Numerical Analysis Hilary Term 2025
Lecture 14–15: Runge–Kutta methods

Recall that last lecture, we derived the explicit Euler, implicit Euler, and implicit midpoint rule for the IVP

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}), \qquad \mathbf{y}(x_0) = \mathbf{y}_0,$$

by applying the mean value theorem to the exact solution

$$\mathbf{y}(x+h) = \mathbf{y}(x) + \int_x^{x+h} \mathbf{f}(\tau, \mathbf{y}(\tau))\,\mathrm{d}\tau = \mathbf{y}(x) + h\int_0^1 \mathbf{f}(x+h\tau, \mathbf{y}(x+h\tau))\,\mathrm{d}\tau \tag{1}$$

to obtain

$$\mathbf{y}(x+h) = \mathbf{y}(x) + h\mathbf{f}(\xi, \mathbf{y}(\xi)) \qquad \xi \in [x, x+h]$$

and approximating ξ by x for explicit Euler, $x+h$ for implicit Euler, and $x+(h/2)$ for implicit midpoint. To obtain higher order schemes, we instead approximate the integral in (1) with a quadrature rule on $[0,1]$: With s quadrature nodes $c_1, \ldots, c_s$ and quadrature weights $b_1, \ldots, b_s$, we have

$$\mathbf{y}(x+h) \approx \mathbf{y}(x) + h\sum_{i=1}^{s} b_i \mathbf{f}(x + c_i h, \mathbf{y}(x + c_i h)). \tag{2}$$

We could, for example, use Gauss quadrature. The only problem is that this approximation requires the values $\mathbf{y}(x + c_i h)$. So, we also need to approximate these values.

Example 1: Consider the case that we use the 1-point Gauss rule:

$$\mathbf{y}(x+h) \approx \mathbf{y}(x) + h\mathbf{f}\left(x + \frac{h}{2}, \mathbf{y}\left(x + \frac{h}{2}\right)\right).$$

If we want an explicit method, then we can approximate the value of $\mathbf{y}\left(x + \frac{h}{2}\right)$ using the explicit Euler method:

$$\mathbf{y}\left(x + \frac{h}{2}\right) \approx \mathbf{y}(x) + \frac{h}{2}\mathbf{f}(x, \mathbf{y}(x)),$$

and so we have

$$\mathbf{y}(x+h) \approx \mathbf{y}(x) + h\mathbf{f}\left(x + \frac{h}{2}, \mathbf{y}(x) + \frac{h}{2}\mathbf{f}(x, \mathbf{y}(x))\right).$$

The above relation then leads to the following scheme:

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}\left(x_n + \frac{h}{2}, \mathbf{y}_n + \frac{h}{2}\mathbf{f}(x_n, \mathbf{y}_n)\right). \tag{3}$$

Example 2: Alternatively, we can approximate the integral in (1) using the trapezium rule:

$$\mathbf{y}(x+h) \approx \mathbf{y}(x) + \frac{h}{2}\mathbf{f}(x, \mathbf{y}(x)) + \frac{h}{2}\mathbf{f}(x+h, \mathbf{y}(x+h)),$$

and again approximate the unknown value $\mathbf{y}(x+h)$ using the explicit Euler method, leading to the scheme

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}h\left[\mathbf{f}(x_n, \mathbf{y}_n) + \mathbf{f}(x_n + h, \mathbf{y}_n + h\mathbf{f}(x_n, \mathbf{y}_n))\right]. \tag{4}$$

We will see below that both of these methods are second order.

We can continue inductively by using high-order quadrature rules to approximate the integral in (1) and using the highest order method constructed thus far to approximate the unknown intermediate values $\mathbf{y}(x_0 + c_i h)$. Of course, we could also construct implicit methods, which is discussed at the end of the notes.

Runge–Kutta methods

The two schemes we have constructed so far share a common structure:

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}\left(x_n + \frac{h}{2}, \mathbf{y}_n + \frac{h}{2}\mathbf{f}(x_n, \mathbf{y}_n)\right),$$
$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}h\left[\mathbf{f}(x_n, \mathbf{y}_n) + \mathbf{f}(x_n + h, \mathbf{y}_n + h\mathbf{f}(x_n, \mathbf{y}_n))\right].$$

First observe that each scheme is of the form $\mathbf{y}_{n+1} = \mathbf{y}_n + h(\text{something})$. Each (something) is a linear combination of other terms that are of the form $\mathbf{f}(x_n + (\text{const})h, \mathbf{y} + h(\text{linear combination of something else}))$. Making this more formal leads to the general definition of a **Runge-Kutta method:**

Definition 1 *The family of* s-stage Runge–Kutta methods *is defined by*

$$\mathbf{\Psi}(x, \mathbf{y}, h, \mathbf{f}) = \mathbf{y} + h\sum_{i=1}^{s} b_i \mathbf{k}_i, \tag{5}$$

where the stages $\mathbf{k}_i$ *(recall that* $\mathbf{y} \in \mathbb{R}^d$*, and also* $\mathbf{k}_i \in \mathbb{R}^d$*) are the solutions of the coupled system of (generally nonlinear) equations*

$$\mathbf{k}_i := \mathbf{f}(x + c_i h, \mathbf{y} + h\sum_{j=1}^{s} a_{ij}\mathbf{k}_j), \quad i = 1, \ldots, s. \tag{6}$$

The coefficients $\{c_i\}_{i=1}^s$ *are always given by*

$$c_i := \sum_{j=1}^{s} a_{ij} \quad i = 1, \ldots, s.$$

Runge–Kutta (RK) methods form a broad class of algorithms for the numerical solution of IVPs. The idea is to achieve a higher order of accuracy than 1 (as in Euler's method) by approximately finding the derivative f at 'intermediate values' (called *stages*) between x_n and x_{n+1} (and similarly $\mathbf{y}_n$ and $\mathbf{y}_{n+1}$). The class includes both explicit and implicit schemes. When applications call for an integrator with some kind of stability or conservation property, there usually exists a suitable RK method. In particular, RK methods can be made arbitrarily high-order without the loss of stability.

Definition 2 *The coefficients of a Runge–Kutta method are commonly summarized in a* Butcher tableau[1]

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^\top \end{array}.$$

Example 1 *The explicit Euler method, the implicit Euler method, and the implicit midpoint rule are Runge–Kutta methods. Their Butcher tables are*

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}, \quad \begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}, \quad \text{and} \quad \begin{array}{c|c} 1/2 & 1/2 \\ \hline & 1 \end{array}, \quad \text{respectively.}$$

Additionally, the Butcher tables for (3) and (4) are

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ \hline & 0 & 1 \end{array} \quad \text{and} \quad \begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}.$$

Example 2 *Taylor expansions are another way to derive RK methods. Consider the following family of methods (where for simplicity we assume $d = 1$, i.e., $\mathbf{y}_n \in \mathbb{R}$ is a scalar and write y_n, k_i etc):*

$$y_{n+1} = y_n + h(b_1 k_1 + b_2 k_2), \tag{7}$$

where

$$\begin{aligned} k_1 &= f(x_n, y_n), & (8)\\ k_2 &= f(x_n + c_2 h, y_n + a_{21} h k_1), & (9) \end{aligned}$$

and where the parameters b_1, b_2, c_2 and a_{21} are to be determined.[2] *The method is consistent iff $b_1 + b_2 = 1$. Further conditions on the parameters are obtained by attempting to maximise the order of accuracy of the method. Indeed, expanding the consistency error τ_n of (7)–(9) in powers of h, after some algebra we obtain*

$$\begin{aligned} \tau_n &= \frac{1}{2} h y''(x_n) + \frac{1}{6} h^2 y'''(x_n) \\ &\quad - b_2 h [c_2 f_x + a_{21} f_y f] - b_2 h^2 \left[\frac{1}{2} c_2^2 f_{xx} + c_2 a_{21} f_{xy} f + \frac{1}{2} a_{21}^2 f_{yy} f^2 \right] + \mathcal{O}(h^3). \end{aligned}$$

Here we have used the abbreviations $f = f(x_n, y(x_n))$, $f_x = \frac{\partial f}{\partial x}(x_n, y(x_n))$, etc. On noting that $y'' = f_x + f_y f$, it follows that $\tau_n = \mathcal{O}(h^2)$ for any f provided that

$$c_2 b_2 = a_{21} b_2 = \frac{1}{2},$$

[1] The use of this tableau was introduced by J. C. Butcher in 1963 with the article *Coefficients for the study of Runge–Kutta integration processes.*

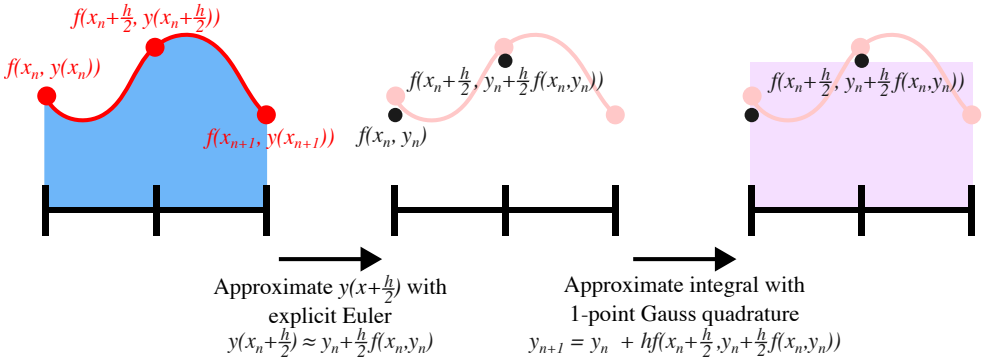
[2] We note in passing that Euler's explicit method is a member of this family of methods, corresponding to $b_1 = 1$ and $b_2 = 0$. However we are now seeking methods that are at least second-order accurate.

which implies that if $a_{21} = c_2$, $b_2 = 1/(2a_2)$ and $b_1 = 1 - 1/(2c_2)$ then the method is second-order accurate; while this still leaves one free parameter, c_2, it is easy to see that no choice of the parameters will make the method generally third-order accurate. There are two well-known examples of second-order explicit Runge–Kutta methods of the form (7), (9):

a) ***The modified Euler method:*** *In this case we take $c_2 = \frac{1}{2}$ to obtain (3):*

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}\left(x_n + \frac{1}{2}h, \mathbf{y}_n + \frac{1}{2}h\mathbf{f}(x_n, \mathbf{y}_n)\right),$$
$$\mathbf{\Psi}(x, \mathbf{y}, h, \mathbf{f}) = x + h\mathbf{f}\left(x + \frac{h}{2}, \mathbf{y} + \frac{h}{2}f(x, \mathbf{y})\right). \tag{10}$$

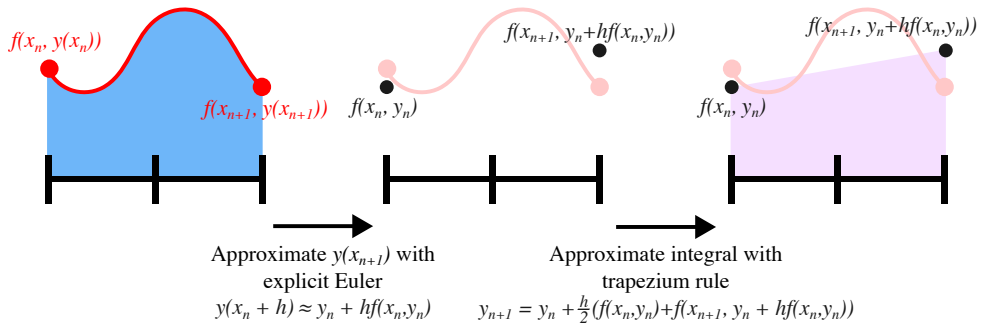
The figure below illustrates this method.



b) ***The improved Euler method:*** *This is obtained by choosing $c_2 = 1$, which gives (4):*

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}h\left[\mathbf{f}(x_n, \mathbf{y}_n) + \mathbf{f}(x_n + h, \mathbf{y}_n + h\mathbf{f}(x_n, \mathbf{y}_n))\right], \tag{11}$$
$$\mathbf{\Psi}(x, \mathbf{y}, h, \mathbf{f}) = \mathbf{y} + \frac{h}{2}\mathbf{f}(x, \mathbf{y}) + \frac{h}{2}\mathbf{f}\left(x + h, \mathbf{y} + h\mathbf{f}(x, \mathbf{y})\right). \tag{12}$$

Here's the corresponding figure.



For these two methods it is easily verified by Taylor series expansion that the consistency error is of the form, respectively,

$$\begin{aligned}\tau_n &= \frac{1}{6}h^2\left[f_yF_1+\frac{1}{4}F_2\right]+\mathcal{O}(h^3),\\ \tau_n &= \frac{1}{6}h^2\left[f_yF_1-\frac{1}{2}F_2\right]+\mathcal{O}(h^3),\end{aligned}$$

where

$$F_1 = f_x + ff_y \quad \textit{and} \quad F_2 = f_{xx} + 2ff_{xy} + f^2f_{yy}.$$

The family (7)–(9) is referred to as the class of explicit two-stage explicit Runge–Kutta methods.

Example 3 *Last but not least, RK4, a 4-stage 4th-order explicit Runge–Kutta method is a famous and very popular choice.*

$$\boldsymbol{\Psi}(x,\mathbf{y},h,\mathbf{f}) = \mathbf{y} + \frac{h}{6}\left(\mathbf{k}_1+2\mathbf{k}_2+2\mathbf{k}_3+\mathbf{k}_4\right),$$

where

$$\begin{aligned}\mathbf{k}_1 &= f(x,\mathbf{y}), \qquad \mathbf{k}_2 = f\left(x+\frac{1}{2}h,\mathbf{y}+\frac{1}{2}h\mathbf{k}_1\right),\\ \mathbf{k}_3 &= f\left(x+\frac{1}{2}h,\mathbf{y}+\frac{1}{2}h\mathbf{k}_2\right), \quad \mathbf{k}_4 = f(x+h,\mathbf{y}+h\mathbf{k}_3).\end{aligned}$$

Its Butcher table reads

0	*0*	*0*	*0*	*0*
1/2	*1/2*	*0*	*0*	*0*
1/2	*0*	*1/2*	*0*	*0*
1	*0*	*0*	*1*	*0*
	1/6	*2/6*	*2/6*	*1/6*

.

We now turn to the convergence properties of RK methods. As mentioned at the end of the last lecture, convergence in the sense that $\mathbf{y}$ converges to the exact solution as $h \to 0$ holds under mild assumptions[3]. We shall thus look at two questions:

- What is the order of accuracy? (Which, as last lecture's theorem showed, is essentially the same as the consistency error)
- Stability: How small does h need to be?

We treat these questions separately.

[3]Strictly speaking, it is not trivial to apply that theorem to general RK methods, which cannot be written easily, as writing a step as $\boldsymbol{\Psi}(s,\mathbf{y},h,\mathbf{f})$ is not trivial, especially for implicit methods. Nonetheless an analogous result still holds; we refer to Section II.3 of the book by Hairer, Norsett, and Wanner for details.

Order of accuracy/consistency order of RK methods

It is convenient at this point to restrict our attention to autonomous IVPs. (Recall that a nonautonomous system can always be made autonomous by increasing its dimension.) The process of making an IVP autonomous commutes with Runge–Kutta discretisation if and only if

$$\sum_{i=1}^{s} b_i = 1, \quad c_k = \sum_{j=1}^{s} a_{kj} \quad k = 1, \dots, s\,,$$

which we assume henceforth. (In other words, if these conditions hold, the RK discretisation of the autonomised system is the autonomisation of the RK discretisation of the original problem.)

By computing appropriate Taylor expansions (as in Example 2), it is possible to derive algebraic conditions the Runge–Kutta coefficients must satisfy for the method to have a targeted consistency order. For example:

Lemma 1 *A Runge–Kutta method is consistent if and only if* $\sum_{i=1}^{s} b_i = 1$ *(which we are already assuming). If the condition*

$$\sum_{i=1}^{s} b_i c_i = \frac{1}{2}$$

is also satisfied, the Runge–Kutta method has consistency order 2, and if the conditions

$$\sum_{i=1}^{s} b_i c_i^2 = \frac{1}{3} \quad \textit{and} \quad \sum_{i=1}^{s} b_i \sum_{j=1}^{s} a_{ij} c_j = \frac{1}{6}$$

are also satisfied, the Runge–Kutta method has consistency order 3.

Nonexaminable: The following table indicates the number of conditions as described above that a Runge–Kutta method must satisfy to have order p:

p	1	2	3	4	5	6	7	8	9	10	20
#conditions	1	2	4	8	17	37	85	200	486	1205	20247374

.

The number of stages of a Runge–Kutta method provides an interesting upper bound on its consistency order.

Lemma 2 *The (consistency) order p of an s-stage Runge–Kutta method is bounded by $p \leq 2s$. If the Runge–Kutta method is explicit, then $p \leq s$.*

Computational considerations. To evolve a numerical solution from x_n to x_{n+1} with a Runge–Kutta method, one needs to compute the stages $\mathbf{k}_i$. If the Runge–Kutta method is explicit, these stages can be computed sequentially (and at a low-cost) starting from $\mathbf{k}_1$ (a Runge–Kutta method is explicit if $a_{ij} = 0$ whenever $j \geq i$, i.e. the matrix $\mathbf{A}$ is strictly lower-triangular). An example of this is the explicit Euler method. If $\mathbf{A}$ is lower-triangular (i.e. possibly $a_{ii} \neq 0$), then the scheme is said to be *diagonally-implicit*;

one can compute the stages $\mathbf{k}_i$ sequentially, solving a sequence of nonlinear problems. The implicit Euler and implicit midpoint rules are examples of diagonally-implicit RK methods. Finally, if $\mathbf{A}$ enjoys neither of these structures, the RK method is said to be fully implicit; one must solve a large coupled nonlinear system for all stages simultaneously.

Barriers. It is possible to construct Runge–Kutta methods that achieve maximal order. So-called *Butcher barriers* quantify the minimal amount of stages that an explicit Runge–Kutta method of order p requires (nonexaminable):

p	1	2	3	4	5	6	7	8	≥ 9
minimal value of s	1	2	3	4	6	7	9	11	$\geq p+3$

This implies that a Runge–Kutta method that has maximal order must be implicit.

Stability of Runge–Kutta methods

As discussed above, if the step size h is sufficiently small the computed solution with a RK method tends to the exact solution. This does not tell us how small h needs to be. Indeed, we have seen that numerical methods for IVPs may encounter stability issues in the last lecture. We now study the appropriate values of h; taking h too small means we'll do many ($O(h^{-1})$) steps to find $y(X)$, which is undesirable.

Fixed points. For simplicity, we only consider autonomous ODEs.

Definition 3 *A* fixed point *of* $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ *is a point* $\mathbf{y}^*$ *such that* $\mathbf{f}(\mathbf{y}^*) = \mathbf{0}$. *A fixed point* $\mathbf{y}^*$ *is* asymptotically stable *(or attractive) if there exists a ball* $B_\delta(\mathbf{y}^*)$ *(of radius* $\delta > 0$ *and centered at* $\mathbf{y}^*$*) such that, whenever* $\mathbf{y}_0 \in B_\delta(\mathbf{y}^*)$*, the solution to* $\mathbf{y}' = \mathbf{f}(\mathbf{y})$, $\mathbf{y}(0) = \mathbf{y}_0$ *satisfies* $\lim_{x\to\infty} \mathbf{y}(x) = \mathbf{y}^*$.

Theorem 1 *A fixed point* $\mathbf{y}^*$ *of an autonomous ODE is asymptotically stable if*

$$\sigma\left(\mathbf{Df}(\mathbf{y}^*)\right) \subset \mathbb{C}^- := \{z \in \mathbb{C} : \mathrm{Re} z < 0\},$$

where $\sigma\left(\mathbf{Df}(\mathbf{y}^*)\right)$ *denotes the set of eigenvalues of the matrix* $\mathbf{Df}(\mathbf{y}^*)$.

This theorem implies that, to study the asymptotic stability of $\mathbf{y}^*$, we can restrict our considerations to the linearised ODE $\mathbf{y}' = \mathbf{Df}(\mathbf{y}^*)(\mathbf{y} - \mathbf{y}^*)$, that is, we can restrict our attention to linear ODEs.

Dahlquist test equation. To further simplify the analysis, we restrict our attention to a single eigenvalue, yielding the *Dahlquist test equation*

$$y' = zy\,, \quad y(0) = 1\,, \quad \text{and} \quad \mathrm{Re} z < 0\,. \tag{13}$$

Clearly, the solution of the Dahlquist test equation is $y(x) = \exp(zx)$, which satisfies $\lim_{x\to\infty} y(x) = 0$. Therefore, $y^* = 0$ is an attractive fixed point.

Do schemes preserve asymptotic stability? In what follows we ask the question: Is the fact that $y \to 0$ respected by the computed solution obtained by a RK method? This can be seen as a 'minimum requirement' for a method to be reasonable; if the exact solution is tending to 0 but the computed one isn't, that surely cannot be a good method. Conversely, if a method (with a given step size h) does satisfy $y_n \to 0$, then it tends to

work well also for more general problems with similar h, in addition to respecting the asymptocic behavior of fixed points as discussed above; this is why we look specifically at the Dahlquist test equation (13).

RK for Dahlquist. The solution of the Dahlquist test equation obtained with a Runge–Kutta method has a special structure:

Definition 4 *Let $\mathbf{\Psi}$ be a one-step method. The function*

$$S : \mathbb{C} \to \mathbb{C}\,, \quad z \mapsto S(z) := \mathbf{\Psi}(0, 1, 1, f : y \mapsto zy),$$

is called the stability function *of $\mathbf{\Psi}$ (defined in (5)). To shorten the notation, we henceforth write $\mathbf{\Psi}(0, 1, 1, z)$ instead of $\mathbf{\Psi}(0, 1, 1, f : y \mapsto zy)$.*

Lemma 3 *If $\mathbf{\Psi}$ is a Runge–Kutta method, then $\mathbf{\Psi}(0, \ell, h, z) = \mathbf{\Psi}(0, 1, 1, zh)\ell$.*

Theorem 2 *Let $\{y_k\}_{k\in\mathbb{N}}$ be the Runge–Kutta solution to the Dahlquist test equation obtained with a time step $h > 0$. Then, $y_k = S(zh)^k$.*

Proof. By direct computation, we can see that

$$y_1 = \mathbf{\Psi}(0, 1, h, z) = \mathbf{\Psi}(0, 1, 1, zh) = S(zh)$$

and that

$$y_2 = \mathbf{\Psi}(0, y_1, h, z) = \mathbf{\Psi}(0, 1, 1, zh)y_1 = S(zh)y_1 = S(zh)^2.$$

Therefore, we conclude that $y_k = S(h\lambda)^k$. □

Notions of stability. As discussed, it is desirable that the discrete solution $\{y_k\}_{k\in\mathbb{N}}$ satisfies $\lim_{k\to\infty} y_k = 0$, mimicking the behavior of the exact solution to the Dahlquist test equation. When this happens, we say that $\{y_k\}_{k\in\mathbb{N}}$ is *asymptotically stable*.

Definition 5 *The region in the complex plane*

$$S_{\mathbf{\Psi}} := \{z \in \mathbb{C} : |S(z)| < 1\}$$

is called the stability region *of the Runge–Kutta method. Clearly, $\{y_k\}_{k\in\mathbb{N}}$ is asymptotically stable if $zh \in S_\Psi$.*

It is not so difficult to see that the stability function of an explicit Runge–Kutta method is a polynomial, which implies that $S_{\mathbf{\Psi}}$ is bounded. Therefore, the numerical approximation computed with an explicit Runge–Kutta method cannot be asymptotically stable if the time step h is too large. This is what we saw in our numerical experiments in the last lecture. However, the stability function of an implicit Runge–Kutta method is a rational function, and hence may not suffer from this limitation.

Definition 6 *A Runge–Kutta method is said to be* A-stable[4] *if $\mathbb{C}^- \subset S_{\mathbf{\Psi}}$.*

[4]This concept was introduced by G. Dahlquist in 1963 with the article *A special stability problem for linear multistep methods.*

A-stability guarantees that $\{y_k\}_{k\in\mathbb{N}}$ will eventually converge to zero. However, the decay can be very slow compared to that of the exact solution.

Example 4 *Let $\{y_k\}$ be the approximate solution to the Dalhquist test equation obtained with the implicit midpoint rule and a fixed step size h. By direct computation, we can see that stability function of the implicit midpoint rule is*

$$S(z) = \frac{2+z}{2-z}.$$

The exact solution converges exponentially to zero with rate $\mathrm{Re}z$*; the smaller (more negative) the* $\mathrm{Re}z$*, the quicker the convergence. On the other hand, $\{y_k\}$ is a geometric sequence with ratio $S(zh)$. This also converges to zero, but the more negative the* $\mathrm{Re}z$*, the closer $|S(zh)|$ to 1, and the slower the decay of $\{y_k\}$. This implies that, if* $\mathrm{Re}z \ll 0$*, the qualitative behavior of $\{y_k\}$ can be very different from the one of the exact solution.*

Therefore, if the initial value problem has a strongly attractive fixed point, it is advisable to further ensure that $\lim_{\mathrm{Re}z\to-\infty} |S(z)| = 0$.

Definition 7 *An A-stable method that further satisfies $\lim_{\mathrm{Re}z\to-\infty} |S(z)| = 0$ is said to be* L-stable.

One can verify that the implicit Euler method is L-stable, but it is not the only one.

The remainder is nonexaminable.

Construction of implicit RK methods

We have seen that s-stage explicit Runge–Kutta methods have at most order s, while implicit RK methods have at most order $2s$. We now construct s-stage implicit Runge–Kutta methods whose order is at least s.

Definition 8 *Let $c_1, \ldots, c_s \in [0,1]$ be (pairwise distinct) collocation points. The corresponding collocation method is the one-step method defined by*

$$\boldsymbol{\Psi}(x, \mathbf{y}, h, \mathbf{f}) = \tilde{\mathbf{y}}(h),$$

where $\tilde{\mathbf{y}}$ is the unique polynomial of degree s that satisfies

$$\tilde{\mathbf{y}}(0) = \mathbf{y} \quad \text{and} \quad \tilde{\mathbf{y}}'(c_i h) = \mathbf{f}(x + c_i h, \tilde{\mathbf{y}}(c_i h)), \quad \text{for } i = 1, \ldots, s. \tag{14}$$

Lemma 4 *Let Q be the highest-order quadrature rule on $[0,1]$ that can be constructed using the nodes $c_1, \ldots, c_s$, and let p_Q be its order (p_Q = 1 + the maximal degree of polynomials it integrates exactly). If $\mathbf{f}$ is sufficiently smooth and $h > 0$ is sufficiently small, the collocation method associated to $c_1, \ldots, c_s$ has order p_Q.*

Corollary 1 *If $\mathbf{f}$ is sufficiently smooth and $h > 0$ is sufficiently small, the order of the collocation method associated to $c_1, \ldots, c_s$ is at least s and at most $2s$ (Gauss quadrature).*

It is not obvious, but collocation methods are indeed Runge–Kutta methods.

Lemma 5 *Collocation methods are Runge–Kutta methods. Their coefficients are*

$$a_{ij} = \int_0^{c_i} L_j(\tau)\,\mathrm{d}\tau\,, \quad b_i = \int_0^1 L_i(\tau)\,\mathrm{d}\tau\,, \tag{15}$$

where $\{L_i\}_{i=1}^s$ *are the Lagrange polynomials associated to* $c_1, \ldots, c_s$.

The Gauss collocation methods form a family of arbitrarily high-order A-stable methods whose stability region is exactly $\mathbb{C}^-$.

An example of a family of L-stable RK methods is the Gauss–Radau family. This is a family of collocation methods where the final quadrature point is fixed to $c_s = 1$ and the remaining points $c_1, \ldots, c_{s-1}$ are chosen to obtain an associated quadrature rule of maximal order $2s - 1$.

For more detailed discussions on RK methods, we refer to the books

- Süli and Mayer, "Introduction to Numerical Analysis"
- Hairer, Norsett, and Wanner, "Solving Ordinary Differential Equations"
- Butcher, "Numerical Methods for Ordinary Differential Equations"