Numerical Analysis Hilary Term 2025
Lecture 1: Lagrange Interpolation

Numerical analysis is the study of computational algorithms for solving problems in scientific computing. It combines mathematical beauty, rigor and numerous applications; we hope you'll enjoy it! In this course we will cover the basics of three key fields in the subject:

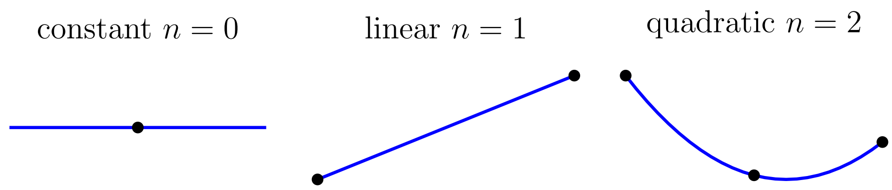
- Approximation Theory (lectures 1, 9–11); recommended reading: L. N. Trefethen, Approximation Theory and Approximation Practice, and E. Süli and D. F. Mayers, An Introduction to Numerical Analysis.
- Numerical Linear Algebra (lectures 2–8); recommended reading: L. N. Trefethen and D. Bau, Numerical Linear Algebra.
- Numerical Solution of Differential Equations (lectures 12–16); recommended reading: E. Süli and D. F. Mayers, An Introduction to Numerical Analysis.

This first lecture comes from Chapter 6 of Süli and Mayers.

Notation: $\Pi_n \stackrel{\text{def}}{=} \{\text{real polynomials of degree} \le n\}$

Setup: Given data f_i at distinct x_i, $i = 0, 1, \ldots, n$, with $x_0 < x_1 < \cdots < x_n$, can we find a polynomial p_n such that $p_n(x_i) = f_i$? Such a polynomial is said to **interpolate** the data, and (as we shall see) can approximate f at other values of x if f is smooth enough. This is the most basic question in approximation theory.

E.g.:



Theorem. The interpolating polynomial of degree $\le n$ exists. That is, there exists $p_n \in \Pi_n$ such that $p_n(x_i) = f_i$ for $i = 0, 1, \ldots, n$.

Proof. Consider, for $k = 0, 1, \ldots, n$, the "cardinal polynomial"

$$L_{n,k}(x) = \frac{(x - x_0) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_0) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)} \in \Pi_n. \tag{1}$$

Then $L_{n,k}(x_i) = \delta_{ik}$, that is,

$$L_{n,k}(x_i) = 0 \ \text{ for } \ i = 0, \ldots, k-1, k+1, \ldots, n \ \text{ and } \ L_{n,k}(x_k) = 1.$$

So now define

$$p_n(x) = \sum_{k=0}^{n} f_k L_{n,k}(x) \in \Pi_n, \tag{2}$$

so that

$$p_n(x_i) = \sum_{k=0}^{n} f_k L_{n,k}(x_i) = f_i \quad \text{for} \quad i = 0, 1, \ldots, n. \qquad \square$$

The polynomial (2) is the **Lagrange interpolating polynomial**.

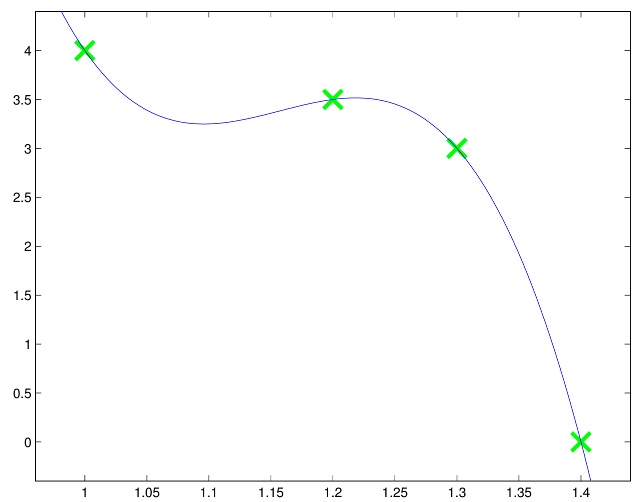
Theorem. The interpolating polynomial of degree $\leq n$ is unique. That is, if there exists a polynomial $p \in \Pi_n$ satisfying $p(x_i) = f_i$ for $i = 0, 1, \ldots, n$, then p is unique (and equal to the Lagrange interpolating polynomial).

Proof. Consider two interpolating polynomials $p_n, q_n \in \Pi_n$. Their difference $d_n = p_n - q_n \in \Pi_n$ satisfies $d_n(x_k) = 0$ for $k = 0, 1, \ldots, n$. That is, d_n is a polynomial of degree at most n and has at least $n + 1$ distinct roots. Algebra $\Longrightarrow$ $d_n \equiv 0 \Longrightarrow p_n = q_n$. $\square$

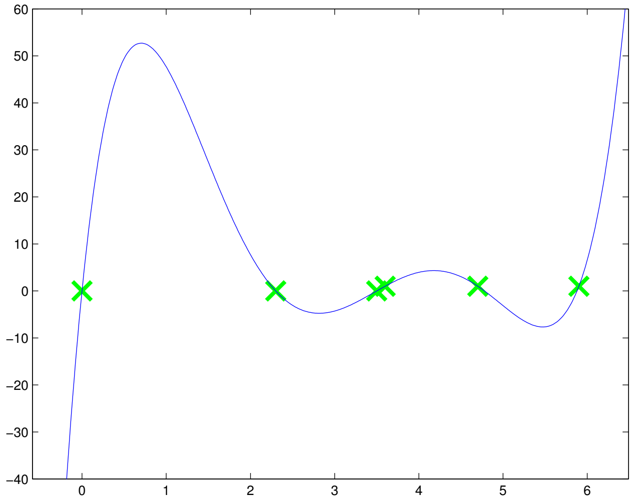
Matlab: `lagrange.m`

```
>> help lagrange
 LAGRANGE  Plots the Lagrange polynomial interpolant for the
 given DATA at the given KNOTS

>> lagrange([1,1.2,1.3,1.4],[4,3.5,3,0]);
```




```
>> lagrange([0,2.3,3.5,3.6,4.7,5.9],[0,0,0,1,1,1]);
```



Data from an underlying smooth function: Suppose that $f(x)$ has at least $n+1$ smooth derivatives in the interval (x_0, x_n). Let $f_k = f(x_k)$ for $k = 0, 1, \ldots, n$, and let p_n be the Lagrange interpolating polynomial for the data (x_k, f_k), $k = 0, 1, \ldots, n$.

Error: How large can the error $f(x) - p_n(x)$ be on the interval $[x_0, x_n]$?

Theorem. For every $x \in [x_0, x_n]$ there exists $\xi = \xi(x) \in (x_0, x_n)$ such that

$$e(x) \stackrel{\text{def}}{=} f(x) - p_n(x) = (x - x_0)(x - x_1) \cdots (x - x_n) \frac{f^{(n+1)}(\xi)}{(n+1)!}, \tag{3}$$

where $f^{(n+1)}$ is the $(n+1)$-st derivative of f.

Proof. For $x = x_k$, $k = 0, 1, \ldots, n$, the error $e(x) = 0$ by construction, and so we can take e.g. $\xi(x_k) \stackrel{\text{def}}{=} x_k$. Now suppose that $x \notin \{x_0, x_1, \ldots, x_n\}$. Let

$$\phi(t) \stackrel{\text{def}}{=} e(t) - \frac{e(x)}{\pi(x)} \pi(t),$$

where

$$\begin{aligned} \pi(t) &\stackrel{\text{def}}{=} (t - x_0)(t - x_1) \cdots (t - x_n) \\ &= t^{n+1} - \left(\sum_{i=0}^{n} x_i \right) t^n + \cdots (-1)^{n+1} x_0 x_1 \cdots x_n \in \Pi_{n+1}. \end{aligned}$$

Note that ϕ vanishes at $n+2$ points: x and x_k, $k = 0, 1, \ldots, n$. $\Longrightarrow$ ϕ' vanishes at $n+1$ points $\xi_0, \ldots, \xi_n$ between these points (i.e. $\xi_k \in (x_k, x_{k+1})$, $k = 0, 1, \ldots, n-1$.) $\Longrightarrow$ ϕ'' vanishes at n points between these new points, and so on until $\phi^{(n+1)}$ vanishes at an (unknown) point ξ in (x_0, x_n). But

$$\phi^{(n+1)}(t) = e^{(n+1)}(t) - \frac{e(x)}{\pi(x)} \pi^{(n+1)}(t) = f^{(n+1)}(t) - \frac{e(x)}{\pi(x)} (n+1)!$$

since $p_n^{(n+1)}(t) \equiv 0$ and because $\pi(t)$ is a monic polynomial of degree $n+1$. The result then follows immediately from this identity since $\phi^{(n+1)}(\xi) = 0$.

□

The above proof may seem ingenious/mysterious. It is perhaps helpful to observe the connections and similarity to Taylor's theorem with remainder, and its proof. Indeed the latter can be seen as a special case of the theorem when x_i all tend to a single point $x_i \to x_*$.

Example: $f(x) = \log(1+x)$ on $[0,1]$. Here, $|f^{(n+1)}(\xi)| = n!/(1+\xi)^{n+1} < n!$ on $(0,1)$. So $|e(x)| < |\pi(x)|n!/(n+1)! \le 1/(n+1)$ since $|x - x_k| \le 1$ for each x, x_k, $k = 0, 1, \dots, n$, in $[0,1] \implies |\pi(x)| \le 1$. This is probably pessimistic for many x, e.g. for $x = \frac{1}{2}$, $\pi(\frac{1}{2}) \le 2^{-(n+1)}$ as $|\frac{1}{2} - x_k| \le \frac{1}{2}$.

This shows the important fact that the error can be large at the end points when samples $\{x_k\}$ are equispaced points, an effect known as the "Runge phenomena" (Carl Runge, 1901), which we return to in lecture 4. More generally, as the expression (3) suggests, the location of the samples $\{x_k\}$ is very important; if one can choose them arbitrarily, it is best to not use equispaced points (see the lecture notes on Gauss quadrature).

Generalisation: Given data f_i and g_i at distinct x_i, $i = 0, 1, \dots, n$, with $x_0 < x_1 < \cdots < x_n$, can we find a polynomial p such that $p(x_i) = f_i$ and $p'(x_i) = g_i$? (i.e., interpolate derivatives in addition to values)

Theorem. There exists a unique polynomial $p_{2n+1} \in \Pi_{2n+1}$ such that $p_{2n+1}(x_i) = f_i$ and $p'_{2n+1}(x_i) = g_i$ for $i = 0, 1, \dots, n$.

Construction: Given $L_{n,k}(x)$ in (1), let

$$\begin{aligned} H_{n,k}(x) &= [L_{n,k}(x)]^2(1 - 2(x - x_k)L'_{n,k}(x_k)) \\ \text{and} \quad K_{n,k}(x) &= [L_{n,k}(x)]^2(x - x_k). \end{aligned}$$

Then

$$p_{2n+1}(x) = \sum_{k=0}^{n} [f_k H_{n,k}(x) + g_k K_{n,k}(x)] \tag{4}$$

interpolates the data as required. The polynomial (4) is called the **Hermite interpolating polynomial**. Note that $H_{n,k}(x_i) = \delta_{ik}$ and $H'_{n,k}(x_i) = 0$, and $K_{n,k}(x_i) = 0$, $K'_{n,k}(x_i) = \delta_{ik}$.

Theorem. Let p_{2n+1} be the Hermite interpolating polynomial in the case where $f_i = f(x_i)$ and $g_i = f'(x_i)$ and f has at least $2n+2$ smooth derivatives. Then, for every $x \in [x_0, x_n]$, there exists $\xi = \xi(x) \in (x_0, x_n)$ such that

$$f(x) - p_{2n+1}(x) = [(x - x_0)(x - x_1) \cdots (x - x_n)]^2 \frac{f^{(2n+2)}(\xi)}{(2n+2)!},$$

where $f^{(2n+2)}$ is the $(2n+2)$nd derivative of f.

Proof (non-examinable): see Süli and Mayers, Theorem 6.4. □

We note that as $x_k \to 0$ in (3), we essentialy recover Taylor's theorem with $p_n(x)$ equal to the first $n+1$ terms in Taylor's expansion. Taylor's theorem can be regarded as a special case of Lagrange interpolation where we interpolate high-order derivatives at a single point.