Numerical Analysis Hilary Term 2025
Lecture 12–13: Introduction to numerical methods for initial-value problems

Initial value problems (IVP)[1]: These arise everywhere in mathematics where we wish to model the evolution in time of a given system.

Definition 1 *Let $I = [x_0, X] \subset \mathbb{R}$ be a (time) interval and $D \subset \mathbb{R}^d$ be an open subset, where $d \in \mathbb{N}^+$ denotes the space dimension.*

- *A* first-order ordinary differential equation *(ODE) is an equation of the form*

$$\mathbf{y}'(x) = \mathbf{f}(x, \mathbf{y})$$

  where the righthand side is a function $\mathbf{f} : I \times D \to \mathbb{R}^d$.

- *An* initial value problem *(IVP) is an ODE with an initial condition, that is,*

$$\mathbf{y}'(x) = \mathbf{f}(x, \mathbf{y}) \qquad \mathbf{y}(x_0) = \mathbf{y}_0.$$

  The goal is to find $\mathbf{y}(x)$, either at the final (time) value $x = X$, or any $x \in [x, X]$.

We'll focus on IVPs in this course, but another important class of ODEs is boundary value problems (BVP), where the condition(s) are imposed at e.g. both the initial and final values $x = x$ and X (and more intricate conditions in domains $I \subseteq \mathbb{R}^2$ and $I \subseteq \mathbb{R}^3$).

Picard's Theorem gives sufficient conditions to ensure that the IVP admits a unique solution[2].

Picard's Theorem. Suppose that $\mathbf{f}$ is continuous in a neighborhood $U \subset \mathbb{R}^{1+d}$ of $(x_0, \mathbf{y}_0)$ that contains the (closed) cylinder

$$R = \{(x, \mathbf{y}) : x_0 \le x \le X_M, \|\mathbf{y} - \mathbf{y}_0\| \le Y_M\},$$

where $X_M > x_0$ and $Y_M > 0$ are constants. Suppose also that there exists a positive constant L such that

$$\|\mathbf{f}(x, \mathbf{y}) - \mathbf{f}(x, \mathbf{z})\| \le L\|\mathbf{y} - \mathbf{z}\|$$

holds whenever $(x, \mathbf{y})$ and $(x, \mathbf{z})$ lie in R. Finally, letting

$$M := \max\{\|\mathbf{f}(x, \mathbf{y})\| : (x, \mathbf{y}) \in R\},$$

suppose that $M(X_M - x_0) \le Y_M$. Then, there exists a unique continuously differentiable function

$$[x_0, X_M] \ni x \mapsto \mathbf{y}(x) \in \mathbb{R}^d$$

that is the solution to our IVP.

[1] The remaining lecture notes on IVP are based on notes by Patrick Farrell, Yuji Nakatsukasa, Alberto Paganini, and Endre Süli. Many of the figures presented are courtesy of Maike Meier.

[2] In the remainder, we only deal with problems that satisfy the conditions in Picard's theorem, and assume $\mathbf{y}$ is differentiable as many times as we take derivatives. For more details about Picard's theorem, including the proof, we refer to the lecture notes of A1 Differential Equations. Another source is chapter 11 of Nick Trefethen's book *Exploring ODEs*, which is freely available at http://people.maths.ox.ac.uk/trefethen/Exploring.pdf

Remarks about Picard's theorem.

- The solution, if it exists, can be expressed by the integral equation

$$\mathbf{y}(x) = \mathbf{y}(x_0) + \int_{x_0}^{x} \mathbf{f}(t, \mathbf{y}(t))\mathrm{d}t\,, \tag{1}$$

  which is obtained by integrating the IVP, and where the integration is to be understood componentwise. Almost all of the numerical schemes we examine will be based on this identity.

- Picard's theorem guarantees the existence of a solution only up to a finite time X_M; consider : $y' = y^2$, $y(0) = 1$, which has solution $y(x) = (1-x)^{-1}$ and blows up at $x = 1$.

- If an IVP satisfies the assumptions of Picard's theorem, its solution is stable on the bounded interval $[x_0, X]$. This means that if $\mathbf{y} : [x_0, X] \to D$ solves the IVP

$$\mathbf{y}'(x) = \mathbf{f}(x, \mathbf{y})\,, \qquad \mathbf{y}(x_0) = \mathbf{y}_0\,, \tag{2}$$

  and $\tilde{\mathbf{y}} : [x_0, X] \to D$ solves the same ODE with a perturbed initial condition $\tilde{\mathbf{y}}_0$, that is,

$$\tilde{\mathbf{y}}'(x) = \mathbf{f}(x, \tilde{\mathbf{y}})\,, \qquad \tilde{\mathbf{y}}(x_0) = \tilde{\mathbf{y}}_0\,,$$

  then

$$\|\mathbf{y}(x) - \tilde{\mathbf{y}}(x)\| \leq e^{L(X-x_0)}\|\mathbf{y}_0 - \tilde{\mathbf{y}}_0\| \quad \forall x \in [x_0, X]. \tag{3}$$

  (We'll prove a related result in the theorem below.) This implies that a small error in the initial condition does not compromise dramatically the solution of the IVP. This is an important property—small errors in $\mathbf{y}$, either at x_0 or some other $x \in [x_0, X)$, would not stop us from getting an accurate $\mathbf{y}$ at $x = X$. However, note that the constant $e^{L(X-x_0)}$ in the above bound grows exponentially as the final time X increases.

Note. Any higher-order IVP can be reformulated as a larger first-order IVP. The simplest way to do so is to define $\hat{\mathbf{y}} := \begin{bmatrix} y & y' & y'' & \cdots & y^{(k)} \end{bmatrix}^\top$ and solve the IVP with respect to $\hat{\mathbf{y}}$. We therefore mainly consider numerical methods for first-order problems. It is also possible to reformulate nonautonomous problems as larger autonomous ones (wherein $\mathbf{y}' = \mathbf{f}(\mathbf{y})$, by working with $\hat{\mathbf{y}} := \begin{bmatrix} x \\ \mathbf{y} \end{bmatrix}$), so sometimes we will restrict ourselves to autonomous ones when it is convenient to do so.

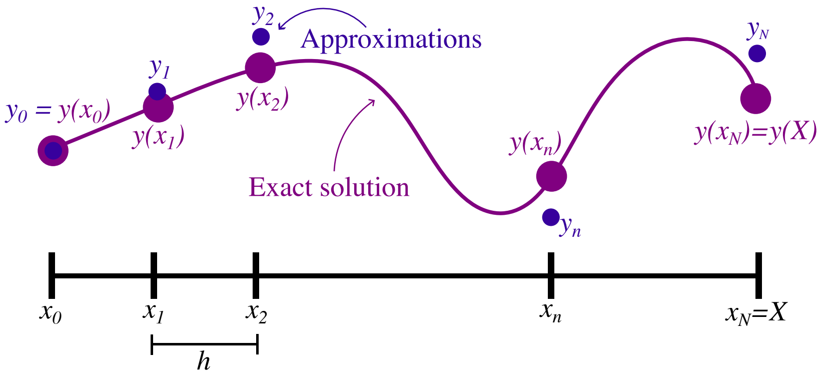
Numerical methods for IVPs: basic idea (Explicit and Implicit methods)

An inconvenient truth is that most IVPs (and most differential equations) cannot be solved analytically (i.e., exactly, to obtain closed-form solutions). It therefore becomes necessary to find approximate solutions with a numerical algorithm. Fortunately, a number of reliable and efficient methods are available for such solution. The remainder of this course is devoted to these methods and their analysis.

Assume that the IVP

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}) \,, \qquad \mathbf{y}(x_0) = \mathbf{y}_0 \,,$$

admits a unique stable solution $\mathbf{y} : [x_0, X] \to D$ that is defined on the bounded interval $[x_0, X]$. How can we compute a numerical approximation of $\mathbf{y}$ that can be made arbitrarily accurate? A simple idea is to first divide the interval $[x_0, X]$ into $N \in \mathbb{N}^+$ subintervals defined by the equidistant points $x_n = x_0 + nh$, $n = 0, \ldots, N$, where the *step size* h is $h = (X - x_0)/N$. To each time step x_n, we associate an approximation $\mathbf{y}_n$ of $\mathbf{y}(x_n)$.



To define how to compute these approximations, we take inspiration from a variant of the integral equation (1):

$$\mathbf{y}(x_{n+1}) = \mathbf{y}(x_n) + \int_{x_n}^{x_{n+1}} \mathbf{f}(x, \mathbf{y}(x)) \, \mathrm{d}x. \tag{4}$$

This equality suggests that, if we have already computed an approximation $\mathbf{y}_n$ of $\mathbf{y}(x_n)$, we could compute $\mathbf{y}_{n+1}$ by adding to $\mathbf{y}_n$ an approximation of the integral appearing on the right-hand side. Therefore, there is a deep connection between quadrature and the solution of IVPs. Indeed if $\mathbf{f}(x, \mathbf{y}) = \mathbf{f}(x)$, i.e., $\mathbf{f}$ does not depend on $\mathbf{y}$, then computing $\mathbf{y}$ is a standard quadrature problem, and can be solved by e.g. Gauss quadrature. Starting with $n = 0$, we could iterate such a strategy to compute the entire sequence $\{\mathbf{y}_n\}_{n=0}^N$. In what follows, we construct three different schemes based on three different (and still very similar) approximations of the integral and investigate the impact that this choice has on the properties of the resulting numerical method.

Explicit Euler, implicit Euler, and implicit midpoint

To construct an approximation of the integral in (4), we recall that by the mean value theorem there is a $\xi \in [x_n, x_{n+1}]$ such that

$$\int_{x_n}^{x_{n+1}} \mathbf{f}(x, \mathbf{y}(x)) \, \mathrm{d}x = h\mathbf{f}(\xi, \mathbf{y}(\xi)).$$

Therefore, we can construct an approximation by replacing ξ with a value s we like. The resulting numerical approximation rule is called a *rectangle rule.* A consequence is that, for any $s \in [x_n, x_{n+1}]$, the approximation error is at most

$$\int_{x_n}^{x_{n+1}} \mathbf{f}(x, \mathbf{y}(x))\,\mathrm{d}x - h\mathbf{f}(s, \mathbf{y}(s)) = h(\mathbf{f}(\xi, \mathbf{y}(\xi)) - \mathbf{f}(s, \mathbf{y}(s)))$$
$$\leq h \max_{r \in [x_n, x_{n+1}]} |\mathbf{f}(r, \mathbf{y}(r)) - \mathbf{f}(s, \mathbf{y}(s))|.$$

For instance, we can choose $s = x_n$, so that

$$\int_{x_n}^{x_{n+1}} \mathbf{f}(x, \mathbf{y}(x))\,\mathrm{d}x \approx h\mathbf{f}(x_n, \mathbf{y}(x_n)).$$

Inserting this gives

$$\mathbf{y}(x_{n+1}) \approx \mathbf{y}(x_n) + h\mathbf{f}(x_n, \mathbf{y}(x_n))\,,$$

which motivates the definition of the *explicit Euler method*[3]

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_n, \mathbf{y}_n). \tag{5}$$

Two other interesting choices are $\xi = x_{n+1}$ and $\xi = (x_n + x_{n+1})/2$, which give rise to the *implicit Euler method*

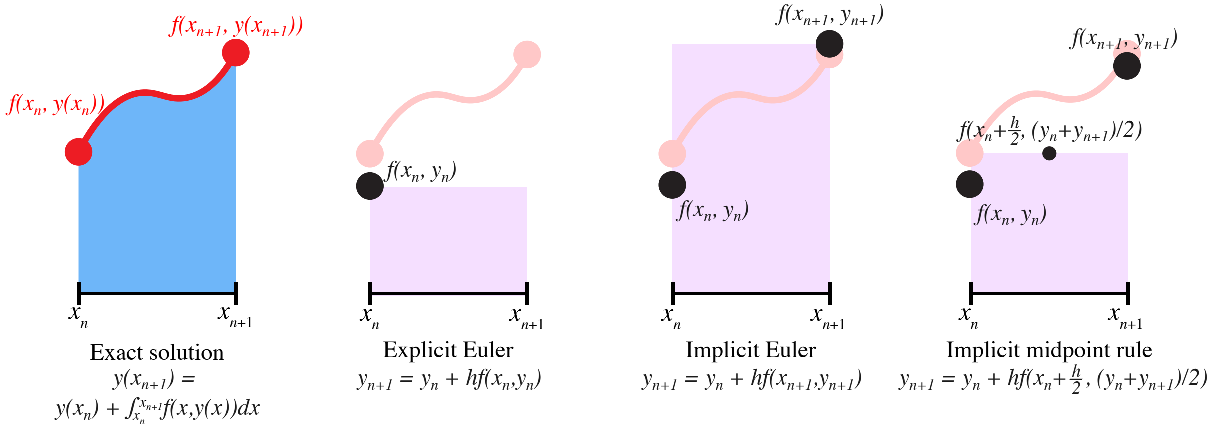
$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_{n+1}, \mathbf{y}_{n+1}) \tag{6}$$

and the *implicit midpoint rule*

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}\left(x_n + \frac{h}{2}, \frac{\mathbf{y}_n + \mathbf{y}_{n+1}}{2}\right), \tag{7}$$

respectively.

Here is an illustration of the three methods.



Note the occurrence of $\mathbf{y}_{n+1}$ on the right-hand side of these last two methods. These numerical methods are called *implicit*, because computing $\mathbf{y}_{n+1}$ requires solving a (generally nonlinear) system, which makes them more computationally expensive than explicit Euler.

[3] The explicit and the implicit Euler methods have been known since 1768! Due, of course, to the great Leonhard Euler.

The arising equations are typically solved with Newton's method, which you saw in M4 Constructive Mathematics. Explicit methods are faster per timestep, but as we will see often suffer from severe timestep restrictions to retain stability, and implicit methods are usually faster for such problems.

Numerical examples

We test these methods on two different examples. First, we consider the linear test case

$$y' = \lambda y\,, \quad y_0 = 1\,, \quad x \in [0, 1] \tag{8}$$

with $N = 10$. The numerical solutions are illustrated in Figure 1. For $\lambda = 3$, we observe that all three methods compute a qualitatively correct solution, although the one computed with the implicit midpoint rule is way more accurate. Next, we investigate what happens for negative values of λ. This case is interesting because the exact solution converges to 0 exponentially as $x \to \infty$. For $\lambda = -5$, we see that all methods provide a qualitatively correct solution. For $\lambda = -20$, we see that the explicit Euler solution is equioscillatory, while the implicit methods still produce a qualitatively correct solution.

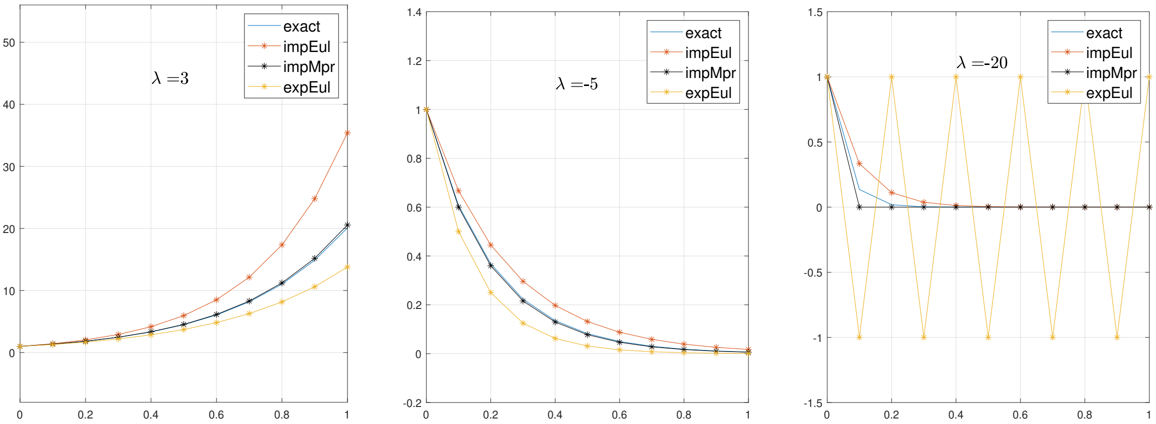


Figure 1: Solving (8) using explicit Euler, midpoint and implicit Euler methods.

If you run the code with $\lambda < -20$, you will see that the explicit Euler solution diverges to $\pm\infty$ and the implicit midpoint rule also starts to oscillate, although the level of these oscillations cannot be compared with the ones of the explicit Euler method. In fact, the implicit midpoint rule does not diverge for any $\lambda < 0$ (not even for $\lambda = -9000$). On the other hand, it is surprising to see that the implicit Euler method provides excellent solutions for any negative number of λ. This example shows that the stability of a numerical method can vary drastically. We will explore this further in the upcoming lectures.

The second test case we consider is the following IVP:

$$\mathbf{y}' = \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix}, \quad \mathbf{y}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x \in [0, 2\pi]\,, \tag{9}$$

whose analytical (exact) solution is $\mathbf{y}(x) = \binom{\cos(x)}{\sin(x)}$. This case is interesting because the quantity $Q(\mathbf{y}) := \|\mathbf{y}(x)\|$ is constant in time. We fix $N = 40$ and plot the orbit (that is,

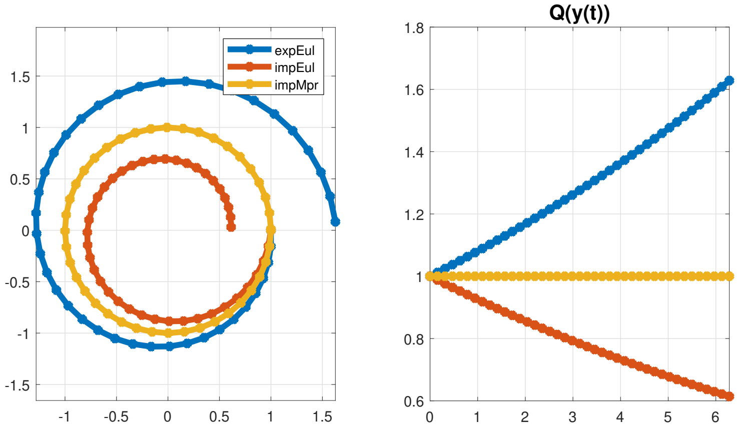


Figure 2: Numerical solutions for (9).

the curve $x \mapsto \mathbf{y}(x)$) of the numerical solutions computed with the three methods above and the evolution of their quantity Q.

The numerical solutions are illustrated in Figure 2. We see that the implicit midpoint rule is the only method that preserves Q, and that it does it up to machine precision! The laws of physics are typically formulated by considering the conservation laws of energy, mass, momentum, etc., and numerical methods that exactly preserve key structural properties of the underyling models are now of prime importance. This line of thinking has led to a beautiful confluence of numerical analysis with geometry and topology, leading to the field called geometric numerical integration (which is off-syllabus).

Consistency of a one-step method

To discuss the convergence and convergence speed of methods, we formally define the class of one-step methods.

Definition 2 *A* one-step method *is a function* $\mathbf{\Psi}$ *that takes the triplet* $(s, \mathbf{y}, h) \subset \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ *and a function* $\mathbf{f}$*, and computes an approximation* $\mathbf{\Psi}(s, \mathbf{y}, h, \mathbf{f})$ *of* $\mathbf{y}(s+h)$*, which is the solution at* $s+h$ *of the IVP*

$$\mathbf{y}'(x) = \mathbf{f}(x, \mathbf{y})\,, \quad \mathbf{y}(s) = \mathbf{y}.$$

Here, we tacitly assume that $\mathbf{y}(s+h)$ *exists. Additionally, the timestep* h *may need to be sufficiently small for* $\mathbf{\Psi}$ *to be well defined.*

In particular, the discrete scheme has the form

$$\mathbf{y}_{n+1} = \mathbf{\Psi}(x_n, \mathbf{y}_n, h, \mathbf{f}) \quad n = 0, 1, \dots, \quad \text{and} \quad \mathbf{y}_0 = \mathbf{y}(x_0). \tag{10}$$

Each of the three schemes we have considered thus far are one-step methods:

- Explicit Euler (5):

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_n, \mathbf{y}_n) \implies \mathbf{\Psi}(x, \mathbf{y}, h, \mathbf{f}) = \mathbf{y} + h\mathbf{f}(x, \mathbf{y}).$$

- Implicit Euler (6):

  $$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_{n+1}, \mathbf{y}_{n+1}) \implies \boldsymbol{\Psi}(x, \mathbf{y}, h, \mathbf{f}) = \mathbf{y} + h\mathbf{f}(x + h, \boldsymbol{\Psi}(x, \mathbf{y}, h, \mathbf{f})).$$

  Note that $\boldsymbol{\Psi}$ is implicitly defined by the above relation.

  (Non-examinable): The existence of $\boldsymbol{\Psi}$ as a function of x and $\mathbf{y}$ can be proven using the implicit function theorem. For simplicity, we assume that $\mathbf{f}$ is a smooth function. If we view h and $\mathbf{f}$ as fixed parameters and set

  $$\mathbf{G}(x, \mathbf{y}, \boldsymbol{\Psi}) := \boldsymbol{\Psi} - h\mathbf{f}(x + h, \boldsymbol{\Psi}) - \mathbf{y},$$

  then $\boldsymbol{\Psi}$ is defined implicitly by the relation

  $$\mathbf{G}(x, \mathbf{y}, \boldsymbol{\Psi}) = 0.$$

  The implicit function theorem states that if the Jacobian matrix $\frac{\partial}{\partial \boldsymbol{\Psi}}\mathbf{G}(x, \mathbf{y}, \boldsymbol{\Psi})$ is invertible (say for all x and $\mathbf{f}$), then there exists a function $\boldsymbol{\Psi} = \boldsymbol{\Psi}(x, \mathbf{y})$ such that $\mathbf{G}(x, \mathbf{y}, \boldsymbol{\Psi}(x, \mathbf{f})) = 0$ (that is, the one-step function is well-defined). Note that

  $$\frac{\partial}{\partial \boldsymbol{\Psi}}\mathbf{G}(x, \mathbf{y}, \boldsymbol{\Psi}) = I - h\frac{\partial}{\partial \mathbf{y}}\mathbf{f}(x + h, \boldsymbol{\Psi}),$$

  which means that the Jacobian matrix $\frac{\partial}{\partial \boldsymbol{\Psi}}\mathbf{G}$ is a perturbation of the identity matrix. Thus, if h is sufficiently small (relative to the Jacobian $\frac{\partial}{\partial \mathbf{y}}\mathbf{f}$), then $\frac{\partial}{\partial \boldsymbol{\Psi}}\mathbf{G}$ is invertible.

- Implicit midpoint (7):

  $$\begin{aligned} \mathbf{y}_{n+1} &= \mathbf{y}_n + h\mathbf{f}\left(x_n + \frac{h}{2}, \frac{\mathbf{y}_n + \mathbf{y}_{n+1}}{2}\right) \\ \implies \boldsymbol{\Psi}(x, \mathbf{y}, h, \mathbf{f}) &= \mathbf{y} + h\mathbf{f}\left(x + \frac{h}{2}, \frac{\mathbf{y} + \boldsymbol{\Psi}(x, \mathbf{y}, h, \mathbf{f})}{2}\right), \end{aligned}$$

  and so $\boldsymbol{\Psi}$ is again implicitly defined. One can play a similar game as above to show that if h is sufficiently small, then $\boldsymbol{\Psi}$ exists as a well-defined function.

We now ask the basic question: what are the bare minimum requirements for a reasonable scheme? Two ideas are the following:

(i) If we take a step length $h = 0$, then the scheme should do nothing: $\boldsymbol{\Psi}(s, \mathbf{y}, 0, \mathbf{f}) = \mathbf{y}$.

(ii) In the limit as $h \to 0$, the one-step method solves the same ODE that we are approximating. In particular, we have

$$\mathbf{y}(x + h) \approx \boldsymbol{\Psi}(x, \mathbf{y}(x), h, \mathbf{f}) \implies \mathbf{f}(x, \mathbf{y}(x)) = \mathbf{y}'(x) \approx \frac{\boldsymbol{\Psi}(x, \mathbf{y}(x), h, \mathbf{f}) - \mathbf{y}(x)}{h}.$$

If our method is reasonable, then the above approximation of the derivative should be an equality in the limit $h \to 0$. That is, we ask that

$$\begin{aligned}
\mathbf{f}(x, \mathbf{y}(x)) &= \lim_{h\to 0} \frac{\boldsymbol{\Psi}(x, \mathbf{y}(x), h, \mathbf{f}) - \mathbf{y}(x)}{h} \\
&= \lim_{h\to 0} \left(\frac{\boldsymbol{\Psi}(x, \mathbf{y}, 0, \mathbf{f}) - \mathbf{y}(x)}{h} + \frac{\partial}{\partial h}\boldsymbol{\Psi}(x, \mathbf{y}, h, \mathbf{f})|_{h=0} + \mathcal{O}(h) \right) \\
&= \lim_{h\to 0} \frac{\boldsymbol{\Psi}(x, \mathbf{y}, 0, \mathbf{f}) - \mathbf{y}(x)}{h} + \frac{\partial}{\partial h}\boldsymbol{\Psi}(x, \mathbf{y}, h, \mathbf{f})|_{h=0} \\
&= \frac{\partial}{\partial h}\boldsymbol{\Psi}(x, \mathbf{y}, h, \mathbf{f})|_{h=0} \qquad \text{by (i).}
\end{aligned}$$

Motivated by these conditions, we define the notion of *consistency*:

Definition 3 *A one-step method* $\boldsymbol{\Psi}$ *is said to be* consistent *if*

$$\boldsymbol{\Psi}(s, \mathbf{y}, 0, \mathbf{f}) = \mathbf{y} \tag{11}$$

and

$$\frac{\partial}{\partial h}\boldsymbol{\Psi}(s, \mathbf{y}, h, \mathbf{f})\Big|_{h=0} = \mathbf{f}(s, \mathbf{y}) \tag{12}$$

It is also useful to quantify how consistent a method is. To this end, we define

Definition 4 *The* consistency error *(also known as the local trunction error)* $\boldsymbol{\tau}$ *is defined as*

$$\boldsymbol{\tau}(s, \mathbf{y}, h, \mathbf{f}) := \frac{\mathbf{y}(s+h) - \mathbf{y}}{h} - \frac{\boldsymbol{\Psi}(s, \mathbf{y}, h, \mathbf{f}) - \mathbf{y}}{h} = \frac{\mathbf{y}(s+h) - \boldsymbol{\Psi}(s, \mathbf{y}, h, \mathbf{f})}{h}, \tag{13}$$

where $\mathbf{y}(s+h)$ *is the solution at* $s+h$ *of the IVP.*

Comments on consistency error:

- One can view the consistency error as measuring how close $\boldsymbol{\Psi}(s, \mathbf{f}, h, \mathbf{f})$ is to $\mathbf{y}(s+h)$. Whether to divide the difference $\mathbf{y}(s+h) - \boldsymbol{\Psi}(s, \mathbf{y}, h, \mathbf{f})$ by h is a stylistic choice. As we will see below, this choice ensures that, under appropriate conditions, the consistency error and the global error converge to zero at the same rate as $h \to 0$.
- The *consistency order* is the largest $\tilde{p}$ such that the consistency error $\|\boldsymbol{\tau}(s, \mathbf{y}, h, \mathbf{f})\| \leq \tilde{C}h^{\tilde{p}}$ for some $\tilde{C}$. We typically compute the consistency order using Taylor expansions.
- The consistency order $\tilde{p}$ measures the local error. Roughly speaking, each step of the scheme incurs an error on the order of $\tilde{p} + 1$.
- In many textbooks, the consistency error (local truncation error) is defined as $\mathbf{y}(s+h) - \boldsymbol{\Psi}(s, \mathbf{y}, h, \mathbf{f})$ (making the choice to not divide by h) so that the consistency error corresponds to the error incurred by the scheme in one step. This definition results in a consistency order one higher than the definition use here (and avoids the need to add 1 in the previous comment).

The following lemma gives additional insight about the definition of consistent one-step method. The gist of it is that a one-step method is consistent if the consistency error can be made arbitrarily small by reducing h.

Lemma. Assume that $h \mapsto \boldsymbol{\Psi}(s, \mathbf{y}, h, \mathbf{f})$ is continuously differentiable in a neighborhood of 0. Then, $\boldsymbol{\Psi}$ is consistent if and only if, for any fixed $\mathbf{f}$,

$$\|\boldsymbol{\tau}(\tilde{s}, \tilde{\mathbf{y}}, h, \mathbf{f})\| \to 0 \quad \text{as } h \to 0$$

locally uniformly in $(\tilde{s}, \tilde{\mathbf{y}}) \in R$, where R is the cylinder from Picard's Theorem.

It is sometimes convenient to represent an abstract one-step method via its *increment function.*

Lemma. Assume that $h \mapsto \boldsymbol{\Psi}(s, \mathbf{y}, h, \mathbf{f})$ is continuously differentiable in a neighborhood of 0. Then, $\boldsymbol{\Psi}$ is consistent if and only if there is a continuous increment function $h \mapsto \boldsymbol{\psi}(s, \mathbf{y}, h, \mathbf{f})$ such that

$$\boldsymbol{\Psi}(s, \mathbf{y}, h, \mathbf{f}) = \mathbf{y} + h\boldsymbol{\psi}(s, \mathbf{y}, h, \mathbf{f}), \quad \boldsymbol{\psi}(s, \mathbf{y}, 0, \mathbf{f}) = \mathbf{f}(s, \mathbf{y}).$$

- Explicit Euler (5):

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_n, \mathbf{y}_n) \implies \boldsymbol{\psi}(x, \mathbf{y}, h, \mathbf{f}) = \mathbf{f}(x, \mathbf{y}).$$

- Implicit Euler (6):

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_{n+1}, \mathbf{y}_{n+1}) \implies \mathbf{y} + h\boldsymbol{\psi}(x, \mathbf{y}, h, \mathbf{f}) = \mathbf{y} + h\mathbf{f}(x + h, \mathbf{y} + h\boldsymbol{\psi}(x, \mathbf{y}, h, \mathbf{f})),$$

  and so $\boldsymbol{\psi}$ is defined implicitly be the relation

$$\boldsymbol{\psi}(x, \mathbf{y}, h, \mathbf{f}) = \mathbf{f}(x + h, \mathbf{y} + h\boldsymbol{\psi}(x, \mathbf{y}, h, \mathbf{f})).$$

  Again using the implicit function theorem, we can show that $\boldsymbol{\psi}$ is well-defined provided h is sufficiently small.

- Implicit midpoint (7):

$$\begin{aligned} \mathbf{y}_{n+1} &= \mathbf{y}_n + h\mathbf{f}\left(x_n + \frac{h}{2}, \frac{\mathbf{y}_n + \mathbf{y}_{n+1}}{2}\right) \\ \implies \mathbf{y} + h\boldsymbol{\psi}(x, \mathbf{y}, h, \mathbf{f}) &= \mathbf{y} + h\mathbf{f}\left(x + \frac{h}{2}, \mathbf{y} + \frac{h}{2}\boldsymbol{\psi}(x, \mathbf{y}, h, \mathbf{f})\right) \\ \implies \boldsymbol{\psi}(x, \mathbf{y}, h, \mathbf{f}) &= \mathbf{f}\left(x + \frac{h}{2}, \mathbf{y} + \frac{h}{2}\boldsymbol{\psi}(x, \mathbf{y}, h, \mathbf{f})\right), \end{aligned}$$

  where we still require that h is sufficiently small for $\boldsymbol{\psi}$ to be well-defined.

Convergence of a one-step method

We define the *global error*

$$e_n := \|\mathbf{y}(x_n) - \mathbf{y}_n\|.$$

Then $e := e_N = \|\mathbf{y}(x_N) - \mathbf{y}_N\|$ is the (global) error in the solution at $x = X$. With a good method, we hope $e \to 0$ as the step size $h \to 0$ (and hence $\boldsymbol{\tau} \to 0$). Note that the global error accounts for the accumulation of errors over all steps, wheres the consistency error measures the error in a single step.

Theorem. Let $\boldsymbol{\Psi}$ be a consistent one-step method and assume that its increment function $\boldsymbol{\psi}$ is Lipschitz continuous with respect to $\mathbf{y}$, that is, that there exists a positive constant $L_{\boldsymbol{\psi}}$ such that, for $0 \le h \le h_0$ and for the same region R of Picard's theorem,

$$\|\boldsymbol{\psi}(x, \mathbf{y}, h, \mathbf{f}) - \boldsymbol{\psi}(x, \mathbf{z}, h, \mathbf{f})\| \le L_{\boldsymbol{\psi}}\|\mathbf{y} - \mathbf{z}\| \qquad \text{for}(x, \mathbf{y}), (x, \mathbf{z}) \in R. \tag{14}$$

Then, assuming that $(x_n, \mathbf{y}_n)$ remains in R, it follows that

$$e \le \left(\frac{\exp\left(L_{\boldsymbol{\psi}}(x_N - x_0)\right) - 1}{L_{\boldsymbol{\psi}}}\right) \max_{n=0,\ldots,N-1} \|\boldsymbol{\tau}(x_n, \mathbf{y}(x_n), h, \mathbf{f})\|\,.$$

Proof. For a generic $n \in \{1, \ldots, N-1\}$,

$$\begin{aligned}
e_{n+1} &= \|\mathbf{y}(x_{n+1}) - \mathbf{y}_{n+1}\|\,,\\
&= \|\mathbf{y}(x_{n+1}) - \boldsymbol{\Psi}(x_n, \mathbf{y}_n, h, \mathbf{f})\|\,,\\
&= \|\mathbf{y}(x_{n+1}) - \boldsymbol{\Psi}(x_n, \mathbf{y}(x_n), h, \mathbf{f}) + \boldsymbol{\Psi}(x_n, \mathbf{y}(x_n), h, \mathbf{f}) - \boldsymbol{\Psi}(x_n, \mathbf{y}_n, h, \mathbf{f})\|\,,\\
&\le \|\mathbf{y}(x_{n+1}) - \boldsymbol{\Psi}(x_n, \mathbf{y}(x_n), h, \mathbf{f})\| + \|\boldsymbol{\Psi}(x_n, \mathbf{y}(x_n), h, \mathbf{f}) - \boldsymbol{\Psi}(x_n, \mathbf{y}_n, h, \mathbf{f})\|\,,\\
&= h\|\boldsymbol{\tau}(x_n, \mathbf{y}(x_n), h, \mathbf{f})\| + \|\left(\mathbf{y}(x_n) + h\boldsymbol{\psi}(x, \mathbf{y}(x_n), h, \mathbf{f})\right) - \left(\mathbf{y}_n + h\boldsymbol{\psi}(x, \mathbf{y}_n, h, \mathbf{f})\right)\|\,,\\
&\le h\|\boldsymbol{\tau}(x_n, \mathbf{y}(x_n), h, \mathbf{f})\| + \|\mathbf{y}(x_n) - \mathbf{y}_n\| + h\|\boldsymbol{\psi}(x, \mathbf{y}(x_n), h, \mathbf{f}) - \boldsymbol{\psi}(x, \mathbf{y}_n, h, \mathbf{f})\|\,,\\
&= h\|\boldsymbol{\tau}(x_n, \mathbf{y}(x_n), h, \mathbf{f})\| + e_n + h\|\boldsymbol{\psi}(x, \mathbf{y}(x_n), h, \mathbf{f}) - \boldsymbol{\psi}(x, \mathbf{y}_n, h, \mathbf{f})\|\,,\\
&\le h\|\boldsymbol{\tau}(x_n, \mathbf{y}(x_n), h, \mathbf{f})\| + e_n + hL_{\boldsymbol{\psi}}\|\mathbf{y}(x_n) - \mathbf{y}_n\|\,,\\
&= h\|\boldsymbol{\tau}(x_n, \mathbf{y}(x_n), h, \mathbf{f})\| + (1 + hL_{\boldsymbol{\psi}})e_n.
\end{aligned}$$

Iterating recursively, this implies that (note that $e_0 = 0$)

$$\begin{aligned}
e_{n+1} &\le (1 + hL_{\boldsymbol{\psi}})^{n+1}e_0 + h\sum_{k=0}^{n}(1 + hL_{\boldsymbol{\psi}})^k \max_{m=0,\ldots,n} \|\boldsymbol{\tau}(x_m, \mathbf{y}(x_m), h, \mathbf{f})\|\\
&= \frac{(1 + hL_{\boldsymbol{\psi}})^{n+1} - 1}{L_{\boldsymbol{\psi}}} \max_{m=0,\ldots,n} \|\boldsymbol{\tau}(x_m, \mathbf{y}(x_m), h, \mathbf{f})\|.
\end{aligned}$$

To conclude the proof, note that $1 + hL_{\boldsymbol{\psi}} \le \exp hL_{\boldsymbol{\psi}}$. □

Stability. Hiding in the hypothesis (14) is a *stability* condition. Recall from the remarks below Picard's theorem that stability of the IVP means that small perturbations of the initial data result in small perturbations of the solution (3). The hypothesis (14) requires that an analogous stability condition holds for the discrete scheme as

$$\|\boldsymbol{\Phi}(x, \mathbf{y}, h, \mathbf{f}) - \boldsymbol{\Phi}(x, \mathbf{z}, h, \mathbf{f})\| \le \|\mathbf{y} - \mathbf{z}\| + h\|\boldsymbol{\psi}(x, \mathbf{y}, h, \mathbf{f}) - \boldsymbol{\psi}(x, \mathbf{z}, h, \mathbf{f})\| \le (1 + hL_{\boldsymbol{\psi}})\|\mathbf{y} - \mathbf{z}\|.$$

In particular, if we define

$$\begin{aligned}
\mathbf{y}_{n+1} &= \boldsymbol{\Phi}(x_n, \mathbf{y}_n, h, \mathbf{f}), \quad n = 0, 1, \ldots, \quad \text{and} \quad \mathbf{y}_0 = \mathbf{y},\\
\mathbf{z}_{n+1} &= \boldsymbol{\Phi}(x_n, \mathbf{z}_n, h, \mathbf{f}), \quad n = 0, 1, \ldots, \quad \text{and} \quad \mathbf{z}_0 = \mathbf{z},
\end{aligned}$$

then

$$\|\mathbf{y}_{n+1} - \mathbf{z}_{n+1}\| = \|\boldsymbol{\Phi}(x_n, \mathbf{y}_n, h, \mathbf{f}) - \boldsymbol{\Phi}(x_n, \mathbf{z}_n, h, \mathbf{f})\| \leq (1 + hL_{\boldsymbol{\psi}})\|\mathbf{y}_n - \mathbf{z}_n\|$$

and so

$$\|\mathbf{y}_n - \mathbf{z}_n\| \leq (1 + hL_{\boldsymbol{\psi}})^n \|\mathbf{y}_0 - \mathbf{z}_0\| \leq e^{hL_{\boldsymbol{\psi}}}\|\mathbf{y}_0 - \mathbf{z}_0\|,$$

which resembles estimate (3).

A common theme in numerical analysis, particularly in the analysis of schemes for ODEs and PDEs, is the idea that

$$\text{“consistency”} + \text{“stability”} \implies \text{“convergence”},$$

which the above theorem makes precise in the context of one-step methods.

Order of accuracy. In the following lectures we will examine methods that can give higher accuracy than Euler's methods. A method is said to have the *order of accuracy* (or just *order*) p if $e \leq Ch^p$ for some constant C; $p > 0$ is usually an integer. As mentioned above, the *consistency order* measures the local error, whereas p does the global error; they are usually the same, as the above theorem suggests. Informally, at each step the computation incurs an error of $O(h^{\tilde{p}+1})$ (note the $+1$ from the definition (13)), and we perform $O(h^{-1})$ steps, so the global error is $O(h^{\tilde{p}}) = O(h^p)$; the theorem above makes this precise.

Outlook

In the next lectures we will study two classes of methods—Runge-Kutta and multistep methods—that can achieve order of accuracy higher than 1. These usually result in more accurate solutions for a fixed computational budget. In studying these methods, the two overarching questions we shall address are:

(i) *Convergence*: Does the method converge to the exact solution as $h \to 0$?

(ii) *Stability*: How small does h need to be for the computed solution to start resembling the exact solution?

The theorem above shows that the answer to (i) is straightforward for one-step methods (including Runge-Kutta methods); roughly, convergence holds as $h \to 0$ if the method is consistent. The second question (ii) is intricate (as we saw already in the discussion of the test problem (8)), and we'll explore this question for Runge-Kutta methods in the next lecture.

For multistep methods, even (i) is not obvious; we'll treat both questions in the final lectures.

Listing 1: l11_ivp1.m

```
clear, set(0,'DefaultFigureWindowStyle','docked')
N = 10; h = 1/N; lambda = -20; %modify these parameters to experiment
expEul = nan(1, N+1); impEul = nan(1, N+1); impMpr = nan(1, N+1);
```

```
y0 = 1; expEul(1) = y0; impEul(1) = y0; impMpr(1) = y0;

for ii = 1:N
    expEul(ii+1) = expEul(ii)*(1+h*lambda);
    impEul(ii+1) = impEul(ii)/(1-h*lambda);
    impMpr(ii+1) = impMpr(ii)*(1+h*lambda/2)/(1-h*lambda/2);
end

t = linspace(0, 1, N+1);figure(1);
plot(t, exp(lambda*t), t, impEul, '*-', t, impMpr, 'k*-',t, expEul, '*-')
legend({'exact', 'impEul', 'impMpr','expEul'})
```

Listing 2: l11_ivp2.m

```
clear, set(0,'DefaultFigureWindowStyle','docked')
N = 40; T = 2*pi; h = T/N; A = [0 1; -1 0];
expEul = nan(2, N+1); impEul = nan(2, N+1); impMpr = nan(2, N+1);
y0 = [1; 0]; expEul(:,1) = y0; impEul(:,1) = y0; impMpr(:,1) = y0;

for ii = 1:N
    expEul(:,ii+1) = (eye(2)+h*A)*expEul(:,ii);
    impEul(:,ii+1) = (eye(2)-h*A)\impEul(:,ii);
    impMpr(:,ii+1) = (eye(2)-h*A/2)\((eye(2)+h*A/2)*impMpr(:,ii));
end
figure(2); subplot(1,2,1);
plot(expEul(1,:), expEul(2,:), '*-', 'linewidth', 4);
plot(impEul(1,:), impEul(2,:), '*-', 'linewidth', 4);
plot(impMpr(1,:), impMpr(2,:), '*-', 'linewidth', 4);
axis equal
legend({'expEul', 'impEul', 'impMpr'})
subplot(1,2,2), t = linspace(0, T, N+1); Q = @(y) sqrt(sum(y.^2, 1));
plot(t, Q(expEul), '*-', t, Q(impEul), '*-', t, Q(impMpr), '*-','linewidth', 4)
title('Q(y(t))','FontSize',24);
fprintf('max(abs(Q(impMpr)-1)) = %e\n', max(abs(Q(impMpr)-1)))
```