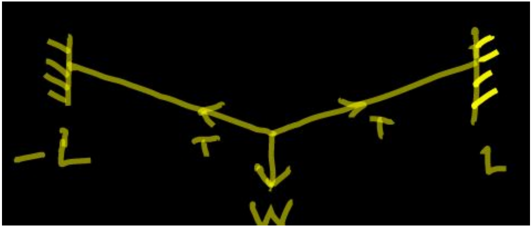
$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = I\delta(t)$$

Again, can we make this more rigorous?

[1] The sharp-eyed will have noticed that in the earlier example we had < 0 and ≥ 0 in the definition of the step function , while here we have ≤ 0 and > 0. As we shall see, this is immaterial.

Example 4c: A point force on a string.

A thin wire is strung at tension T between $x = -L$ and $x = L$. A weight W is hung from it at $x = a$. Displacements are small.



(Prelims): the displacement of the wire, $y(x)$ satisfies

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0$$

for both $-L < x < 0$ and $0 < x < L$, with $y(-L) = 0 = y(L)$.
At $x = a$, the wire is continuous, so

$$[y]_{x=0^-}^{x=0^+} = 0$$

(the $[\,\cdot\,]$ notation means'the jump in'). A force balance shows that

$$\left[T\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{x=0^-}^{x=0^+} = W$$

It's easy to solve for $y(x)$ by joining two straight lines to give

$$(**) \qquad y(x) = \begin{cases} -W(L+x)/2T & -L < x < 0, \\ -W(L-x)/2T & 0 < x < L. \end{cases}$$

Can you convince yourself that, for *all* $-L < x < L$,

$$T\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = W\delta(x)\,?$$

(Hint: what are the first and second derivatives of a piecewise linear function such as (**)?) This is the representation of a point force as a delta function.

1.2 Towards a definition of $\delta(x)$

Suppose there is an object $\delta(x)$ with

$$\delta(x) = 0 \qquad x \neq 0, \qquad \int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1$$

(even though we don't really know what $\int$ means here). Take a continuous function $\phi(x)$. We expect that, for $\Delta > 0$,

$$\begin{aligned}
\int_{-\infty}^{\infty} \delta(x)\phi(x)\,\mathrm{d}x &= \int_{-\Delta}^{\Delta} \delta(x)\phi(x)\,\mathrm{d}x \quad \text{as } \delta(x) = 0 \text{ for } x \neq 0 \\
&= \int_{-\Delta}^{\Delta} \delta(x)\,(\phi(x) - \phi(0) + \phi(0))\,\mathrm{d}x \\
&= \int_{-\Delta}^{\Delta} \delta(x)\,(\phi(x) - \phi(0))\,\mathrm{d}x + \phi(0)\int_{-\Delta}^{\Delta} \delta(x)\,\mathrm{d}x \\
&= \int_{-\Delta}^{\Delta} \delta(x)\,(\phi(x) - \phi(0))\,\mathrm{d}x + \phi(0) \\
&\to \phi(0) \quad \text{as } \Delta \to 0.
\end{aligned}$$

In the last step, we've used the continuity of $\phi(x)$: for any $\epsilon > 0$, for all Δ small enough,

$$\left|\phi(x) - \phi(0)\right| < \epsilon \qquad \text{for all } -\Delta < x < \Delta$$

so we hope that we can make

$$\int_{-\Delta}^{\Delta} \delta(x)\,(\phi(x) - \phi(0))\,\mathrm{d}x$$

as small as we wish.

We conclude that

$$\int_{-\infty}^{\infty} \delta(x)\phi(x)\,\mathrm{d}x = \phi(0).$$

HOW CAN WE MAKE THIS RIGOROUS???

1.3 Test functions and actions on them

Plan:

- • Define a class of ***test functions***, generically called $\phi(x)$.
- • Then for any continuous[2] function $f(x)$, define the ***action*** of f on ϕ by the map
  $$\phi \mapsto \langle f, \phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x.$$
  Note it looks like an inner product; in fact it is a *functional*: a map from the space of test functions to $\mathbb{R}$. Think of it as a weighted average of f, with weight ϕ.
- • Define ***distributions*** by their actions on test functions in a way consistent with the above.

[2]Integrable is enough, in fact.

We have already seen one example: the delta function $\delta(x)$ is *defined* by its action

$$\phi \mapsto \langle \delta(x), \phi(x) \rangle = \phi(0).$$

Then we'll define derivatives of distributions (yes!) by the integration by parts formula:

$$\begin{aligned} \langle f', \phi \rangle &= \int_{-\infty}^{\infty} f'(x)\phi(x)\,\mathrm{d}x \\ &= [f(x)\phi(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\phi'(x)\,\mathrm{d}x \\ &= -\langle f, \phi' \rangle, \end{aligned}$$

provided only that $\phi(x)$ vanishes at $x = \pm\infty$ and $\phi'(x)$ is also a test function.

Test functions defined properly

Definition 6. $\phi\colon \mathbb{R} \to \mathbb{R}$ is a ***test function*** if

- • $\phi(x)$ has derivatives $\phi^{(k)}$ of all orders k (also called being C^∞ or 'smooth');
- • There is an X such that $\phi(x) = 0$ for all $|x| > X$ (this is called having *compact support*[3]).

We then have $\phi(x) \to 0$ at $\pm\infty$, and that $\phi'(x)$ is also a test function. We call the space of test functions $\mathcal{D}$.

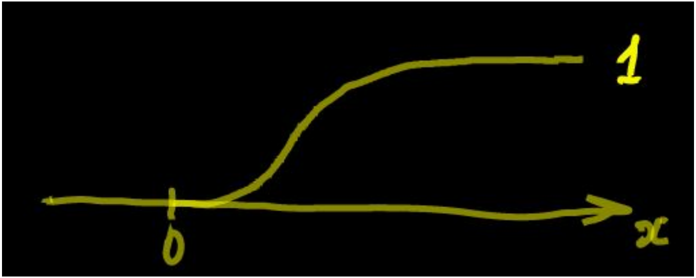
We shall *not* need to know much about test functions beyond these two properties. But we should at least show that they exist.

[3] The support of a function $\phi(x)$ is the smallest closed set containing all the points where $\phi \neq 0$.

Start with the famous function

$$\Phi(x) = \begin{cases} 0 & x \le 0, \\ e^{-1/x} & x > 0. \end{cases}$$

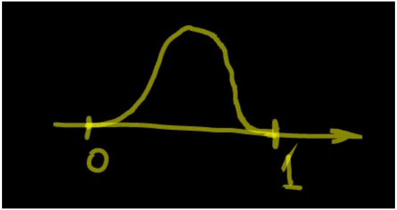
All its derivatives exist (and vanish) at $x = 0$ [because $\Phi^{(n)}(x)$ is a polynomial in $1/x$ times $e^{-1/x}$; put $y = 1/x$ so $y \to \infty$ as $x \to 0$, and you have terms like y^M/e^y which you can L'Hopitalize M times].



We are not yet there: $\Phi(x) > 0$ for $x > 0$ but it doesn't vanish at ∞ so it's not a test function. However,

$$\phi(x) = \Phi(x)\Phi(1-x)$$

is indeed a test function.



As noted above, all we care about test functions is that they exist, have compact support, are C^∞, and there are lots of them.

A function is specified uniquely by its action on all test functions

Theorem 8. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and suppose that

$$\langle f, \phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x = 0$$

for all test functions ϕ. Then $f(x) \equiv 0$.

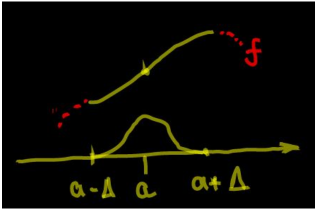
From this, we deduce that if $\langle f_1, \phi \rangle = \langle f_2, \phi \rangle$ for all $\phi \in \mathcal{D}$, then $\langle f_1 - f_2, \phi \rangle = 0$ and so $f_1 = f_2$. This the required uniqueness.

Note this is not a helpful way of specifying a function because it doesn't tell you how to recover f from its weighted averages against test functions. Later we consider weighted averages against other functions (exponentials) which *do* let us recover the function. (The Fourier series is one example of this.)

Proof of Theorem 8. Suppose (WLOG) that $f(a) > 0$. Then, as f is continuous, there is $\Delta > 0$ such that $f(x) > 0$ for all $x \in (a - \Delta, a + \Delta)$. By adapting the example above, we can produce a test function $\phi(x)$ which vanishes outside $(a - \Delta, a + \Delta)$ and is positive inside this interval. But then

$$\langle f, \phi \rangle = \int_{a-\Delta}^{a+\Delta} f(x)\phi(x)\,\mathrm{d}x > 0$$

as the integrand is strictly positive; this is a contradiction.



Convergence of sequences of test functions

We end with a definition of a very strong form of convergence for sequences of test functions (much stronger than pointwise or uniform convergence).

Definition 8a. The sequence $\{\phi_n\}_{n\geq 1}$ of test functions converges to zero, $\phi_n \to 0$, if:

- • $\phi_n(x) = 0$ for all n and x outside some interval $I \subset \mathbb{R}$ (this stops them running away to infinity);
- • for all k, $\phi_n^{(k)} \to 0$ uniformly as $n \to \infty$.

Obviously we say $\phi_n \to \phi$ if $\phi_n - \phi \to 0$. (Note the definition ensures that the limit ϕ is a test function.)

As noted, the action of a locally integrable function f on test functions $\phi \in \mathcal{D}$ is a map from $\mathcal{D}$ to $\mathbb{R}$:

$$\phi \mapsto \langle f, \phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x.$$

This map is

- *Linear*:

$$\langle f, a\phi + b\psi \rangle = a\langle f, \phi \rangle + b\langle f, \psi \rangle$$

  for $a, b \in \mathbb{R}$ and $\phi, \psi \in \mathcal{D}$;
- *Continuous*, in that, if $\{\phi_n\}$ is a sequence and $\phi_n \to 0$ then $\langle f, \phi_n \rangle \to \langle f, 0 \rangle = 0$. This follows easily from uniform convergence of ϕ_n on I as above, and it says that a small input generates a small output.

We can now define a distribution: next lecture.

1.4: Distributions

We now define a distribution by its action on test functions.

Definition 9. A ***distribution*** (also called a *generalised function*) F is a *continuous linear functional (map)* from $\mathcal{D}$ to $\mathbb{R}$,

$$\phi \mapsto \langle F, \phi \rangle \in \mathbb{R}.$$

Here, as in the previous lecture:

- • Continuous: if $\phi_n \to 0$ in $\mathcal{D}$ then $\langle F, \phi_n \rangle \to 0$ in $\mathbb{R}$;
- • Linear: $\langle F, a\phi + b\psi \rangle = a\langle F, \phi \rangle + b\langle F, \psi \rangle$.

Sometimes, for notational clarity, we give our distributions an argument x, for example $\delta(x)$, the delta function.

Regular districutions and the Heaviside function

Proposition 13. A locally integrable function f defines a distribution F_f with action

$$\langle F_f, \phi \rangle = \langle f, \phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x.$$

We call this a *regular* distribution.

Proof. This is what we noted at the end of Lecture 2.
Example 14. The two locally integrable functions

$$H_1(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geq 0, \end{cases} \qquad \text{and} \qquad H_2(x) = \begin{cases} 0 & x \leq 0, \\ 1 & x > 0, \end{cases}$$

define the same regular distribution $\mathcal{H}$, or $\mathcal{H}(x)$, with action

$$\langle \mathcal{H}, \phi \rangle = \int_0^{\infty} \phi(x)\,\mathrm{d}x,$$

called the *Heaviside function*. Note that the action does not care about the value at $x = 0$.

The delta function

Proposition 15. The delta function δ, or $\delta(x)$, with action

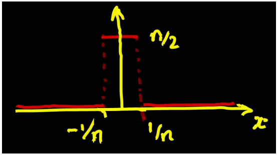
$$\langle \delta, \phi \rangle = \phi(0)$$

is a distribution.

Proof. Linearity is obvious. And if $\phi_n \to 0$ in $\mathcal{D}$, then $\phi_n(0) \to 0$ by uniform convergence. Hence $\langle \delta, \phi_n \rangle \to 0$.

Example 16. Show that the locally integrable functions

$$\delta_n(x) = \begin{cases} n/2 & |x| < 1/n, \\ 0 & |x| \geq 1/n \end{cases}$$



converge to δ as $n \to \infty$. By this, we mean that $\langle \delta_n, \phi \rangle \to \langle \delta, \phi \rangle$ as $n \to \infty$, for all ϕ.

Solution. For any test function $\phi(x)$,

$$\begin{aligned}
\langle \delta_n, \phi \rangle &= \frac{n}{2} \int_{-1/n}^{1/n} \phi(x)\,\mathrm{d}x \\
(*) \qquad &= \frac{n}{2} \phi(\xi_n) \int_{-1/n}^{1/n} \mathrm{d}x \qquad \text{(MVT for integrals, } -1/n < \xi_n < 1/n\text{)} \\
&= \phi(\xi_n) \\
&\to \phi(0) \qquad \text{(continuity of } \phi\text{).}
\end{aligned}$$

The key step is $(*)$: the MVT for integrals is very useful.

Proposition 17. If $f(x)$ is continuous at $x = 0$, then

$$\langle \delta_n, f \rangle \to f(0) \qquad \text{as } n \to \infty.$$

Proof. Same as for Example 16 with f instead of ϕ.

What this says is that the delta function can be used on continuous functions, not just the much more restricted class of test functions:

$$\langle \delta, f \rangle = f(0).$$

1.5: Operations on distributions

We now define a series of operations on distributions. In every case, the definition is consistent with the same operation on a function (equivalent to a regular distribution).

Definition 19. If $F(x)$ is a distribution, we define $F(x-a)$ and $F(ax)$ by the actions

$$\langle F(x-a), \phi(x)\rangle = \langle F(x), \phi(x+a)\rangle, \quad \langle F(ax), \phi(x)\rangle = \frac{1}{|a|}\langle F(x), \phi(x/a)\rangle.$$

This is true for regular distributions:

$$\begin{aligned}\langle f(x-a), \phi(x)\rangle &= \int_{-\infty}^{\infty} f(x-a)\phi(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} f(x)\phi(x+a)\,\mathrm{d}x \\ &= \langle f(x), \phi(x+a)\rangle.\end{aligned}$$

(The other calculation is an exercise.)

This leads to the *sifting property* of the delta function:

$$\langle \delta(x-a), \phi(x) \rangle = \langle \delta(x), \phi(x+a \rangle = \phi(a).$$

That is, $\delta(x-a)$ picks out the value of ϕ at $x = a$ (which is where $x - a = 0$).[4]

Definition 22. If F is a distribution and $f(x)$ is C^∞ then the distribution fF has action

$$\langle fF, \phi \rangle = \langle F, f\phi \rangle,$$

noting that $f\phi$ is a test function (as is $\phi(x+a)$ above).

[4]Compare with the discrete formula $\sum_i \delta_{ij} f_i = f_j$: to what well known object does the discrete version of δ correspond?

Derivative of a distribution

Definition 23. If F is a distribution, its derivative F' has action

$$\langle F', \phi \rangle = -\langle F, \phi' \rangle.$$

We saw that this works for (differentiable) functions earlier, using integration by parts.

Proposition 24. If F is a distribution, so is F'.

Proof. The action of F' is clearly linear. Also (see LN), if $\phi_n \to 0$ then $\phi_n' \to 0$ too, so

$$\langle F', \phi_n \rangle = -\langle F, \phi_n' \rangle \to 0,$$

which shows continuity.

We conclude that (like test functions) distributions can be differentiated infinitely often.

The usual calculus rules work for distributions, for example:

Proposition 23a. If F is a distribution and f is a smooth (C^∞) function, then $(fF)' = fF' + f'F$ (Leibniz).

Proof. For any test function ϕ,

$$\begin{aligned}
\langle (fF)', \phi \rangle &= -\langle fF, \phi' \rangle && \text{(Def 23)} \\
&= -\langle F, f\phi' \rangle && \text{(Def 22)} \\
&= -\langle F, (f\phi)' - f'\phi \rangle && \text{(key step)} \\
&= \langle F', f\phi \rangle + \langle F, f'\phi \rangle && \text{(Def 23)} \\
&= \langle fF', \phi \rangle + \langle f'F, \phi \rangle && \text{(Def 22)} \\
&= \langle fF' + f'F, \phi \rangle.
\end{aligned}$$

Examples 26, 27a.

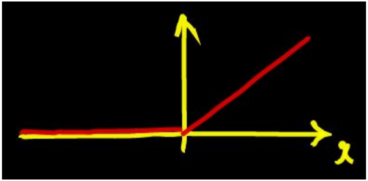
- • $\mathcal{H}' = \delta$, because:

$$\begin{aligned}\langle \mathcal{H}', \phi \rangle &= -\langle \mathcal{H}, \phi' \rangle \\ &= -\int_0^\infty \phi'(x)\,\mathrm{d}x \\ &= -[\phi(x)]_0^\infty \\ &= \phi(0) \\ &= \langle \delta, \phi \rangle\end{aligned}$$

- • This shows that differentiating a function with a jump discontinuity gives a delta (scaled by the magnitude of the jump).
- • $\langle \delta', \phi \rangle = -\langle \delta, \phi' \rangle = -\phi'(0)$, and similarly for higher derivatives of δ.

• If $f(x) = \max(x, 0)$, then

$$f'(x) = \begin{cases} 0 & x < 0, \\ 1 & x > 0, \end{cases}$$

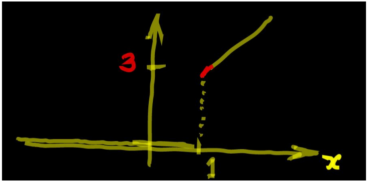


and so,
as a distribution, $f'(x) = \mathcal{H}(x)$ and $f''(x) = \delta(x)$.

• $x\delta(x) = 0$, as $\langle x\delta, \phi\rangle = \langle \delta, x\phi\rangle = 0 \cdot \phi(0) = 0$.

• If $f(x) = (x+2)\mathcal{H}(x-1)$ then

$$\begin{aligned} f'(x) &= (x+2)\delta(x-1) + \mathcal{H}(x-1) \\ &= 3\delta(x-1) + \mathcal{H}(x-1), \end{aligned}$$



as you only need
to evaluate the coefficient of $\delta(x-1)$ at $x = 1$
(where the argument of $\delta(x-1)$ vanishes).

We end with a reassuring calculation.

Example 28a. Let F be a distribution. Then

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}.$$

Proof.

$$\begin{aligned}\left\langle \frac{F(x+h)-F(x)}{h}, \phi \right\rangle &= \left\langle F(x), \frac{\phi(x-h)-\phi(x)}{h} \right\rangle \\ &\to \langle F, -\phi' \rangle \qquad \text{as } h \to 0 \\ &= \langle F', \phi \rangle\end{aligned}$$

as required.

2.1 Laplace Transform: definition and properties

We have seen how a continuous function is uniquely specified by its weighted average against (action on) all test functions. This is not really helpful because it gives us no obvious way to recover the function from this knowledge.

We now use a smaller class of functions as the weight in our weighted average, specifically exponential functions. Because of their special form, knowledge of the weighted average for a family of exponential weights lets us recover the function, a process called *inversion*. It also lets us transform differential equation problems into simpler ones.

This is the celebrated *Laplace Transform*.

The Laplace transform defined

Definition 33. Let $f(x)$ be a real or complex-valued function defined on $[0, \infty)$. The ***Laplace Transform*** of $f(x)$, denoted by $\mathcal{L}f$ or $\bar{f}(p)$, is

$$\mathcal{L}f = \bar{f}(p) = \int_0^\infty f(x)\mathrm{e}^{-px}\,\mathrm{d}x$$

for those $p \in \mathbb{C}$ for which the integral exists.

As noted above, this is a weighted average with weight e^{-px}, a family of exponentials.

Note also that $f(x)$ is only defined on $[0, \infty)$.

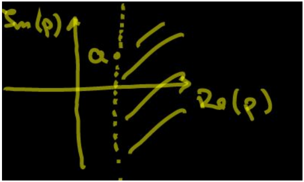
Exponentials

Example 35. If $f(x) = \mathrm{e}^{ax}$ for $a \in \mathbb{C}$, then

$$\begin{aligned}
\bar{f}(p) &= \int_0^\infty \mathrm{e}^{ax}\mathrm{e}^{-px}\,\mathrm{d}x \\
&= \int_0^\infty \mathrm{e}^{-(p-a)x}\,\mathrm{d}x \\
&= -\left[\frac{\mathrm{e}^{-(p-a)x}}{p-a}\right]_0^\infty \\
&= \frac{1}{p-a}
\end{aligned}$$

provided that $\mathrm{Re}(p) > \mathrm{Re}(a)$.

Note: although the integral
does not exist for $\mathrm{Re}(p) < \mathrm{Re}(a)$, the function
$1/(p-a)$ can be holomorphically continued into
all of $\mathbb{C}$ except for $p = a$, where it has a pole.



Powers of x

Example 36. Let $f_n(x) = x^n$ for $n = 0, 1, 2, \ldots$. Then

$$\overline{f_0}(p) = \int_0^\infty 1 \cdot \mathrm{e}^{-px}\,\mathrm{d}x = \frac{1}{p},$$

and for $n \geq 1$,

$$\begin{aligned}
\overline{f_n}(p) &= \int_0^\infty x^n \mathrm{e}^{-px}\,\mathrm{d}x \\
&= \left[-\frac{1}{p}x^n \mathrm{e}^{-px}\right]_0^\infty + \frac{1}{p}\int_0^\infty n x^{n-1}\mathrm{e}^{-px}\,\mathrm{d}x \\
&= \frac{n}{p}\overline{f_{n-1}}(p) \\
&= \frac{n!}{p^{n+1}} \qquad \text{by iteration and } \overline{f_0} = 1/p.
\end{aligned}$$

You will notice this is the Gamma function $\Gamma(\cdot)$ in mild disguise. See Sheet 1 exercise 6 for the result that, for *all* $n > -1$,

$$\mathcal{L}x^n = \frac{\Gamma(n+1)}{p^{n+1}}.$$

Trigonometric functions

Example 37. For all real a, $e^{iax} = \cos(ax) + i\sin(ax)$. So, for $\mathrm{Re}(p) > 0$,

$$\mathcal{L}e^{iax} = \frac{1}{p - ia} = \frac{p + ia}{p^2 + a^2} = \mathcal{L}\cos(ax) + i\mathcal{L}\sin(ax).$$

However, by holomorphic continuation (ie using the Identity Theorem) we can extend this to all $a \in \mathbb{C}$, so that

$$\mathcal{L}\cos(ax) = \frac{p}{p^2 + a^2}, \qquad \mathcal{L}\sin(ax) = \frac{a}{p^2 + a^2}$$

for $\mathrm{Re}(p) > |\mathrm{Im}(a)|$. Note $\cos(ax)$ is even in x but its transform is odd in p, vice versa for $\sin(ax)$ (memory tip).

Heaviside and δ

Examples 38 and 40. For real $a > 0$,

$$\mathcal{L}\delta(x-a) = \mathrm{e}^{-pa} \quad \text{and} \quad \mathcal{L}\mathcal{H}(x-a) = \frac{\mathrm{e}^{-pa}}{p}.$$

The first of these is sifting: $\langle \delta(x-a), \mathrm{e}^{-px} \rangle = \mathrm{e}^{-px}|_{x=a} = \mathrm{e}^{-pa}$ (recall that δ works on contiunuous functions). The second is by integration:

$$\int_0^\infty \mathcal{H}(x-a)\mathrm{e}^{-px}\mathrm{d}x = \int_a^\infty \mathrm{e}^{-px}\mathrm{d}x = \left[\frac{\mathrm{e}^{-px}}{-p}\right]_a^\infty = \frac{\mathrm{e}^{-pa}}{p}.$$

2.2 Domain of existence

We now show that the Laplace Transform exists (if at all)[5] then it does so for all large $\mathrm{Re}(p)$. Formally:

Proposition 41/42. If $\bar{f}(p)$ exists for $\mathrm{Re}(p) = p_0$, then

1. $\bar{f}(p)$ exists for all $\mathrm{Re}(p) > p_0$;
2. $\bar{f}(p) \to 0$ as $\mathrm{Re}(p) \to \infty$.

Proof. We exploit the exponential decay of the Laplace kernel e^{-px}. For (1), if $\mathrm{p} > p_0$, then

$$\left| f(x)\mathrm{e}^{-px} \right| < \left| f(x)\mathrm{e}^{-p_0 x} \right|$$

so the integral exists by comparison.

[5]Exercise: think of a function that is continuous on $[0, \infty)$ for which the Laplace Transform integral does not exist.

For (2), write $p = p_0 + t$. For $\mathrm{Re}(t) > 0$, take any real $a > 0$. Then

$$|\bar{f}(p_0+t)| = \left| \int_0^a + \int_a^\infty f(x) \mathrm{e}^{-(p_0+t)x} \mathrm{d}x \right| \qquad (*)$$

$$\le \left| \int_0^a f(x)\mathrm{e}^{-(p_0+t)x}\mathrm{d}x \right| + \left| \int_a^\infty f(x)\mathrm{e}^{-p_0 x}\mathrm{e}^{-tx}\mathrm{d}x \right|$$

$$(*) \qquad \le \int_0^a \left| f(x)\mathrm{e}^{-(p_0+t)x} \right| \mathrm{d}x + \mathrm{e}^{-a\mathrm{Re}(t)} \int_a^\infty \left| f(x)\mathrm{e}^{-p_0 x}\right| \mathrm{d}x$$

$$(*) \qquad \le M(a) \int_0^a \left| \mathrm{e}^{-(p_0+t)x} \right| \mathrm{d}x + \mathrm{e}^{-a\mathrm{Re}(t)} I$$

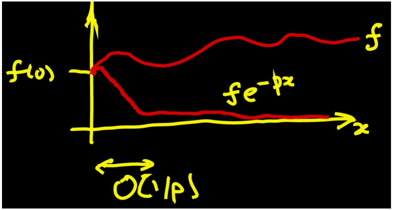
(here $M(a)$ is a bound for $f(x)$ on $[0, a]$ and I exists because $\bar{f}$ does)

$$= M(a) \int_0^a \mathrm{e}^{-\mathrm{Re}(p_0+t)x} \mathrm{d}x + \mathrm{e}^{-a\mathrm{Re}(t)} I$$

$$= M(a) \frac{1 - \mathrm{e}^{-a\mathrm{Re}(p_0+t)}}{\mathrm{Re}(p_0+t)} + \mathrm{e}^{-a\mathrm{Re}(t)} I.$$

As $\mathrm{Re}(t) \to \infty$, the first term decays algebraically and the second exponentially. Key steps are $(*)$. See SN for intuitiove digression on

Digression. The proof above is interesting and typical of results that estimate integrals as a parameter (here, p) varies. Although $\mathrm{e}^{-px} \to 0$ as $p \to \infty$ it only does so for $x \in (0, \infty)$: ie, nonuniformly. Because $f(x)\mathrm{e}^{-px} = f(0)$ when $x = 0$ for all p, the small region round $x = 0$ contributes $O(f(0)/p)$ to the integral, while the rest of the range of integration makes an exponentially small contribution. Letting $a \to 0$ above gives $\bar{f}(p) \sim f(0)/p$ as $\mathrm{Re}(p) \to \infty$, known as *Watson's Lemma*.

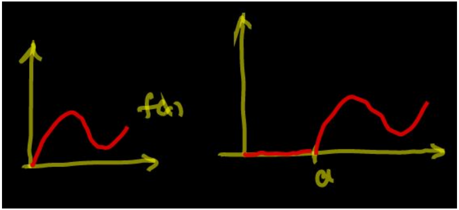


2.3 More properties of the Laplace Transform

Propositions 43 & 49. Suppose that $a > 0$ and that the Laplace Transform of $f(x)$ converges for $\mathrm{Re}(p) > p_0$. Then:

1. $\mathcal{L}\Big(f(x)\mathrm{e}^{-ax}\Big) = \bar{f}(p+a)$;
2. $\mathcal{L}\Big(f(x-a)\mathcal{H}(x-a)\Big) = \mathrm{e}^{-ap}\bar{f}(p)$.

(The latter is the transform of the translation of $f(x)$ by a.)



Proof. For (1),

$$\int_0^\infty f(x)\mathrm{e}^{-ax}\mathrm{e}^{-px}\,\mathrm{d}x = \int_0^\infty f(x)\mathrm{e}^{-(p+a)x}\,\mathrm{d}x$$
$$= \bar{f}(p+a)$$

For (2),

$$\begin{aligned}\int_0^\infty f(x-a)\mathcal{H}(x-a)\mathrm{e}^{-px}\mathrm{d}x &= \int_a^\infty f(x-a)\mathrm{e}^{-px}\mathrm{d}x \\ &= \int_0^\infty f(t)\mathrm{e}^{-p(a+t)}\mathrm{d}x \qquad \text{(by } x-a=t\text{)} \\ &= \mathrm{e}^{-ap}\bar{f}(p).\end{aligned}$$

Note thst these results appear related. We shall see this more clearly when we get to the Fourier transform.

These results are useful in identifying functions from their transforms (inversion), *provided* that the transform determines the function uniquely. We shall see later that it does (compare the way the MGF determines the PDF in probability).

Examples 47a & 48a. Find the inverses of:

- • $\bar{f}(p) = 1/[p(p-1)]$. By partial fractions,

$$\bar{f}(p) = \frac{-1}{p} + \frac{1}{(p-1)}, \quad \text{so} \quad f(x) = -1 + \mathrm{e}^x.$$

  Note that $\bar{f}(p)$ converges for $\mathrm{Re}(p) > 1$.
- • $\bar{f}(p) = \mathrm{e}^{-p}/p^2$. Here we know that $1/p^2$ is the LT of x, so e^{-p}/p^2 is the LT of $(x-1)\mathcal{H}(x-1)$.
- • $\bar{f}(p) = p/(p^2 - 2p + 5)$. You can do this by partial fractions but the roots are complex. It's easier to see that

$$\bar{f}(p) = \frac{p}{(p-1)^2 + 4} = \frac{p-1}{(p-1)^2 + 4} + \frac{1}{(p-1)^2 + 4}$$

  so, as

$$\mathcal{L}\cos(ax) = \frac{p}{p^2 + a^2}, \qquad \mathcal{L}\sin(ax) = \frac{a}{p^2 + a^2},$$

  we set $a = 2$ and combine the results above to get $f(x) = \mathrm{e}^x \cos 2x + \frac{1}{2}\mathrm{e}^x \sin 2x$. [Exercise: what if $a = -2$?]

2.4 The Laplace Transform of a derivative

The way the LT acts on a derivative makes it a powerful tool for solving differential equations.

Proposition 44 (the LT of a derivative). Provided that the LTs of $f(x)$ and $f'(x)$ converge, and that $f(x)\mathrm{e}^{-px} \to 0$ as $x \to \infty$,[6] all holding for $\mathrm{Re}(p) > p_0$,

$$\mathcal{L}f' = \overline{f'}(p) = p\bar{f}(p) - f(0).$$

Proof. We integrate by parts:

$$\begin{aligned}\mathcal{L}f' &= \int_0^\infty f'(x)\mathrm{e}^{-px}\,\mathrm{d}x \\ &= \left[f(x)\mathrm{e}^{-px}\right]_0^\infty + \int_0^\infty f(x)\cdot p\mathrm{e}^{-px}\,\mathrm{d}x \\ &= p\bar{f}(p) - f(0).\end{aligned}$$

[6] Exercise: think of an integrable function which does not satisfy this condition for any p. The point? It is not true that if a function is integrable then it must vanish at infinity. [Hint: think of narrow top-hats near integer values of x.]

Higher-order derivatives

Corollary 45. Provided all the LTs exist, and obvious technical conditions at $x = \infty$ are satisfied,

$$\mathcal{L}f'' = \overline{f''}(p) = p^2\bar{f}(p) - pf(0) - f'(0),$$

with similar formulae for higher derivatives (see LN).

Proof. Put $f' = g$; then $f'' = g'$. Now $\bar{g}(p) = p\bar{f}(p) - f(0)$ and $\overline{g'}(p) = p\bar{g}(p) - g(0) = p(p\bar{f}(p) - f(0)) - f'(0)$, which gives the result.

You can also do this by integrating by parts twice. For higher derivatives, apply the idea above recursively.

Example 46a. Solve the differential equation (DE)

$$f'' - 2f' - 3f = e^x \quad \text{for } x > 0, \quad \text{with} \quad f(0) = 1, \quad f'(0) = -1.$$

Solution. You can do this by standard methods. They are a pain. Take the LT of the DE to get

$$\underbrace{p^2\bar{f} - pf(0) - f'(0)}_{\text{LT of } f''} - 2\underbrace{(p\bar{f} - f(0))}_{\text{LT of } f'} - 3\bar{f} = p^2\bar{f} - p + 1 - 2(p\bar{f} - 1) - 3\bar{f}$$

$$= 1/(p-1) \quad \text{(LT of RHS)}.$$

Tidying up,

$$(p^2 - 2p - 3)\bar{f} = (p-3)(p+1)\bar{f} = (p-3) + 1/(p-1)$$

so

$$\begin{aligned}\bar{f}(p) &= \frac{1}{p+1} + \frac{1}{(p-3)(p+1)(p-1)} \\ &= \frac{1}{p+1} + \frac{1}{8(p-3)} + \frac{1}{8(p+1)} - \frac{1}{4(p-1)},\end{aligned}$$

Copying $\bar{f}$ over,

$$\bar{f}(p) = \frac{1}{p+1} + \frac{1}{8(p-3)} + \frac{1}{8(p+1)} - \frac{1}{4(p-1)},$$

and inversion gives

$$f(x) = \frac{9}{8}e^{-x} + \frac{1}{8}e^{3x} - \frac{1}{4}e^{x}.$$

Worth doing a check: at $x = 0$,

$$\frac{9}{8} + \frac{1}{8} - \frac{1}{4} = 1, \qquad -\frac{9}{8} + 3 \cdot \frac{1}{8} - \frac{1}{4} = -1.$$

[Note: $p^2 - 2p - 3 = 0$ is the auxiliary equation.]

Example 50a. Solve $f' + f = x,\ 0 < x < \infty$, with $f(0) = 1$. **Solution.** The LT in x gives

$$p\bar{f} - 1 + \bar{f} = 1/p^2$$

so

$$\bar{f} = \left(1 + 1/p^2\right)/(p+1) = \frac{1}{p^2} - \frac{1}{p} + \frac{2}{p+1},$$

from which $f(x) = x - 1 + 2\mathrm{e}^{-x}$.

Example 51a. Solve $f'' = \delta(x-1)$ with $f(0) = 0$, $f'(0) = 0$. **Solution.** Taking the LT in x gives $p^2\bar{f} = \mathrm{e}^{-p}$ so

$$\bar{f}(p) = \frac{\mathrm{e}^{-p}}{p^2}, \qquad \text{giving} \qquad f(x) = (x-1)\mathcal{H}(x-1)$$

(we saw this transform above). This is an 'impulse' at $x = 1$.

Example 51b. Solve $f'' - f = \delta(x-1)$ with $f(0) = 0$ and $f(x) \to 0$ as $x \to \infty$. Here the positive feedback in the equation ($f'' = f + \cdots$) creates an exponentially growing solution which we have to eliminate.
Solution. The key point is that $f(x) \to 0$ at infinity means that $\bar{f}(p)$ is holomorphic for $\mathrm{Re}(p) > 0$. Take the LT of the DE to get

$$p^2\bar{f} - p(\text{zero}) - f'(0) - \bar{f} = \mathrm{e}^{-p}, \quad \text{so} \quad \bar{f}(p) = \frac{f'(0) + \mathrm{e}^{-p}}{p^2 - 1}.$$

This function has a pole at $p = 1$ *unless* we choose $f'(0) = -\mathrm{e}^{-1}$. This gives

$$\bar{f}(p) = \frac{\mathrm{e}^{-p} - \mathrm{e}^{-1}}{p^2 - 1} = (\mathrm{e}^{-p} - \mathrm{e}^{-1}) \cdot \underbrace{\frac{1}{2}\left(-\frac{1}{p+1} + \frac{1}{p-1}\right)}_{\text{LT of } -\sinh x}.$$

From

$$\bar{f}(p) = (\mathrm{e}^{-p} - \mathrm{e}^{-1}) \cdot \underbrace{\frac{1}{2}\left(-\frac{1}{p+1} + \frac{1}{p-1}\right)}_{\text{LT of } \sinh x}$$

we see that

$$f(x) = \sinh(x-1)\mathcal{H}(x-1) - \mathrm{e}^{-1}\sinh x.$$

This can also be written as

$$f(x) = \begin{cases} -\mathrm{e}^{-1}\sinh x & 0 < x < 1, \\ -\mathrm{e}^{-x}\sinh 1 & 1 < x < \infty. \end{cases}$$

It's not hard to check that f is continuous at $x = 1$, and that f' has a jump of 1 there, both consistent with $\delta(x-1)$ in the DE ($f'' = \delta(x-1) + \cdots$). Those doing DEs 2 will recognise f as the Green's function for $f'' - f$ with zero boundary conditions at 0 and ∞.

A PDE example

Example. This example is a little harder and more interesting. You are to solve for waves in a semi-infinite string $x > 0$, which is initially (time $t = 0$) straight and at rest, while the end $x = 0$ is moved up and down with amplitude $f(t)$. The problem for the displacement $u(x, t)$ is:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad x > 0, t > 0,$$

with

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = 0 \quad \text{and} \quad u(0, t) = f(t).$$

We expect a signal to propagate away from $x = 0$ at the wavespeed c.

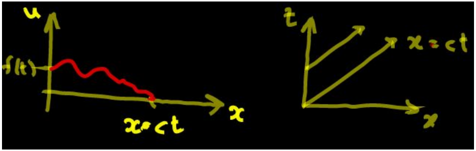
Take the LT in t so that $u(x,t) \mapsto \overline{u}(x,p) = \int_0^\infty u(x,t)\mathrm{e}^{-pt}\mathrm{d}t$. This gives

$$p^2\overline{u} \quad [+\text{ zero from the IC}] = c^2\frac{\partial^2\overline{u}}{\partial x^2} \quad \text{with} \quad \overline{u}(0,p) = \overline{f}(p).$$

Solutions of this equation are $\mathrm{e}^{\pm px/c}$, but only the minus sign gives decay for $\mathrm{Re}(p) \to \infty$. Using $\overline{u}(0,p) = \overline{f}(p)$,

$$\overline{u}(x,p) = \overline{f}(p)\mathrm{e}^{-px/c}, \quad \text{so} \quad u(x,t) = f(t - x/c)\mathcal{H}(t - x/c)$$

which is indeed the boundary amplitude moving to the right at speed c.[7]



[7] You can, of course, get this from the general solution of the wave equation in the form $F(t - x/c) + G(t + x/c)$; here $F(t - x/c) = f(t - x/c)\mathcal{H}(t - x/c)$ and the fact that $G = 0$ corresponds to there being no incoming waves (no exponential growth in the LT).

LT of $xf(x)$

We end with a counterpart to the formula for the transform of a derivative.[8]

Proposition 52. If the LT of $f(x)$ exists, then

$$\mathcal{L}(xf) = \overline{xf(x)} = -\frac{\mathrm{d}\bar{f}}{\mathrm{d}p}.$$

Proof. Start from the right-hand side:

$$\begin{aligned}
-\frac{\mathrm{d}\bar{f}}{\mathrm{d}p} &= -\frac{\mathrm{d}}{\mathrm{d}p}\int_0^\infty f(x)\mathrm{e}^{-px}\,\mathrm{d}x \\
&= \int_0^\infty -\frac{\partial}{\partial p}\left(f(x)\mathrm{e}^{-px}\right)\mathrm{d}x \\
&= \int_0^\infty f(x)\cdot x\mathrm{e}^{-px}\,\mathrm{d}x \\
&= \mathcal{L}(xf).
\end{aligned}$$

[8] The relationship is clearer for the Fourier Transform, as we shall see.

[Aside: why did we not need to state that the LT of $xf(x)$ exists as well as that of $f(x)$? The key is in the exponential decay of e^{-px}. Suppose that the LT of f exists for $\mathrm{Re}(p) > p_0$. Take $\epsilon > 0$ and look at

$$\int_0^\infty xf(x)\mathrm{e}^{-(p+\epsilon)x}\,\mathrm{d}x = \int_0^\infty x\mathrm{e}^{-\epsilon x} \cdot f(x)\mathrm{e}^{-px}\,\mathrm{d}x.$$

The function $x\mathrm{e}^{-\epsilon x}$ is continuous and bounded on $[0, \infty)$, so this integral exists because the LT of f does; then let $\epsilon \to 0$.]

Example 53a. Invert $\bar{f}(p) = 1/(p+1)^2$. (We already know how to do this.)
Solution. We have

$$\bar{f}(p) = \frac{1}{(p+1)^2} = -\frac{\mathrm{d}}{\mathrm{d}p}\frac{1}{p+1},$$

and as $1/(p+1)$ is the LT of e^{-x}, we have $f(x) = x\mathrm{e}^{-x}$.

Example 53b. The LT of 1 is $1/p$, so the LT of x is $-\mathrm{d}/\mathrm{d}p(1/p) = 1/p^2$; of x^2 is $-\mathrm{d}/\mathrm{d}p(1/p^2) = 2/p^3$; and so on.

Example 53c. Solve $f' + xf = 0$ with $f(0) = 1$ and deduce the LT of $\mathrm{e}^{-x^2/2}$. (This gives the MGF of $Y = |X|$ when $X \sim N(0, 1)$.)

Solution. The DE has solution $f(x) = \mathrm{e}^{-x^2/2}$ (use the integrating factor $\mathrm{e}^{x^2/2}$). Taking LT of the DE,

$$p\bar{f} - 1 - \frac{\mathrm{d}\bar{f}}{\mathrm{d}p} = 0.$$

We have amost the same integrating factor:

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\bar{f}\mathrm{e}^{-p^2/2}\right) = -\mathrm{e}^{-p^2/2}.$$

The solution that decays as $\mathrm{Re}(p) \to \infty$ is[9]

$$\bar{f}(p) = \mathrm{e}^{p^2/2}\int_p^\infty \mathrm{e}^{-s^2/2}\mathrm{d}s.$$

[9]Note how we incorporate decay at infinity via the upper limit and the minus by the lower one. Use L'Hopital to show that $\bar{f}(p)$ indeed decays at infinity.

A final PDE example

Example 53d. Suppose that $u(x,t)$ satisfies

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad x > 0,\ t > 0$$

with $u(x,0) = 0$ and $u(0,t) = 1$ for $t > 0$. Find $g(t) = -\partial u/\partial x|_{x=0}$. This models heat flow in a semi-infinite bar, initially at temperature zero, when the temperature at the end $x = 0$ is raised to 1 and held at that value. The question asks for the heat flux into the bar at $x = 0$ as a function of time.

Solution. Take the LT in t:

$$\overline{u}(x,p) = \int_0^\infty u(x,t)\mathrm{e}^{-pt}\mathrm{d}t.$$

Note that we require $\overline{u}(x,p)$ to decay to zero as $x \to \infty$.

Take the LT of the PDE and the BC at $x = 0$:

$$\overline{\frac{\partial u}{\partial t}} = p\overline{u} - (\text{zero}) = \frac{\partial^2 \overline{u}}{\partial x^2}, \qquad \overline{u(0,t)} = \overline{u}(0,p) = \frac{1}{p}$$

(as the transform of 1 is $1/p$).

The solution of $\partial^2\overline{u}/\partial x^2 = p\overline{u}$ that is bounded[10] at $x = \infty$ is $\overline{u}(x,p) = A(p)\mathrm{e}^{-x\sqrt{p}}$ and using the transformed BC we get

$$\overline{u}(x,p) = \frac{\mathrm{e}^{-x\sqrt{p}}}{p} \qquad \text{so} \qquad \overline{\frac{\partial u}{\partial x}} = -\sqrt{p}\mathrm{e}^{-x\sqrt{p}}/p.$$

Setting $x = 0$ we get $\overline{g}(p) = 1/\sqrt{p}$. Now we know $\mathcal{L}t^\alpha = \Gamma(\alpha+1)/p^{\alpha+1}$ so

$$g(t) = \frac{1}{\Gamma(\frac{1}{2})t^{\frac{1}{2}}} = \frac{1}{\sqrt{\pi t}} \qquad \text{as } \Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

[10]What assumption does this imply about the branch for $\sqrt{p}$?

3.1 The Laplace convolution

The LT of a product of functions has no simple relation to those of the functions themselves (it may not even exist — can you think of an example?). However, there is a close relative of the product: the *convolution*.

Definition 56. If f and g are defined on $[0,\infty)$, their (Laplace) convolution $h = f * g$ is

$$h(x) = (f * g)(x) = \int_0^x f(t)g(x-t)\,\mathrm{d}t.$$

Remark 57. We have $f * g = g * f$ (exercise: put $t - x = u$ above).

Example 58a. If $f(x) = 1$ and $g(x) = x$, then

$$(f * g)(x) = \int_0^x 1 \cdot (x - t)\,\mathrm{d}x = \left[xt - \frac{1}{2}t^2\right]_0^x = \frac{1}{2}x^2$$

Note: $\bar{f} = 1/p$, $\bar{g} = 1/p^2$, and $\overline{f * g} = \frac{1}{2} \cdot 2/p^3 = 1/p^3 = \bar{f}\,\bar{g}$. This is *not a coincidence*.

Example 59a. Let $f(x) = \lambda \mathrm{e}^{-\lambda x} = g(x)$ for $x > 0$. Then

$$(f * g)(x) = \lambda^2 \int_0^x \mathrm{e}^{-\lambda t} \mathrm{e}^{-\lambda(x-t)}\mathrm{d}x = \lambda^2 \mathrm{e}^{-\lambda x} \int_0^x 1\,\mathrm{d}x = \lambda^2 x \mathrm{e}^{-\lambda x}.$$

And now

$$\bar{f}(p) = \frac{\lambda}{\lambda + p} = \bar{g}(p),$$

while

$$\overline{f * g}(p) = \lambda^2 \mathcal{L}\left(x\mathrm{e}^{-\lambda x}\right) = \lambda^2 \left(-\frac{\mathrm{d}}{\mathrm{d}p}(\bar{f}/\lambda)\right) = \frac{\lambda^2}{(\lambda + p)^2} = \bar{f}\,\bar{g}.$$

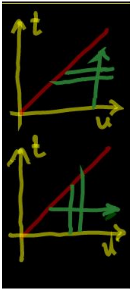
Both examples illustrate the following famous theorem.

The Laplace Convolution Theorem

Theorem 60. Let f and g have LTs $\bar{f}$ and $\bar{g}$ for $\mathrm{Re}(p) > p_0$. Then

$$\overline{f * g} = \bar{f}\,\bar{g}.$$

Proof. Start with the RHS:



$$\begin{aligned}
\bar{f}(p)\bar{g}(p) &= \int_0^\infty f(t)\mathrm{e}^{-pt}\,\mathrm{d}t \int_0^\infty g(s)\mathrm{e}^{-ps}\,\mathrm{d}s \\
&= \int_0^\infty \int_0^\infty f(t)g(s)\mathrm{e}^{-p(s+t)}\,\mathrm{d}s\,\mathrm{d}t \\
(*) \qquad &= \int_0^\infty \int_t^\infty f(t)g(u-t)\mathrm{e}^{-pu}\,\mathrm{d}u\,\mathrm{d}t \quad (s+t=u) \\
(*) \qquad &= \int_0^\infty \int_0^u f(t)g(u-t)\mathrm{e}^{-pu}\,\mathrm{d}t\,\mathrm{d}u \quad \text{(swap order)} \\
&= \int_0^\infty \int_0^u f(t)g(u-t)\,\mathrm{d}t\,\mathrm{e}^{-pu}\,\mathrm{d}u \\
&= \int_0^\infty (f * g)(u)\mathrm{e}^{-pu}\,\mathrm{d}u = \overline{f * g}(p).
\end{aligned}$$

Technical note: we need a theorem to justify changing the order of integration en route; this is Fubini's theorem (if we are using Lebesgue integration).

Example 62a. Find $f(x)$ satisfying

$$f'' + f = g(x), \quad 0 < x < \infty, \quad \text{with } f(0) = 0,\ f'(0) = 0.$$

Solution. Take the LT:

$$p^2\bar{f} - (\text{zero}) + \bar{f} = \bar{g}, \quad \text{so} \quad \bar{f}(p) = \frac{\bar{g}(p)}{p^2 + 1}.$$

Now $1/(p^2 + 1)$ is the LT of $\sin x$, so

$$f(x) = \int_0^x g(t)\sin(x - t)\,\mathrm{d}t.$$

(Talking point: why is the case $g(x) = \sin x$ special?)

Example 62b. We continue the heat equation example 53d of lecture 5. This is the first example we have seen that you cannot do by 'elementary' methods (you can write down the solution to the PDE problem 53d by trying the particular form $u(x,t) = v(x/\sqrt{t})$, kown as a *similarity solution*).

In the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x > 0,\ t > 0 \quad \text{with} \quad u(x,0) = 0, \quad u(0,t) = f(t).$$

for $u(x,t)$, find the relationship between the boundary heat flux $g(t) = -\partial u/\partial x|_{x=0}$ and the boundary temperature $f(t)$.

(In example 53d we took $f = 1$. Here we apply a general temperature $f(t)$ at $x = 0$.)

Solution. As in Example 53d, taking the LT in t gives

$$p\overline{u}(x,p) = \frac{\partial^2 \overline{u}}{\partial x^2} \quad \text{with} \quad \overline{u}(0,p) = \overline{f}(p),$$

so $\overline{u}(x,p) = \overline{f}(p)\mathrm{e}^{-x\sqrt{p}}$. (Earlier, we had $\overline{f}(p) = 1/p$.)

Differentiating in x,

$$\overline{-\frac{\partial u}{\partial x}} = -\frac{\partial \overline{u}}{\partial x} = \sqrt{p}\,\overline{f}(p)\mathrm{e}^{-x\sqrt{p}}.$$

Putting $x = 0$ gives

$$\overline{g}(p) = \sqrt{p}\,\overline{f}(p), \quad \text{so} \quad \overline{f}(p) = \overline{g}(p)/\sqrt{p}.$$

But we saw earlier that $1/\sqrt{p} = \mathcal{L}(1/\sqrt{\pi t})$. Using the convolution theorem, we have

$$f(t) = g(t) * \frac{1}{\sqrt{\pi t}} = \frac{1}{\sqrt{\pi}} \int_0^t \frac{g(s)}{\sqrt{t-s}} \mathrm{d}s.$$

You may ask how to find g from f: this amounts to solving the integral equation, known as Abel's equation,

$$f(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{g(s)}{\sqrt{t-s}} \mathrm{d}s$$

for g, given f. There is a neat trick. Go back to

$$\bar{g}(p) = \sqrt{p}\,\bar{f}(p) \quad \text{and write it as} \quad \bar{g}(p) = \frac{p}{\sqrt{p}}\bar{f}(p).$$

We recognise $p\bar{f}(p)$: it is $f(0) + \overline{f'}(p)$. So

$$\bar{g}(p) = \frac{f(0)}{\sqrt{p}} + \frac{\overline{f'}(p)}{\sqrt{p}}.$$

By inverting the first term, and convolution as above on the second,

$$g(t) = \frac{f(0)}{\sqrt{\pi t}} + \frac{1}{\sqrt{\pi}} \int_0^t \frac{f'(s)}{\sqrt{t-s}} \mathrm{d}s.$$

(Talking point: what is the physical interpretation of the first term on the RHS (see Ex. 53d)? The second term?)

3.2 Uniqueness and inversion

We now state two key theorems; for proofs see LN.[11]

Theorem 63. Let f be continuous on $[0,\infty)$ with $|f| < M\mathrm{e}^{cx}$ for some $M > 0$ and $c \in \mathbb{R}$. If $\bar{f}(p) \equiv 0$ then $f(x) \equiv 0$.

That is, provided a function is continuous and grows no more than exponentially at infinity, it is uniquely determined by its Laplace Transform.

Theorem 64. (Laplace Inversion Theorem.) Suppose f is continuous on $[0,\infty)$ and has LT $\bar{f}(p)$ for $\mathrm{Re}(p) > p_0$. Then, for $x > 0$, $f(x)$ is given by the contour integral representation

$$f(x) = \frac{1}{2\pi \mathrm{i}} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \bar{f}(p)\mathrm{e}^{px}\,\mathrm{d}p,$$

for any real $\sigma > p_0$ (the precise value of σ is unimportant).

[11] We'll see a slick proof of the Inversion Theorem later.

The proof of the inversion theorem in LN is for a special case in which $\bar{f}(p)$ is a rational function with just one pole, at (WLOG) $p = 0$ (so $p_0 = 0$), meaning that $\bar{f}$ has the form $g(p)/p^n$ where g is a polynomial of degree $< n$. The key steps are:

- • Close the contour with a semicircle to the left and show the contribution from the arc vanishes in the limit.
- • Use the Residue Theorem to show that the integral is equal to $\mathrm{Res}_{p=0} g(p)\mathrm{e}^{px}/p^n$ (the $2\pi\mathrm{i}$'s cancel).
- Combine the formula for the residue of a pole of this form, and the Leibniz rule, to show that the LT of the integral is $\bar{f}(p)$ and so the integral is equal to $f(x)$.

Example 66a. Invert $\bar{f}(p) = 1/(p^2+1)$. (We can do this by partial fractions and we know the answer is $f(x) = \sin x$.)

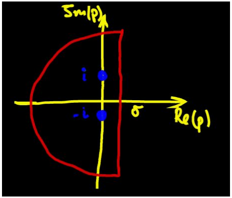
Solution. The function $\bar{f}(p)\mathrm{e}^{px}$ has poles at $p = \pm\mathrm{i}$. We have

$$\mathrm{Res}\left(\frac{\mathrm{e}^{px}}{p^2+1}; p = \pm\mathrm{i}\right) = \frac{\mathrm{e}^{\pm\mathrm{i}x}}{\pm 2\mathrm{i}}$$

and adding these gives

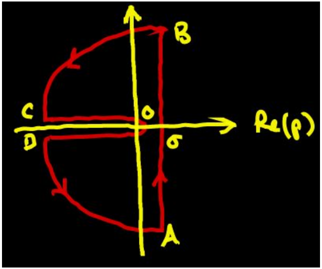
$$f(x) = \frac{1}{2\pi\mathrm{i}} \cdot 2\pi\mathrm{i}\left(\frac{\mathrm{e}^{\mathrm{i}x}}{2\mathrm{i}} + \frac{\mathrm{e}^{-\mathrm{i}x}}{-2\mathrm{i}}\right) = \sin x.$$

(Working out the residues — for simple poles — by differentiating p^2+1 and then putting $p = \pm\mathrm{i}$.)



Example 67a. Invert $p^{-\frac{1}{2}}$ (defined as $r^{-\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\theta/2}$, $r = |p|$, $-\pi < \theta < \pi$, so that the branch cut lies along the negative real p-axis).

Solution.
The branch-cut means we need to use a keyhole contour Γ. By Cauchy's theorem,



$$\frac{1}{2\pi\mathrm{i}}\int_\Gamma p^{-\frac{1}{2}}\mathrm{e}^{px}\mathrm{d}p = 0.$$

So

$$\frac{1}{2\pi\mathrm{i}}\int_A^B \cdots\,\mathrm{d}p = -\frac{1}{2\pi\mathrm{i}}\int_C^O \cdots\,\mathrm{d}p - \frac{1}{2\pi\mathrm{i}}\int_O^D \cdots\,\mathrm{d}p$$

as the semicircle (of radius R) gives no contribution as $R \to \infty$.

- On CO, put $p = r\mathrm{e}^{\mathrm{i}\pi}$ so $p^{-\frac{1}{2}} = r^{-\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\pi/2} = -\mathrm{i}r^{-\frac{1}{2}}$.
- On OD, $p = r\mathrm{e}^{-\mathrm{i}\pi}$ and $p^{-\frac{1}{2}} = r^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\pi/2} = \mathrm{i}r^{-\frac{1}{2}}$.

Therefore, as $R \to \infty$,

$$\frac{1}{2\pi \mathrm{i}} \int_C^O p^{-\frac{1}{2}} \mathrm{e}^{px} \mathrm{d}p \to \frac{1}{2\pi \mathrm{i}} \int_\infty^0 (-\mathrm{i}) r^{-\frac{1}{2}} \mathrm{e}^{-rx} (-\mathrm{d}r)$$
$$= -\frac{i}{2\pi \mathrm{i}} \int_0^\infty r^{-\frac{1}{2}} \mathrm{e}^{-rx} \mathrm{d}r$$
$$= -\frac{1}{2\pi} x^{-\frac{1}{2}} \Gamma(\tfrac{1}{2}).$$

Similarly

$$\frac{1}{2\pi \mathrm{i}} \int_O^D \cdots \mathrm{d}p \to -\frac{1}{2\pi} x^{-\frac{1}{2}} \Gamma(\tfrac{1}{2})$$

so adding gives that

$$\frac{1}{2\pi \mathrm{i}} \int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} p^{-\frac{1}{2}} \mathrm{e}^{px} \mathrm{d}p = \frac{2}{2\pi} x^{-\frac{1}{2}} \Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{\pi} x^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi}} x^{-\frac{1}{2}}.$$

3.3 Laplace Transforms and power series

Theorem 68. Suppose the LT of $f(x)$ is the convergent sum

$$\bar{f}(p) = \sum_{n=0}^{\infty} a_n/p^{n+1}, \qquad \mathrm{Re}(p) > p_0.$$

(Recognise this as a Laurent series.) Then

$$f(x) = \sum_{n=0}^{\infty} a_n x^n/n!.$$

Proof. See LN. The key point is that $1/p^{n+1}$ is the LT of $x^n/n!$. The technical issue is to justify taking the summation outside the contour integral.

Note: As this is a Laurent series for $\bar{f}(p)$, its domain of convergence is in fact the annulus $|p| > p_0$. That means that the singularities of $\bar{f}(p)$ all lie within $|p| \le p_0$, so, for example, the transform $p^{-\frac{1}{2}}$ is not covered because of its branch cut.

Example 69a. Let

$$\bar{f}(p) = \frac{1}{p-1} = \frac{1}{p(1-1/p)} = \frac{1}{p}\sum_0^\infty \frac{1}{p^n} = \sum_0^\infty \frac{1}{p^{n+1}},$$

so $f(x) = \sum_0^\infty x^n/n! = \mathrm{e}^x$ (of course).

Final LT example: 55 & 70a. Our final example illustrates the range of LT ideas. The Bessel function of order zero, $J_0(x)$, satisfies

$$xJ_0'' + J_0' + xJ_0 = 0, \quad 0 < x < \infty, \qquad J_0(0) = 1, \quad J_0'(0) = 0.$$

See LN for a plot. This function crops up widely, eg when you separate variables in the radially symmetric 2-D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right);$$

its solution tells you modes of radially symmetric oscillations of a drum (like trig functions for 1-D waves).

Take the LT of the DE

$$xJ_0'' + J_0' + xJ_0 = 0, \qquad J_0(0) = 1, \quad J_0'(0) = 0,$$

to give

$$-\frac{\mathrm{d}}{\mathrm{d}p}\left(p^2\overline{J_0} - p\right) + p\overline{J_0} - 1 - \frac{\mathrm{d}\overline{J_0}}{\mathrm{d}p} = 0,$$

which tidies up to

$$(p^2+1)\frac{\mathrm{d}\overline{J_0}}{\mathrm{d}p} + p\overline{J_0} = 0.$$

Separating this equation gives

$$\overline{J_0}(p) = \frac{A}{\sqrt{p^2+1}};$$

but what is te integration constant A? There is a trick: we know that $\overline{J_0'}(p) \to 0$ as $\mathrm{Re}(p) \to \infty$, and as

$$\overline{J_0'}(p) = p\overline{J_0}(p) - 1 = A\frac{p}{\sqrt{p^2+1}} - 1,$$

we see that $A = 1$ and so $\overline{J_0}(p) = 1/\sqrt{p^2+1}$.

Now by the Binomial theorem (exponent $-\frac{1}{2}$),

$$\begin{aligned}
\overline{J_0}(p) &= (p^2+1)^{-\frac{1}{2}} \\
&= \frac{1}{p}(1+1/p^2)^{-\frac{1}{2}} \quad \text{(we need to expand for large } p\text{)} \\
&= \frac{1}{p}\sum_{k=0}^{\infty} \frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{1-2k}{2})}{k!}\left(\frac{1}{p^2}\right)^k \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{2^{2k}(k!)^2}\frac{1}{p^{2k+1}}
\end{aligned}$$

so, as $(2k)!/p^{2k+1} \leftrightarrow x^{2k}$, the power series for $J_0(x)$ is

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}(k!)^2} x^{2k}.$$

We end with a very pretty result. Note that

$$\left(\overline{J_0}(p)\right)^2 = \left(\frac{1}{\sqrt{p^2+1}}\right)^2 = \frac{1}{p^2+1} = \mathcal{L}\sin x.$$

But by the convolution theorem, $\left(\overline{J_0}\right)^2$ is the LT of $J_0 * J_0$. Thus,

$$\int_0^x J_0(t)J_0(x-t)\,\mathrm{d}t = \sin x,$$

a beautiful and unexpected result with which to end our coverage of the LT.

4.1 The Fourier Transform: definition

We now turn to functions defined on all of $\mathbb{R}$ and use trig functions to take weighted averages (cf Fourier series on a finite interval).

Definition 71. If $f(x)\colon \mathbb{R} \to \mathbb{C}$ is integrable, its *Fourier Transform* $\widehat{f}(s)$, also written $\mathcal{F}f(s)$, is

$$\widehat{f}(s) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}x.$$

Note. The class of functions that have a FT is smaller than thise with a LT because the function must be integrable (eg, e^{x} is not allowed).

Some examples

Examples 73, 74a, 75, 76. In all these examples, $a > 0$ is real.

- Suppose that $f(x)$ is the indicator function

$$f(x) = \mathbb{1}_{[-a,a]} = \begin{cases} 0 & |x| > a, \\ 1 & |x| \leq a. \end{cases}$$

Then

$$\widehat{f}(s) = \int_{-a}^{a} \mathrm{e}^{-\mathrm{i}sx} \mathrm{d}x = \left[\frac{\mathrm{e}^{-\mathrm{i}sx}}{-\mathrm{i}s} \right]_{-a}^{a} = \frac{2 \sin as}{s}.$$

- If $f(x) = \delta(x - a)$, then by sifting $\widehat{f}(s) = \mathrm{e}^{-\mathrm{i}as}$ (for all real a).
- If $f(x) = \mathrm{e}^{-a|x|}$, then

$$\widehat{f}(s) = \int_{-\infty}^{0} \mathrm{e}^{ax - \mathrm{i}sx} \mathrm{d}x + \int_{0}^{\infty} \mathrm{e}^{-ax - \mathrm{i}sx} \mathrm{d}x = \frac{1}{a - \mathrm{i}s} + \frac{1}{a + \mathrm{i}s} = \frac{2a}{s^2 + a^2}.$$

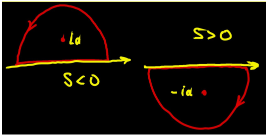
- If $f(x) = a/(x^2 + a^2)$, then

$$\widehat{f}(s) = \int_{-\infty}^{\infty} \frac{a}{x^2 + a^2} \mathrm{e}^{-\mathrm{i}sx} \mathrm{d}x.$$

This is a routine contour integral (like the Laplace inversion of $1/(p^2 + 1)$ earlier). The function

$$f(z) = \frac{a\mathrm{e}^{-\mathrm{i}sz}}{z^2 + a^2}$$

has poles at $z = \pm\mathrm{i}a$ at which the residues are $\pm\mathrm{e}^{\pm ax}/(2\mathrm{i})$. Close with a semicircular contour in the UHP if $s < 0$ and in the LHP if $s > 0$ to get exponential decay of $\mathrm{e}^{-\mathrm{i}sz}$. The result is



$$\widehat{f}(s) = \begin{cases} \pi\mathrm{e}^{as} & s < 0, \\ \pi\mathrm{e}^{-as} & s > 0, \end{cases} \quad \text{which is} \quad \widehat{f}(s) = \pi\mathrm{e}^{-a|s|}.$$

Is this a coincidence? NO!

- If $f(x) = \mathrm{e}^{-a^2x^2}$, then

$$\begin{aligned}
\widehat{f}(s) &= \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}sx - a^2x^2}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{-s^2/4a^2}\mathrm{e}^{-a^2(x+\mathrm{i}s/2a^2)^2}\,\mathrm{d}x \quad \text{(completing the square)} \\
&= \mathrm{e}^{-s^2/4a^2}\int_{-\infty}^{\infty} \mathrm{e}^{-a^2u^2}\,\mathrm{d}u \quad (\text{by } x+\mathrm{i}s/2a^2 = u) \\
&= \frac{\sqrt{\pi}}{a}\mathrm{e}^{-s^2/4a^2}.
\end{aligned}$$

The change of variable $x + \mathrm{i}s/2a^2 = u$ is equivalent to moving the integration contour up by using Cauchy's theorem on a rectangular contour; see LN.

We shall return to these examples in the next lecture.

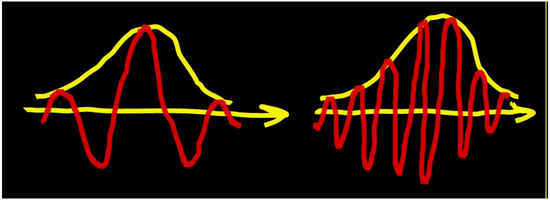
4.2 Properties of the Fourier Transform

There is a clear similarity between properties of the FT and those of the LT (after all, both use an exponential weight). The FT ones are simpler, because it is defined on all of $\mathbb{R}$.

Proposition 79 (Riemann–Lebesgue Lemma). If $f(x)$ is integrable, then

$$\lim_{s\to\pm\infty}\int_{-\infty}^{\infty} f(x)\cos sx\,[\text{or } f(x)\sin sx]\,\mathrm{d}x = 0.$$

Proof. Covered in Part A Integration. It works by cancellation of positive and negative areas of a rapidly oscillating signal.



Theorem 80a. Let $f(x)$ be integrable, vanish at infinity, and assume that all FTs below exist. Then:

(a) $\widehat{f}(s) \to 0$ as $s \to \pm\infty$.

(b) $\widehat{f}(0) = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ (and if f is a PDF then $\widehat{f}(0) = 1$).

(c) $\widehat{f'}(s) = \mathrm{i}s\widehat{f}(s)$.

(d) $\widehat{xf}(s) = \mathrm{i}\mathrm{d}\widehat{f}/\mathrm{d}s$.

(e) $(\mathrm{e}^{\mathrm{i}ax} f(x))\hat{}(s) = \widehat{f}(s-a)$.

(f) $(f(x-a))\hat{}(s) = \mathrm{e}^{-\mathrm{i}as}\widehat{f}(s)$.

Note that (c) and (d), and (e) and (f), go together. Compare these with the corresponding LT results: the FT is ‘cleaner’. Note also that (c) and (d) remind us of position and momentum operators in QM.

Proof.

(a) is the Riemann–Lebesgue Lemma.

(b) Put $s = 0$ in the definition of $\widehat{f}(s)$.

(c) Integrate by parts:

$$\begin{aligned}\widehat{f'}(s) &= \int_{-\infty}^{\infty} f'(s)\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}x \\ &= \left[f(x)\mathrm{e}^{-\mathrm{i}sx}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-\mathrm{i}s)\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}x \\ &= \mathrm{i}s\widehat{f}(s).\end{aligned}$$

(d) Differentiate under the integral:

$$\mathrm{i}\frac{\mathrm{d}\widehat{f}}{\mathrm{d}s} = \mathrm{i}\int_{-\infty}^{\infty} f(x)(-\mathrm{i}x)\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}x = \widehat{xf}(s).$$

(e) and (f), left as exercises.

The Fourier convolution theorem

Definition 81. The convolution on $\mathbb{R}$ of two functions f and g is

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,\mathrm{d}t.$$

Notes. (a) In probablility, the PDF of the sum of two independent random variables X and Y is $f_{X+Y}(x) = (f_X * f_Y)(x)$.
(b) When we looked at the LT, we defined functions on $[0, \infty)$ only. You can get to the Laplace form of the convolution by using the definition above on functions of the form $f(x)\mathcal{H}(x)$ (exercise).

No surprise: there is a convolution theorem for the FT.

Theorem 81. If $f, g\colon \mathbb{R} \to \mathbb{C}$, and the relevant FTs exist, then

$$\widehat{f * g}(s) = \widehat{f}(s)\widehat{g}(s).$$

Proof. As for the LT:

$$\begin{aligned}
\widehat{f}(s)\widehat{g}(s) &= \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}sx}\mathrm{d}x \int_{-\infty}^{\infty} g(y)\mathrm{e}^{-\mathrm{i}sy}\mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)\mathrm{e}^{-\mathrm{i}s(x+y)}\mathrm{d}x\mathrm{d}y \\
(*) \qquad &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(u-x)\mathrm{e}^{-\mathrm{i}su}\mathrm{d}x\mathrm{d}u \quad (x+y=u) \\
&= \int_{-\infty}^{\infty}\underbrace{\int_{-\infty}^{\infty} f(x)g(u-x)\,\mathrm{d}x}_{(f*g)(u)}\,\mathrm{e}^{-\mathrm{i}su}\mathrm{d}u \\
&= \widehat{f * g}(s).
\end{aligned}$$

Example 83a. The convolution of $\delta(x-a)$ and $f(x)$ is

$$\int_{-\infty}^{\infty} \delta(t-a) f(x-t)\,\mathrm{d}t = f(x-a);$$

the FT of $\delta(x-a)$ is $\mathrm{e}^{-\mathrm{i}as}$ (sifting), and the FT of the convolution $f(x-a)$ is (see earlier) $\mathrm{e}^{-\mathrm{i}as}\widehat{f}(s)$.

Towards the inversion theorem

Recall two examples of FT pairs from Lecture 7. We had

$$\widehat{e^{-a|x|}}(s) = \frac{2a}{s^2 + a^2} = 2\pi \cdot \underbrace{\frac{a}{\pi(s^2 + a^2)}}_{\text{integrates to 1}} .$$

But as $a \to 0$,

$$e^{-a|x|} \to 1 \quad \text{and} \quad \frac{a}{\pi(s^2 + a^2)} \to \delta(s)$$

(recall Sheet 1 Exercise 8). Similarly, as $a \to 0$,

$$e^{-a^2x^2} \to 1 \quad \text{and} \quad \widehat{e^{-a^2x^2}} = \frac{\sqrt{\pi}}{a} e^{-s^2/4a^2} = 2\pi \frac{e^{-s^2/4a^2}}{\sqrt{4\pi a^2}} \to 2\pi\delta(s)$$

(note that $e^{-s^2/4a^2}/\sqrt{2\pi \cdot 2a^2}$ is the PDF of $N(0, 2a^2)$).

This strongly suggests the amazing result that

$$\widehat{1}(s) = 2\pi\delta(s).$$

Corollary.

The FT of $\mathrm{e}^{\mathrm{i}ax}$ is $2\pi\delta(s-a)$.

Proof (of corollary). The FT of $\mathrm{e}^{\mathrm{i}ax}f(x)$ is $\widehat{f}(s-a)$. Put $f(x) = 1$.

We don't have a complete framework for the FT of distributions, as needed to show properly that $\widehat{1}(s) = 2\pi\delta(s)$. We'll accept that it is correct, and we'll use it to great effect before long. See the last few pages of SN for a sketch of how to do this.

Note: these results do *not* say that functions like $\mathrm{e}^{\mathrm{i}ax}$ are integrable in any classical sense. They are statements about distributions and their FTs in the same way that $\delta(x)$ is not a classical function.

The Fourier inversion theorem

Theorem 84. Let $f(x)$ be continuous and integrable. Then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(s)\mathrm{e}^{\mathrm{i}xs}\,\mathrm{d}s \quad \text{where} \quad \widehat{f}(s) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}x.$$

Proof. We have

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(s)\mathrm{e}^{\mathrm{i}xs}\,\mathrm{d}s &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(y)\mathrm{e}^{-\mathrm{i}sy}\,\mathrm{d}y\right)\mathrm{e}^{\mathrm{i}xs}\,\mathrm{d}s \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)\mathrm{e}^{\mathrm{i}s(x-y)}\,\mathrm{d}y\mathrm{d}s \\
(*) \qquad &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\underbrace{\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}xs}\mathrm{e}^{-\mathrm{i}ys}\,\mathrm{d}s}_{\text{FT (in } s\text{) of } \mathrm{e}^{\mathrm{i}xs}}\mathrm{d}y \\
(**) \qquad &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\cdot 2\pi\delta(y-x)\,\mathrm{d}y \\
&= f(x).
\end{aligned}$$

Notes. (1) We swapped the order of integration in key step $(*)$; in the term labelled 'FT (in s) of e^{ixs}', we used the result $\widehat{e^{iax}} = \delta(s-a)$ and relabelled the variables $a, x, s \mapsto x, s, y$: this is, I agree, a bit confusing (but inevitable).

(2) There are lots of classical proofs, all of which boil down to approximating the delta function in steps $(*)$-$(**)$.

(3) If $f(x)$ has a jump, the inverse FT gives the average of the left- and right-hand limits (like Fourier series).

Example. Invert $\widehat{f}(s) = e^{-a|s|}$ (here $a > 0$).

Solution. By the inversion formula,

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-a|s|}e^{ixs}\,ds \\ &= \frac{1}{2\pi}\int_{-\infty}^{0} e^{as+ixs}\,ds + \frac{1}{2\pi}\int_{0}^{\infty} e^{-as+ixs}\,ds \\ &= \frac{1}{2\pi}\frac{2a}{x^2+a^2} = \frac{a}{\pi(x^2+a^2)}. \end{aligned}$$

Aside: Fourier Transforms and Fourier Series

At the heart of the FT is the idea of expanding in an orthogonal basis. You have seen this:

• In finite dimensions: linear algebra, a basis of vectors eg $\{\mathbf{e}_i\}$ in $\mathbb{R}^n$, orthogonality

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \quad \text{(Kronecker Delta; over } \mathbb{C}^n \text{ it is } \mathbf{e}_i \cdot \overline{\mathbf{e}_j} = \delta_{ij}\text{);}$$

• In a countably infinite setting: eg Fourier series, basis of trig functions (in complex form) $\{e^{inx}\}$, with orthogonality

$$\int_{-\pi}^{\pi} e^{imx} e^{-inx} \, dx = 2\pi\delta_{mn};$$

• Now in an uncountably infinite setting: FT, basis functions $\{e^{isx}\}$, orthogonality as the 'integral'

$$\int_{-\infty}^{\infty} e^{ixs} e^{-iys} \, ds = 2\pi\delta(y - x).$$

Clearly we need theory to back this up! But not in this course.

Now compare the FT and Fourier series:

- If $f(x)$ is 2π-periodic then it has the complex FS[12]

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{where} \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx.$$

- And for a function on all of $\mathbb{R}$,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(s) e^{ixs}\, ds \quad \text{where} \quad \widehat{f}(s) = \int_{-\infty}^{\infty} f(x) e^{-isx}\, dx.$$

Apart from the position of 2π, the correspondence is clear.

And now if $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$, using $\widehat{e^{inx}} = \delta(s-n)$, we get the frequency decomposition of a periodic function:

$$\widehat{f}(s) = \sum_{n=-\infty}^{\infty} c_n \delta(s-n).$$

[12] This is easily seen to be equivalent to the all-real series in terms of son and cos, but much cleaner for our needs.

The Laplace Inversion Theorem

We end this lecture by deducing the Laplace inversion theorem from Fourier inversion.

Corollary 85 (to Theorem 84). Let the continuous function $f(x)$ have Laplace Transform $\bar{f}(p)$ for $\mathrm{Re}(p) > p_0$; then, for $x > 0$,

$$f(x) = \frac{1}{2\pi\mathrm{i}} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \bar{f}(p)\mathrm{e}^{xp}\,\mathrm{d}p \qquad (\sigma > p_0).$$

Proof. Put $p = \sigma + \mathrm{i}s$. Then

$$\begin{aligned} \bar{f}(\sigma+\mathrm{i}s) &= \int_0^\infty f(x)\mathrm{e}^{-(\sigma+\mathrm{i}s)x}\mathrm{d}x \\ &= \int_0^\infty \left(f(x)\mathrm{e}^{-\sigma x}\right)\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}x \\ &= \left[f(x)\mathrm{e}^{-\sigma x}\mathcal{H}(x)\right]\hat{}. \end{aligned}$$

We have found that $\bar{f}(\sigma + \mathrm{i}s)$ is the FT of $f(x)\mathrm{e}^{-\sigma x}\mathcal{H}(x)$. By Fourier inversion, for $x > 0$ (this is where $\mathcal{H}(x) = 1$)

$$f(x)\mathrm{e}^{-\sigma x} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(\sigma + \mathrm{i}s)\mathrm{e}^{\mathrm{i}xs}\,\mathrm{d}s$$

Take $\mathrm{e}^{-\sigma x}$ to the RHS and insert i twice:

$$f(x) = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \bar{f}(\sigma + \mathrm{i}s)\mathrm{e}^{x(\sigma+\mathrm{i}s)}\,\mathrm{i}\,\mathrm{d}s.$$

But $\sigma + \mathrm{i}s = p$, so

$$f(x) = \frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \bar{f}(p)\mathrm{e}^{xp}\,\mathrm{d}p.$$

Using the inversion theorem

Example 90a. Use the inversion theorem to evaluate

$$I_1 = \int_{-\infty}^{\infty} \frac{\sin s}{s}\,\mathrm{d}s \quad \text{and} \quad I_2 = \int_{-\infty}^{\infty} \frac{1}{s^2+1}\,\mathrm{d}s.$$

Solution. These depend on the fact that $f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(s)\,\mathrm{d}s$ (analogous to $\widehat{f}(0) = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$). For the first integral, set

$$f(x) = \mathbb{1}_{[-1,1]} \quad \text{so} \quad \widehat{f}(s) = \frac{2\sin s}{s}.$$

Hence

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin s}{s} \mathrm{e}^{\mathrm{i}xs}\,\mathrm{d}s$$

and putting $x = 0$ gives $I_1 = \pi$. For I_2 use $f(x) = \mathrm{e}^{-|x|}$.

There are other (more complicated) examples in LN.

Some differential equation examples

Example. Invert $\widehat{f}(s) = \mathrm{e}^{-ts^2}$ $(t > 0)$. We have seen this before, reverse-engineering the transform of e^{-ax^2}.

Solution. A little trick:

$$\frac{\mathrm{d}\widehat{f}}{\mathrm{d}s} = -2ts\mathrm{e}^{-ts^2} = -2ts\widehat{f}.$$

so

$$\mathrm{i}\frac{\mathrm{d}\widehat{f}}{\mathrm{d}s} = \widehat{xf} = -2t(\mathrm{i}s\widehat{f}) = -2t\widehat{f'}.$$

Inverting,

$$xf = -2tf', \quad \text{or} \quad f' = -\frac{x}{2t}f.$$

This separable ODE has the solution $f(x) = f(0)\mathrm{e}^{-x^2/4t}$. And

$$f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(s)\,\mathrm{d}s = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-ts^2}\,\mathrm{d}t = \frac{1}{2\sqrt{\pi t}},$$

so $f(x) = \mathrm{e}^{-x^2/4t}/2\sqrt{\pi t}$.

Example. Solve the differential equation problem

$$f'' + xf' + f = 0, \qquad f(x) \to 0 \quad \text{as} \quad x \to \pm\infty, \quad \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$$

(the integral condition normalises f).

Solution. Take the FT in x:

$$(\mathrm{i}s)^2 \widehat{f} + \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{i}s\widehat{f}) + \widehat{f} = 0,$$

tidying up to

$$\frac{\mathrm{d}\widehat{f}}{\mathrm{d}s} = -s\widehat{f}, \quad \text{giving} \quad \widehat{f}(s) = \widehat{f}(0)\mathrm{e}^{-s^2/2}.$$

But $\widehat{f}(0) = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$, so $\widehat{f}(s) = \mathrm{e}^{-s^2/2}$ and then (use previous example with $t = \frac{1}{2}$) we get $f(x) = \mathrm{e}^{-x^2/2}/\sqrt{2\pi}$ (this is the PDF of $N(0,1)$).

Heat equation initial-value problem

Example 95a. Find $u(x,t)$ satisfying

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad t > 0, \quad -\infty < x < \infty, \quad \text{with} \quad u(x,0) = f(x),$$

with both $f(x)$ and $u(x,t)$ vanishing at $x = \pm\infty$. This is the evolution of temperature in an infinite bar from its initial distribution $f(x)$. The conditions at infinity are technical.

Solution. Take the FT in x, $\widehat{u}(s,t) = \int_{-\infty}^{\infty} u(x,t)\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}s$, to give

$$\frac{\partial \widehat{u}}{\partial t} = (\mathrm{i}s)^2\widehat{u} = -s^2\widehat{u} \qquad \text{with} \quad \widehat{u}(s,0) = \widehat{f}(s).$$

The solution is $\widehat{u}(s,t) = \widehat{f}(s)\mathrm{e}^{-ts^2}$, so by convolution

$$u(x,t) = \left(f(x) * \frac{1}{2\sqrt{\pi t}}\mathrm{e}^{-x^2/4t} \right) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(y)\mathrm{e}^{-(x-y)^2/4t}\,\mathrm{d}y \quad (\dagger)$$

Note: when $f(x) = \delta(x)$, $u(x,t) = \mathrm{e}^{-x^2/4t}/2\sqrt{\pi t}$, representing a point unit amount of heat diffusing away from $x = 0$, called the *fundamental solution*. Eqn ($\dagger$) at $t \to 0$ illustrates $\delta_\epsilon \to \delta$...

Laplace's equation in a half-space

Example 93. Find $u(x, y)$ satisfying

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty,\ y > 0, \quad \text{with} \quad u(x, 0) = f(x)$$

and the decay conditions $u, |\nabla u| = O(1/\sqrt{x^2 + y^2})$ at infinity.[13]

This problem models the steady heat-flow in a half-plane with a prescribed temperature $f(x)$ on $y = 0$.

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[13] These conditions (more important than the corresponding ones for the heat equation) let you use a standard proof (integrate $\nabla \cdot (u \nabla u)$ over the inside of a large semicircle) to show uniqueness. Without them, you can add solutions such as $u = y$ or $u = xy$.

Solution. Take the FT in x to get

$$-s^2\widehat{u} + \frac{\partial^2\widehat{u}}{\partial y^2} = 0, \qquad \widehat{u}(s,0) = \widehat{f}(s).$$

Solutions are of the form $A\mathrm{e}^{sy} + B\mathrm{e}^{-sy}$ or, equivalently, $C\mathrm{e}^{|s|y} + D\mathrm{e}^{-|s|y}$. Remember that $y > 0$: the solution that decays as $y \to \infty$ is $\widehat{u}(s,y) = \widehat{f}(s)\mathrm{e}^{-|s|y}$. But we know that

$$\mathrm{e}^{-y|s|} \quad \text{is the FT (in } x\text{) of} \quad g(x,y) = \frac{y}{\pi(x^2+y^2)}.$$

so, by the convolution theorem (in x),

$$u(x,y) = (f * g)(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{(x-t)^2+y^2}\,\mathrm{d}t.$$

Parseval and Plancherel

We end with two important results.

Example 84 (Parseval's theorem). If $f(x)$ and $g(x)$ have FTs $\widehat{f}(s)$ and $\widehat{g}(s)$, then (using an overline for complex conjugate)

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(s)\overline{\widehat{g}(s)}\,\mathrm{d}s.$$

Corollary (Plancherel's theorem). Putting $g = f$, we have

$$\int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f}(s)|^2\,\mathrm{d}s.$$

This says that the 'energy' in f and its FT are the same (up to a scaling of 2π). It is the analogue of saying that if $f(x)$ has Fourier series $\sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}nx}$ then

$$\int_{-\pi}^{\pi} |f(x)|^2 = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2.$$

Parseval says the 'angle' between f and g is the same as that between $\widehat{f}$ and $\widehat{g}$.

Proof (of Parseval). Intuitive version: start with

$$\begin{aligned}
\int_{-\infty}^{\infty} \widehat{f}(s)\overline{\widehat{g}(s)}\,\mathrm{d}s &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}x\right)\left(\int_{-\infty}^{\infty}\overline{g(y)}\mathrm{e}^{\mathrm{i}sy}\,\mathrm{d}y\right)\mathrm{d}s \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\overline{g(y)}\mathrm{e}^{\mathrm{i}s(y-x)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\overline{g(y)}\left(\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}s(y-x)}\,\mathrm{d}s\right)\mathrm{d}x\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\overline{g(y)}\cdot 2\pi\delta(y-x)\,\mathrm{d}x\,\mathrm{d}y \qquad (*) \\
&= \int_{-\infty}^{\infty} f(y)\overline{g(y)}\,\mathrm{d}y.
\end{aligned}$$

(For the manouevre giving $(*)$, see the corresponding step in our proof of the Fourier inversion theorem.)

Although the proof lacks full rigour (step $(*)$, changing order of integration etc) it is in the spirit of the course!

Another proof (sketch; fill in the gaps). This is a standard proof combining convolution and inversion.

- Let $h(x) = \overline{g(-x)}$. Then show $\int f(t)\overline{g(t)}\,\mathrm{d}t$ is the same as $f * h$ evaluated at $x = 0$.
- Note $\widehat{h} = \overline{\widehat{g}}$.
- By convolution, $\widehat{f * h} = \widehat{f}\,\widehat{h} = \widehat{f}\,\overline{\widehat{g}}$.
- Invert: $(f * h)(x) = \frac{1}{2\pi}\int \widehat{f}(s)\overline{\widehat{g}(s)}\mathrm{e}^{\mathrm{i}xs}\,\mathrm{d}s$.
- Put $x = 0$ on both sides.

Final example. Evaluate $\int_{-\infty}^{\infty} \sin^2 s/s^2\,\mathrm{d}s$.

Solution. Set $f(x) = \mathbb{1}_{[-1,1]}$, so $\widehat{f}(s) = 2\sin s/s$ and use Plancherel.

Final comments

I hope you have liked this course as much as I have. It brings together a number of deep ideas. It reaches back to earlier courses (especially complex analysis) and opens the door to a huge range of further topics.

The final chapter of LN is a quick overview of where you can go next. Possible destinations range from a proper theoretical (functional analysis) treatment of all aspects of the course to a wide variety of applications.

The SN contain quite a few asides, especially on the FT, and a little appendix showing how to take rge FT of a distribution rigorously. None of these are examinable but do read them.

Enjoy!

Postscript: Distributions and the FT

This is NOT examinable and it is FOR INTEREST ONLY. But it is a low-hanging fruit.

How do we define the FT of a distribution properly? We need to define the action on a test function.

For an ordinary integrable function $f(x)$ with FT $\widehat{f}(s)$, and with $\phi(\cdot)$ a test function, we have

$$
\begin{aligned}
\langle \widehat{f}, \phi \rangle &= \int_{-\infty}^{\infty} \widehat{f}(s) \phi(s) \, \mathrm{d}s \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-\mathrm{i}sx} \, \mathrm{d}x \, \phi(s) \, \mathrm{d}s \\
&= \int_{-\infty}^{\infty} f(x) \int_{-\infty}^{\infty} \phi(s)) \mathrm{e}^{-\mathrm{i}sx} \, \mathrm{d}s \, \mathrm{d}x \\
&= \langle f, \widehat{\phi} \rangle.
\end{aligned}
$$

And this is our definition: for any distribution F, its FT is another distribution $\widehat{F}$ whose action on a test function ϕ is

$$\langle \widehat{F}, \phi \rangle = \langle F, \widehat{\phi} \rangle.$$

You see how we define the new oblect $\widehat{F}$ in terms of something we already know about, namely F. Note also that this is consistent with the result for integrable functions above, and note how it depends on the symmetry of xs in the kernel of the FT.

The inverse FT of a distribution is defined in the same way (replace taking the FT by taking the inverse in the above).

One rarely uses these definitions verbatim but they are always there to underpin more intuitive short cuts, as shown by the following examples.

Example. The delta function has FT with action

$$\begin{aligned}\langle \widehat{\delta}, \phi \rangle &= \langle \delta, \widehat{\phi} \rangle \\ &= \widehat{\phi}(0) \qquad \text{(ordinary action of } \delta) \\ &= \int_{-\infty}^{\infty} \phi(x)\,\mathrm{d}x \\ &= \langle 1, \phi \rangle,\end{aligned}$$

so $\widehat{\delta} = 1$ as expected. Notice that we only write down an ordinary integral when it is meaningful.

And similarly

$$\begin{aligned}\langle \widehat{1}, \phi \rangle &= \langle 1, \widehat{\phi} \rangle \\ &= \int_{-\infty}^{\infty} \widehat{\phi}(s)\,\mathrm{d}s \\ &= 2\pi\phi(0) \qquad \text{by the usual inversion formula} \\ &= 2\pi\langle \delta, \phi \rangle,\end{aligned}$$

so $\widehat{1} = 2\pi\delta$, again as expected.[14]

[14] There is a certain circularity here, as we used the formula $\widehat{1} = 2\pi\delta$ to derive the inversion formula. But we can derive it by other (standard) means so this is not a worry.

Did you spot the technical issue above? If we are to write down

$$\langle \widehat{F}, \phi \rangle = \langle F, \widehat{\phi} \rangle,$$

we need $\widehat{\phi}$ to be a test function whenever ϕ is. Unfortunately this is not the case for the compact-support test functions we used in this course. However, one can use a variation on the definition of a test function: $\phi(x)$ is a test function if it is C^∞ and, as $x \to \pm\infty$, $\phi(x)\mathrm{e}^{c|x|} \to 0$ for all real $c > 0$, with a similar condition on all the derivatives of ϕ. That is, $\phi(x)$ and its derivatives decay faster than exponentially at infinity. Example: $\phi(x) = \mathrm{e}^{-x^2}$. It is then not hard to show that if ϕ is a test function, then so is its FT $\widehat{\phi}$.

With this modification the theory goes through fine and the resulting distributions are called *tempered distributions*, as opposed to the *Schwarz distributions* we used earlier. For all practical purposes the two are the same. That is why I have written this note in small type!