Part A1: Differential Equations I

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Contents

*partly based on notes by Janet Dyson, Peter Grindrod, Colin Please, Paul Tod, Lionel Mason and others

Introduction

The solution of problems in most parts of applied mathematics and many areas of pure can more often than not be reduced to the problem of solving some differential equations. Indeed, many parts of pure maths were originally motivated by issues arising from differential equations, including large parts of algebra and much of analysis, and differential equations are a central topic in research in both pure and applied mathematics to this day. From Prelims, and even from school, you know how to solve some differential equations. Indeed most of the study of differential equations in the first year consisted of finding explicit solutions of particular ODEs or PDEs. However, for many differential equations which arise in practice one is unable to give explicit solutions and, for the most part, this course will consider what information one can discover about solutions without actually finding the solution. Does a solution exist? Is it unique? Does it depend continuously on the initial data? How does it behave asymptotically? What is appropriate data?

So, first we will develop techniques for proving Picard's theorem for the existence and uniqueness of solutions of ODEs; then we will look at how phase plane analysis enables us to estimate the long term behaviour of solutions of plane autonomous systems of ODEs. We will then turn to PDEs and show how the method of characteristics reduces the solution of a first order semi-linear PDE to solving a system of non-linear ODEs. Finally we will look at second order semi-linear PDEs: We classify them and investigate how the different types of problem require different types of boundary data if the problem is to be well posed. We then look at how the maximum principle enables us to prove uniqueness and continuous dependence on the initial data for two very special problems: Poisson's equation and the inhomogeneous heat equation – each with suitable data.

Throughout, we shall use the following convenient abbreviations: we shall write

DEs: for differential equations.

ODEs: for ordinary DEs, i.e. differential equations with only ordinary derivatives.

PDEs: for partial DEs, i.e. differential equations with partial derivatives.

The course contains four topics, with a section devoted to each. The chapters are:

1. ODEs and Picard's Theorem (for existence/uniqueness of solutions/continuous dependence on initial data).
2. Plane autonomous systems of ODEs
3. First order semi-linear PDEs: the method of characteristics.
4. Second-order semi-linear PDEs: classification; well posedness; the Maximum Principle and its consequences

Remarks on lecture in MT 24

There has been a change of syllabus from the course in previous years. Please keep this in mind when using past papers etc for revision. Key changes are that

- the alternative proof of Picard's theorem via contraction mapping theorem is no longer part of the syllabus, and is hence only included as an optional appendix .
- there is a new part concerning existence of solutions of ODE until potential blow up and the comparison principle (sections 1.7 and 1.8)

Books

The main text is P J Collins *Differential and Integral Equations*, O.U.P. (2006), which can be used for the whole course (Chapters 1-7, 14, 15).

Other good books which cover parts of the course include

W E Boyce and R C DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 7th edition, Wiley (2000).

E Kreyszig, *Advanced Engineering Mathematics, 8th Edition*, Wiley (1999).

G F Carrier and C E Pearson, *Partial Differential Equations – Theory and Technique*, Academic (1988).

J Ockendon, S Howison, A Lacey and A Movchan, *Applied Partial Differential Equations*, Oxford (1999) [a more advanced text].

Acknowledgements

I am grateful to colleagues for allowing me to use and adapt their notes from earlier courses.

PART I Ordinary Differential Equations

1 ODEs and Picard's Theorem

1.1 Introduction

An ODE is an equation for an unknown function $y(x)$ of one variable x of the form

$$G(x, y(x), y'(x), y''(x),\ \ldots,\ y^{(n)}(x)) = 0$$

where $y^{(n)}(x) = \frac{d^n y}{dx^n}(x)$ denotes n-th order derivative of $y(x)$. Often this can be solved for the highest derivative of y and written in the form

$$y^{(n)}(x) = F(x, y, y',\ \ldots,\ y^{(n-1)}).$$

The *order* n of the ODE is the order of the highest derivative which appears.
Given an ODE, certain obvious questions arise. We could ask:

- Does it have solutions? Can we find them (explicitly or implicitly)? If not, can we at least say something about their qualitative behaviour?
- Given data e.g. the values $y(a),\ y'(a),\ \ldots y^{(n-1)}(a)$ of $y(x)$ and its first $n-1$ derivatives at some initial $x = a$, does there exist a solution? If so, is this solution unique? And does the solution depend continuously on the given data?

We shall consider these questions in Part I.

For simplicity, we begin with a first-order ODE with data

$$y'(x) = f(x, y(x)) \text{ with } y(a) = b. \tag{1.1}$$

We call such a combination of an ODE and the prescribed value at some $x = b$ an *initial value problem* or IVP, since we are given y at an initial, or starting, value of x.

You know how to solve a variety of equations like this but there are many functions f for which you cannot explicitly solve such a problem. Nonetheless, you might expect that a solution *exists*, i.e. that there is some function that satisfies both the ODE and that has $y(a) = b$ (even if you cannot find a formula for it) and perhaps you expect that the solution is *unique* (i.e. that there is only one function that satisfies the ODE and the initial data) as you may not have encountered the following difficulties.

Warning examples:

Example 1: Non-uniqueness of solutions Consider the IVP

$$y'(x) = 3y(x)^{2/3}; \quad y(0) = 0. \tag{1.2}$$

You have seen in prelims that you can find a solution of the ODE by separating the variables

$$\int \frac{dy}{3y^{2/3}} = \int dx,$$

to get $y(x) = (x + A)^3$. Combined with the initial condition $y(0) = 0$ you get

(i) There is a solution $y(x) = x^3$;

(ii) But evidently there is another solution: Namely the constant function $y(x) = 0$, $x \in \mathbb{R}$, satisfies both the initial condition $y(0) = 0$ and the ODE since $y'(x) = 0 = 3y(x)^{2/3}$ for all x.

(iii) In fact we can find that there are infinitely many solutions. Pick c, d with $c \leq 0 \leq d$ and define the function $y_{c,d}(x)$ by

$$y_{c,d}(x) := \begin{cases} (x-c)^3 & \text{for } x < c \\ 0 & \text{for } c \leq x < d \\ (x-d)^3 & \text{for } d \leq x \end{cases}$$

These functions all satisfy the initial condition, are differentiable everywhere (including at the points c and d) and satisfy the ODE.

For this initial value problem we hence get that the solution does exist but is not unique (in fact far from it, since we've found infinitely many solutions).

Furthermore, even if a solution of (1.1) exists, it may not exist for all x.

Example 2: Blow up Consider the IVP

$$y'(x) = y^2(x); \quad y(0) = 1. \tag{1.3}$$

Using separation of variables we can see this has solution $y(x) = \frac{1}{1-x}$. As $y(x) \to \infty$ as $x \nearrow 1$ we hence get that the solution only exists until we approach this value of x, so since our initial data is given at $x = 0 < 1$ the solution is defined only for $x < 1$.

On the other hand, this solution is in fact unique (which you will be able to check once you have seen Picard's theorem).

So, if we want to be sure that our problem (1.1) has a unique solution then we must impose conditions on f, and we cannot necessarily expect to have solutions for all x, but instead hope to prove that a solution will exist at least on an interval $[a-h, a+h]$ around the x value $x = a$ at which we are given the data.

So what we are looking for are conditions on the function f that are *sufficient to guarantee existence and uniqueness of a solution of our (IVP)*. This will be the first *existence theorem* which you've encountered. The idea of such theorems is that they give you conditions that you can check (often with very simple calculations) and that will tell you that the problem is solvable (and often give you additional information on the solution like here uniqueness). This is helpful in many contexts in both pure mathematics and in applied mathematics (where such theorems mean e.g. that you will not need to worry about whether there are further solutions than the one that you might e.g. generate with numerics etc). Such theorems usually guarantee that the solution exists on at least a certain interval around the initial point $x = a$, and once you have existence of a solution you can then analyse further properties of this solutions (such as continuous dependence, maximal existence interval, potential blow-up or asymptotic behaviour) with other methods.

We shall seek a solution $y(x)$ of problem (1.1) which is defined (at least) on an interval $[a-h, a+h]$ for a suitable $h > 0$ and whose graph is contained in a rectangle R around the point $(x, y) = (a, b)$ that describes our initial condition $y(a) = b$. For now we consider rectangles of the form

$$R = \{(x, y) : |x-a| \leq h,\ |y-b| \leq k\}, \tag{1.4}$$

see figure 1.1 on which f is defined (later on part of the task to apply Picard will be to find out if we can choose $h, k > 0$ in a suitable way so that all assumptions of the theorem are satisfied).

Our goal is to identify conditions on the function $f : R \to \mathbb{R}$ which *guarantee* the existence and uniqueness of a solution $y(x)$ of the initial value problem (1.1).

IMPORTANT: These conditions will be properties of the function $f : R \to \mathbb{R}$ that assigns to each pair of numbers (x, y) the function value $f(x, y)$. This function $(x, y) \mapsto f(x, y)$ of two variables should not be confused with the function $x \mapsto f(x, y(x))$ that we obtain by composing the function f with a given function of one variable $y(x)$.

At first glance it can be a bit confusing that we use the letter y for two different things but this is standard in the literature: On the one hand we will use y as notation for a number/independent variable if we talk of the properties of the function f, such as in the assumptions of Picard's theorem below or when you are checking whether these assumptions are satisfied in applications. On the other hand we will use $y(x)$ as notation for a function in one variable (such as the desired solution of our IVP).

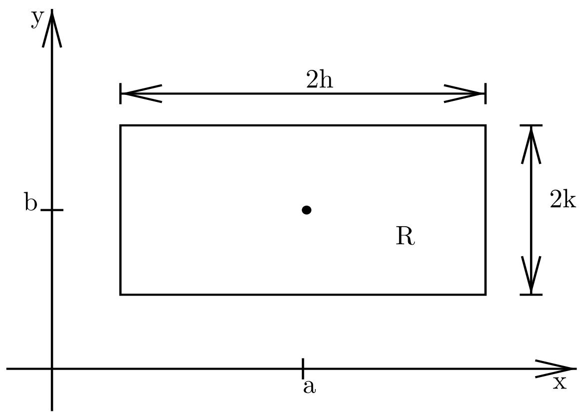


Figure 1.1: The rectangle R

Our first assumption on the function f is that $f : R \to \mathbb{R}$ is continuous in R.

Since R is a rectangle that we obtain from closed and bounded intervals, knowing that $f : R \to \mathbb{R}$ is continuous guarantees that f is bounded on R and achieves its maximum and minimum[1]. In the following we will always set

$$M := \sup_R |f| = \max_R |f|.$$

As the above example 1 shows, only asking that f is continuous is not sufficient to guarantee that the solution of the IVP (1.1) is unique (even on a very small interval $[a - h, a + h]$). [2].

We hence need to impose a second condition on f and the right property to ask for is the following Lipschitz-condition, which asks for Lipschitz continuity in the second variable.

[1]Indeed, R is a so called compact set so this is a special case of the result that continuous functions on compact sets achieve their extrema that you will see in A2 Metric spaces

[2]Continuity of f turns out to be sufficient to guarantee existence of a solution, but the corresponding result, called Peano's Theorem, goes beyond this course and is only covered in C4.6 Fixed point methods for non-linear PDEs

Definition 1.1. *A function $f : [a-h, a+h] \times [b-k, b+k] \to \mathbb{R}$, satisfies a* Lipschitz condition *(with constant L) if there exists a number $L > 0$ such that*

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2| \text{ for all } x \in [a-h, a+h] \text{ and all } y_1, y_1 \in [b-k, b+k]. \quad (1.5)$$

This is a new condition on a function, stronger than being continuous in the second variable but weaker than being continuously differentiable. We will later see that this condition not only allows us to prove that the IVP has a unique solution but also that this solution changes continuously if we change the data b.

Useful Remark: One way to ensure that f satisfies a Lipschitz condition on R is the following: Suppose that, on R, f is differentiable with respect to y , with $|f_y(x, y)| \leq K$ for some K and all $(x, y) \in R$. Then for each $x \in [a-h, a+h]$ we can apply the mean value theorem to the function $[b-k, b+k] \ni y \mapsto f(x, y)$ to see that for any $y, \tilde{y} \in [b-k, b+k]$ there exists ξ between y and $\tilde{y}$ so that

$$|f(x, \tilde{y}) - f(x, y)| = |f_y(x, \xi)(y - \tilde{y})| \leq K|y - \tilde{y}| \quad (1.6)$$

So, f clearly satisfies the Lipschitz condition on such rectangles R with $L = K$.

On the other hand $f(y) = |y|$ is Lipschitz continuous, but is not differentiable at $y = 0$.

Our main result on the existence and uniqueness of solution now asserts that these two conditions are sufficient to guarantee the existence and uniqueness of a solution at least on a suitably small interval $[a-h, a+h]$. In practice this means that to know that a solution exists (and is unique), we don't need to be able to solve the problem, but just need to check the above two properties for the function $(x, y) \mapsto f(x, y)$ (which we stress is a function of two numbers x and y, NOT the composition of f with another function $x \mapsto y(x)$).

1.2 Picard's Theorem

Theorem 1.1. (Picard's existence theorem): *Let $f : R \to \mathbb{R}$ be a function defined on the rectangle $R := \{(x, y) : |x-a| \leq h, |y-b| \leq k\}$ which satisfies*
P(i)*: (a) f is continuous in R with $|f(x, y)| \leq M$ for all $(x, y) \in R$) (b) $Mh \leq k$.*
P(ii)*: f satisfies a Lipschitz condition in R.*

Then the IVP

$$y'(x) = f(x, y(x)) \text{ with } y(a) = b.$$

has a unique solution y on the interval $[a-h, a+h]$.

We give the proof of the theorem below and first explain how it can be applied to discuss existence and uniqueness of solutions for some examples of IVPs.

Example 1: Consider the IVP

$$y'(x) = y^3(x) \sin(x^2 y(x)), \qquad y(0) = 1$$

The function of two variables we have to consider is $(x, y) \mapsto y^3 \sin(x^2 y)$ which is a continuous function on all of $\mathbb{R}^2$ (it's simply a function in two variables that is obtained as product and composition of polynomials and the sine function, so certainly continuous). It is also differentiable on $\mathbb{R}^2$ with $\partial_y f(x, y) = x^2 y^3 \cos(x^2 y) + 3y^2 \sin(x^2 y)$.

If we fix any rectangle $R = [-h, h] \times [1-k, 1+k]$ then we can bound $|f| \leq (1+k)^3 =: M$ and $|\partial_y f| \leq h^2(k+1)^3 + 3(k+1)^2$. We hence have that **P(i)** is satisfied if h and k are so that

$Mh \leq k$, i.e. $h \leq \frac{k}{(1+k)^3}$ and by the MVT know that the Lipschitz condition is satisfied with $L = h^2(k+1)^3 + 3(k+1)^2$ (for any choice of such a rectangle).

Thus, even though we cannot solve the (IVP) explicitly, we know that there exists a unique solution on an interval $[-h, h]$ for h chosen so that $h \leq \frac{k}{(1+k)^3}$ for some $k > 0$.

Remark: To get an example of such an h we can substitute any value $k > 0$ into the above inequality, e.g. $k = 1$ would give $h = \frac{1}{8}$ and the theorem will guarantee that the solution exists and is unique *at least* on the interval $[-\frac{1}{8}, \frac{1}{8}]$.

Note: You can get a better h by choosing $k > 0$ so that the right hand side of $h \leq \frac{k}{(1+k)^3}$ is maximal (giving $k = \frac{1}{2}$ and $h = \frac{4}{27}$).

Picard's theorem ensures that the solution exists and is unique *at least* on the corresponding interval $[-h, h]$, but we note that the "best", i.e. largest, interval on which existence/uniqueness holds will in general be larger (see also section 1.7 below).

Since the warning example 1 doesn't have a unique solution, something goes wrong for it. As an exercise, show that the warning example 1 fails the Lipschitz condition (in any neighbourhood of the initial point).

The following example also fails the Lipschitz condition in any neighbourhood of a point $(a, 0)$. However, the Lipschitz condition does hold on any rectangle which does not contain any point $(x, 0)$.

Example 2: Consider the IVP

$$y'(x) = x^2 y(x)^{1/5}, \quad y(0) = b.$$

So we consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^2 y^{1/5}$ which is clearly continuous. (Note that when we write $y^{1/5}$ here we mean to take the real root: so that if y is negative we will take $-|y|^{1/5}$.)

Case $b = 0$: $f(x, y)$ does not satisfy a Lipschitz condition on any rectangle of the form $R_0 = \{(x, y) : |x| \leq h,\ |y| \leq k\}$, where $h > 0$ and $k > 0$.

Suppose it does, then there exists a finite constant L such that for all $|x| \leq h$ and $|y|, |\tilde{y}| \leq k$

$$|x^2||y^{1/5} - \tilde{y}^{1/5}| \leq L|y - \tilde{y}|$$

so in particular (choosing $\tilde{y} = 0$ and $x = h$)

$$|h^2||y^{-4/5}| \leq L \text{ for every } y \in [-h, h] \setminus \{0\}.$$

But this is a contradiction as $|h^2||y^{-4/5}|$ is unbounded as $y \to 0$ so the function does not satisfy a Lipschitz condition on R_0. So Picard's theorem does not apply if we take $b = 0$. (We saw that f satisfies a Lipschitz condition on any rectangle where its derivative with respect to y exists and is bounded. The problem here is that the derivative of f is unbounded as $y \to 0-$ and does not even exist at $y = 0$.)

Case $b > 0$: However, the assumptions of Picard's theorem will be satisfied if we take as initial condition $y(0) = b > 0$, provided we take a rectangle, R_b, given by $R_b = \{|x| \leq h,\ |y - b| \leq k\}$ with $0 < k < b$, so that y cannot be zero in this rectangle.

On any such rectangle $f_y(x, y) = \frac{x^2 y^{-4/5}}{5}$ is bounded by $|f_y| \leq \frac{1}{5} h^2 (b - k)^{-4/5}$ so by (1.6), f satisfies a Lipschitz condition, and $\mathbf{P(ii)}$ is satisfied.

For $\mathbf{P(i)}$: f is continuous on R_b and

$$\max_{R_b} |x^2 y^{1/5}| = h^2(b+k)^{1/5} =: M,$$

so Picard's theorem applies in the rectangle R provided $h > 0$ satisfies

$$h^2(b+k)^{1/5} h \leq k.$$

That is

$$h^3 \leq \frac{k}{(b+k)^{1/5}}. \tag{1.7}$$

We can of course solve this problem directly using separation of variables if we wish giving

$$y = \left(4x^3/15 + b^{4/5}\right)^{5/4},$$

so actually the solution exists for all x. Note the solution above is valid for $b = 0$ BUT the trivial solution is also valid. So while we still have existence, uniqueness does not hold.

1.3 Rewriting an IVP into an equivalent integral equation

A key idea of not only the proof of Picard's theorem, but of many other methods and proofs in the analysis of differential equations is that it is often easier to work not with the differential equation directly, but with an equivalent integral equation for which we can prove the existence of a solution (and could indeed iteratively construct a solution) using an iteration scheme.

To see that we can equivalently formulate an IVP as an integral equation we will exploit basic properties of integration as well as the assumption that f is continuous:

Namely, we note that if $y(x)$ is differentiable and satisfies (1.1) on an interval $[a-h, a+h]$, then $y(x)$ is certainly continuous so since f is also assumed to be continuous we know that the composition $x \mapsto f(x, y(x))$ is a continuous function on $[a-h, a+h]$ so integrable. We can thus integrate the differential equation $y'(t) = f(t, y(t))$ satisfied by y from a to variable $x \in [a-h, a+h]$ to see that

$$y(x) - y(a) = [y(t)]_a^x = \int_a^x f(t, y(t))dt \text{ for any } x \in [a-h, a+h].$$

Rearranging we get that any solution of the IVP satisfies the integral equation

$$y(x) = b + \int_a^x f(t, y(t))dt \text{ for any } x \in [a-h, a+h]. \tag{1.8}$$

Conversely, if $y : [a-h, a+h] \to [b-k, b+k]$ is a continuous function which satisfies this integral equation (1.8), then $y(a) = b$ and since the integrand $t \mapsto f(t, y(t))$ is continuous we get from the Fundamental theorem of Calculus that y is differentiable in every $x \in [a-h, a+h]$ with $y'(x) = f(x, y(x))$, so y is a solution of the IVP (1.1). Thus (1.1) and (1.8) are equivalent.

In particular, establishing that a unique solution y of the IVP (1.1) exists is equivalent to establishing that a unique solution of the integral equation (1.8) exists.

There is an extensive theory for integral equations associated to both ODEs and PDEs, and two standard ways of proving the existence of solutions of such equations are

- By iteration (or successive approximation): We start with an initial guess y_0 for the solution and successively improve it by computing the next iterate y_{n+1} by inserting the previous iterate into the right hand side of the integral equation, i.e. setting

$$y_{n+1}(x) := b + \int_a^x f(t, y_n(t))dt.$$

  Note, this is equivalent to asking that y_{n+1} satisfies $y'_{n+1}(x) = f(x, y_n(x))$ and $y_{n+1}(a) = b$.

- By viewing the equation as a fixed point problem $Ty(x) = y(x)$ on a suitable space of functions and applying a suitable fixed point theorem. In our context T can be chosen as the operator that assigns to a given function $y(x)$ the function $Ty(x) = b + \int_a^x f(t, y(t))dt$ that corresponds to the right hand side of the integral equation. One can then check that T (when considered on a suitable metric space of functions) satisfies all of the assumptions of the contraction mapping theorem that you will see in A2 Metric spaces. This proof is off sylabus, but we include it in Appendix **??** since similar arguments are commonly used in the theory of not just ODEs but also PDEs to establish the existence of solutions.

For some problems one can prove existence and uniqueness at the same time/with the same method, but often it is easier to separate the two proofs and approach them with different methods. Here we will first establish existence via successive approximation (see section 1.4) below. The standard approach to prove uniqueness of ODEs is to use Gronwall's inequality. We will carry out this part of the proof of Picard's theorem in Section 1.5 and will see that with this approach we not only get uniqueness of solutions (i.e. that two solutions y_1 and y_2 of the same ODE with the same initial condition must agree), but can also deduce that the solution depends continuously on the initial data (i.e. that solutions y_1 and y_2 of the same ODE must be close to each other if their initial values $y_1(a)$ and $y_2(a)$ are close).

1.4 Proof of the existence part of Picard's theorem via method of successive approximation

The idea of obtaining a solution with iteration is to start with a suitable initial guess, which is a function $y_0(x)$, then use $y_0(x)$ to generate the next iterate $y_1(x)$ and continue inductively. The hope is that if these iterates are constructed suitably (and if the problem has the right properties, in our case **P (i)** and **P(ii)**) then these iterates $y_n(x)$ will be defined for every n and will converge to a limit $y_\infty(x)$ that is a solution of our problem.

As an initial guess we start with the simplest function which has the right initial value, i.e. let $y_0(x)$ be the constant function

$$y_0(x) = b \text{ for every } x \in [a-h, a+h] \tag{1.9}$$

and then for $n = 0, 1, \dots$ want to define the next Picard iterate inductively by the function we get by inserting the previous iterate into the right hand side of the integral equation, i.e. by

$$y_{n+1}(x) := b + \int_a^x f(t, y_n(t))dt. \tag{1.10}$$

To prove Picard's theorem we first show that these iterates $y_n(x)$ are all well defined (and continuous) functions with graph in R, then derive suitable estimates on the difference $y_{n+1}(x)-$

$y_n(x)$ between subsequent iterates and finally use these estimates to prove that the iterates converge to a limit y_∞ which solves our problem.

So to begin we show

Claim 1: Each $y_n(x)$ is a well defined and continuous function on $[a-h, a+h]$ which satisfies $|y_n(x) - b| \leq k$ for all $x \in [a-h, a+h]$.

Proof of Claim 1: This is clearly true for $n = 0$, so suppose the claim is true for some $n \geq 0$. Then for $t \in [a-h, a+h]$ we have that $(t, y_n(t)) \in R$ so we can evaluate f at this point as f is defined on R. Also, as $y_n : [a-h, a+h] \to [b-k, b+k]$ is continuous, we know that $t \mapsto (t, y_n(t))$ is a continuous function from $[a-h, a+h]$ to the rectangle R so as f is a continuous function from R to $\mathbb{R}$ we get that the composition $[a-h, a+h] \ni t \mapsto f(t, y_n(t)) \in \mathbb{R}$ is continuous and hence integrable.

Thus the next iterate y_{n+1} is a well defined function from $[a-h, a+h]$ to $\mathbb{R}$ which by properties of integration is differentiable, so in particular also continuous. It remains to check that $|y_{n+1}(x) - b| \leq k$. For this we use the definition of M, the assumption that $Mh \leq k$ and the triangle inequality which allow us to bound

$$|y_{n+1}(x) - b| \quad \leq \quad \left| \int_a^x |f(t, y_n(t))| dt \right| \leq \left| \int_a^x M dt \right| = M|x-a| \leq Mh \leq k$$

for every $x \in [a-h, a+h]$. Note that the modulus outside the integrals is required to cover the case $x \leq a$.

Thus claim 1 is true by induction. $\square$

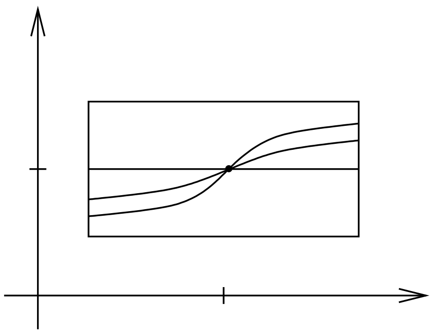


Figure 1.2: successive iterates all given by continous functions with graph in R.

Next we want to use the Lipschitz condition **P(ii)** to prove estimates on the differences $y_{n+1}(x) - y_n(x)$ between subsequent iterates.

The key point here is that if we are given any two functions $y(x)$ and $\tilde{y}(x)$ then the Lipschitz condition allows us to bound the difference between the corresponding right hand sides of the integral equation by

$$\begin{aligned} |\int_a^x f(t, y(t)) dt - \int_a^x f(t, \tilde{y}(t)) dt| &\leq |\int_a^x |f(t, y(t)) - f(t, \tilde{y}(t))| dt| \\ &\leq L| \int_a^x |y(t) - \tilde{y}(t))| dt| \end{aligned} \tag{1.11}$$

where the first step is by triangle inequality and the second by **P(ii)**.

Combined with an induction argument this will allow us to prove:

Claim 2: Let $\hat{M}$ be so that $|f(x,b)| \leq \hat{M}$ for all $x \in [a-h, a+h]$ and let L be so that the Lipschitz condition (1.5) holds.

Then the differences

$$e_n(x) := y_n(x) - y_{n-1}(x), \quad n = 1, 2, \dots \tag{1.12}$$

satisfy

$$|e_n(x)| \leq \frac{L^{n-1}\hat{M}}{n!}|x-a|^n \tag{1.13}$$

for every $x \in [a-h, a+h]$ and every $n \in \mathbb{N}$.

We note that here we can of course choose $\hat{M} = M$ if we want (as a bound on $|f|$ on the whole rectangle will of course also give a bound on the line $[a-h, a+h] \times \{b\} \subset R$).

Proof of Claim 2. We prove the claim by induction. For the base case $n = 1$ we can use that since $y_0(x) = b$ for all x we have

$$e_1(x) = y_1(x) - y_0(x) = b + \int_a^x f(t, y_0(t))dt - b = \int_a^x f(t,b)dt$$

and thus

$$|e_1(x)| \leq |\int_a^x |f(t,b)|dt| \leq |\int_a^x \hat{M}dt| \leq \hat{M}|x-a|$$

as claimed.

So suppose that the claim holds for some $n \in \mathbb{N}$. Then we use the definition of the Picard iterates (1.10) to write

$$\begin{aligned} e_{n+1}(x) = y_{n+1}(x) - y_n(x) &= b + \int_a^x f(t, y_n(t))dt - \big[b + \int_a^x f(t, y_{n-1}(t))dt\big] \\ &= \int_a^x f(t, y_n(t)) - f(t, y_{n-1}(t))dt. \end{aligned}$$

The important point is now that the Lipschitz condition P(ii), combined with the fact that the graphs of y_n and y_{n-1} are in the rectangle R, implies that for all $|t-a| \leq h$

$$|f(t, y_n(t)) - f(t, y_{n-1}(t))| \leq L|y_n(t) - y_{n-1}(t)| = L|e_n(t)|.$$

Combined with the induction assumption we thus get that

$$|\, e_{n+1}(x)| \leq L\left|\int_a^x |e_n(t)|dt\right| \leq L\left|\int_a^x \frac{L^{n-1}\hat{M}}{n!}|t-a|^n dt\right| = \frac{L^n\hat{M}}{(n+1)!}|x-a|^{n+1},$$

so that (1.13) is true by induction.

□

We now use these two claims to prove the *existence* of a solution to the integral equation (1.8).

The key observation here is that we can express our iterates as

$$\begin{aligned} y_n(x) &= y_n(x) - y_{n-1}(x) + y_{n-1}(x) - .... + y_1(x) - y_0(x) + y_0(x) \\ &= \sum_{k=1}^{n} e_k(x) + y_0(x) \end{aligned} \tag{1.14}$$

i.e. as a sum of the fixed function $y_0(x) = b$ and a sum of the differences e_k. The estimates of Claim 2 will then allow us to apply the Weierstrass M-test to conclude that the series of functions $\sum e_k(x)$, and hence $y_n(x)$, converges uniformly.

As a last step in the proof of the existence part of Picard we can hence prove:

Claim 3: The iterates $y_n(x) = y_0(x) + \sum_{j=1}^{n} e_j(x)$ converge uniformly to a continuous function $y_\infty(x)$ on the interval $[a-h, a+h]$ and $y_\infty(x)$ is a solution of the integral equation (1.8).

Proof of Claim 3: We note that Claim 2 implies that for every $n \in \mathbb{N}$

$$|e_n(x)| \leq \frac{L^{n-1}\hat{M}}{n!} h^n =: M_n \text{ for all } x \in [a-h, a+h]. \tag{1.15}$$

As $\sum_{n=1}^{\infty} M_n$ converges (e.g. by ratio test since $\frac{M_{n+1}}{M_n} \to 0$) we hence know from the Weierstrass M-test that $\sum_{j=1}^{n} e_j(x)$ converges uniformly on $[a-h, a+h]$ as $n \to \infty$. As $y_0(x)$ is just a fixed function (i.e. independent of n) we hence know that also $y_n(x) = y_0(x) + \sum_{j=1}^{n} e_j(x)$ converges uniformly to a limiting function $y_\infty(x)$ on $[a-h, a+h]$. The uniform convergence combined with the continuity of the functions y_n ensures that y_∞ is also continuous.

Importantly, we also get that the functions $f(t, y_n(t))$ that we use to define the next iterates converge uniformly to $f(t, y_\infty(t))$. Indeed, thanks to the Lipschitz-condition and the uniform convergence of the $y_n(t)$ we have

$$\sup_{t \in [a-h, a+h]} |f(t, y_n(t)) - f(t, y_\infty(t))| \leq L \sup_{t \in [a-h, a+h]} |y_n(t) - y_\infty(t)| \to 0,$$

where the second step follows since y_n converges uniformly.

Since we can exchange limit and integrals if we are dealing with uniformly convergent sequences of functions we hence know that for every $x \in [a-h, a+h]$

$$\lim_{n\to\infty} \int_a^x f(t, y_n(t))dt = \int_a^x \lim_{n\to\infty} f(t, y_n(t))dt = \int_a^x f(t, y_\infty(t))dt.$$

This allows us to pass to the limit in the recursion (1.10) that we used to define y_{n+1} to see that for every $x \in [a-h, a+h]$

$$y_\infty(x) = \lim_{n\to\infty} y_{n+1}(x) = b + \lim_{n\to\infty} \int_a^x f(t, y_n(t))d = b + \int_a^x f(t, y_\infty(t))dt. \tag{1.16}$$

Hence y_∞ is indeed a solution of the integral equation and as the integral equation is equivalent to the IVP we have proven the existence part of Picard's theorem.

$\square$

Remark: Sometimes we it is useful to consider the above problem on a rectangle which is not symmetric but e.g. of the form $R = [a - h_1, a + h_2] \times [b - k_1, b + k_2]$ for some $h_{1,2} > 0$ and $k_{1,2} > 0$.

You can easily check that the above proof still applies provided the condition $Mh \leq k$ is replaced with the condition that $M \max(h_1, h_2) \leq \min(k_1, k_2)$.

Similarly, the proof of uniqueness we carry out later also applies in these settings so we get the analogue of Picard's theorem also for such rectangles.

Picard's theorem as stated above is a *local* result in that it guarantees existence of a solution on an interval $[a - h, a + h]$ with $Mh \leq k$ around the point $x = a$ where the data is given. We note that this interval that we obtain from Picard's theorem will in general not be the maximal interval on which the solution is defined. Instead the condition $Mh \leq k$ is chosen so that we are guaranteed a priori (that is before even trying to solve the problem/knowing anything about existence and uniqueness of solutions) that any solution of our problem will be so that its graph is in the rectangle R. To be more precise we have

Lemma 1.2. *Let $f : R \to \mathbb{R}$ be so that **P(i)** holds and let $y(x)$ be any solution of the IVP* (1.1) *which is defined on an interval $[a - h_1, a + h_2]$ for some $0 < h_{1,2} \leq h$. Then*

$$|y(x) - b| \leq k \text{ for all } x \in [a - h_1, a + h_2].$$

Of course we already knew that the solution we constructed above via iteration has this property. The point of this lemma is that it applies also to any other potential solution and that it does not require the Lipschitz condition. We can hence use this in two ways:

- In situations where we do not have the Lipschitz condition and hence might have to deal with multiple solutions of the same problem, we get that all of these solutions will satisfy the above estimate
- In the setting of Picard's theorem we will use this as an important part of the uniqueness proof: It tells us that there cannot be any solution whose graph leaves the rectangle and hence to show uniqueness it will be enough to prove that any solution with graph in the rectangle must be the one that we obtain by the above iteration.

Proof of Lemma 1.2. We argue by contradiction, so suppose the claim is wrong, i.e. that there exists a solution $y : [a - h_1, a + h_2] \to \mathbb{R}$ of the IVP (1.1) so that there is some $x_1 \in [a - h_1, a + h_2]$ with $|y(x_1) - b| > k$. By symmetry we can assume without loss of generality that $x_1 > a$.

As the graph 'starts out' in the rectangle $R = [a - h, a + h] \times [b - k, b + k]$ but contains a point $(x_1, y(x_1))$ with $x_1 \in [a - h, a + h]$ that is outside of the rectangle and as f is continuous, there must be a first point $(x_0, y(x_0))$, where the graph intersects the (upper or lower) boundary of the rectangle. That is there is some $x_0 \in (a, x_1)$ so that

$$|y(x_0) - b| = k \text{ while } |y(x) - b| < k \text{ for all } x \in [a, x_0).$$

Since this means that for $t \in [a, x_0]$ the points $(t, y(t))$ are all in the rectangle where $|f|$ is bounded by M and since y satisfies the IVP and hence the integral equation we hence get that

$$\begin{aligned} |y(x_0) - b| = \left| \int_a^{x_0} f(t, y(t)) ds \right| \leq \left| \int_a^{x_0} |f(t, y(t))| ds \right| &\leq M|x_0 - a| \\ < M(x_1 - a) \leq Mh_1 \leq Mh \leq k \end{aligned} \tag{1.17}$$

a contradiction. □

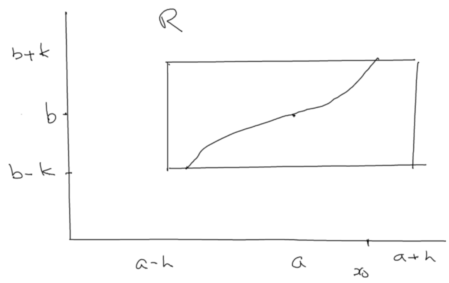


Figure 1.3: x_0 is the first x where the graph meets the boundary of the rectangle

1.5 Gronwall's inequality

We will prove the uniqueness part of Picard's theorem using Gronwall's inequality, which is one of the most helpful tools to control the growth of a quantity, in our case the distance between two solutions.

There are several different versions of Gronwall's inequality, both for integral inequalities and differential inequalities, and for us the following simple version with be sufficient:

Theorem 1.3. (Gronwall's inequality) : *Suppose $A \geq 0$ and $B \geq 0$ are constants and v is a non-negative continuous function satisfying*

$$v(x) \leq B + A \left| \int_a^x v(s)ds \right| \tag{1.18}$$

for all x in an interval $[a - h_1, a + h_2]$, $h_{1,2} \geq 0$. Then

$$v(x) \leq Be^{A|x-a|} \text{ for all } x \in [a - h_1, a + h_2].$$

Note that the modulus in (1.18) is needed to take care of the case $x \leq a$.

Proof: For $x \geq a$ let $V(x) = \int_a^x v(s)ds$, so that $V'(x) = v(x)$. As $x \geq a$ and $v \geq 0$ also $V(x) \geq 0$ and we have

$$V'(x) \leq B + AV(x).$$

Multiply through by the integrating factor e^{-Ax} so

$$\begin{aligned}
(V'(x) - AV(x))e^{-Ax} &\leq Be^{-Ax} \text{ that is} \\
\frac{d}{dx}(V(x)e^{-Ax}) &\leq Be^{-Ax}, \text{ so, integrating and noting that } V(a) = 0 \\
V(x)e^{-Ax} &\leq \int_a^x Be^{-As}ds = \frac{B}{A}(e^{-Aa} - e^{-Ax}), \text{ so} \\
V(x) &\leq \frac{B}{A}(e^{A(x-a)} - 1).
\end{aligned}$$

Finally, using (1.18)

$$v(x) \le B + A\int_a^x v(s)ds = B + AV(x) \le B + A\frac{B}{A}(e^{A(x-a)} - 1) = Be^{A(x-a)},$$

as required. Similarly if $x \le a$. □

Remark: Gronwall's inequality says that for $x \ge a$ the function v, which satisfies an integral inequality, is bounded above by the solution of the integral equation one obtains when there is equality in (1.18). For, if we we consider $x \ge a$ and differentiate

$$v(x) = B + A\int_a^x v(s)ds,$$

we get

$$v'(x) = Av(x), \quad v(a) = B,$$

which has solution $v(x) = Be^{A(x-a)}$.

1.6 Uniqueness and continuous dependence on the initial data

We want to use Gronwall's inequality to prove the uniqueness part of Picard's theorem. To this end we have to show that if $y(x)$ and $\tilde{y}(x)$ are two solutions of the ODE to the same initial data then these functions must coincide. Actually with the same argument we obtain a more general and very useful fact, namely that solutions whose initial values are close must have small distance from each other (and you can see uniqueness as a special case of this in that it says that solutions whose initial data is the same must have distance zero from each other, i.e. must be the same). For this the main ingredient is the Lipschitz condition and we will obtain these estimates for all solutions whose graph is in the corresponding rectangle. As we have already shown that (Pi) implies that any solution of the IVP has graph in the rectangle this will be enough to get uniqueness.

So suppose that f is a continuous function which satisfies a Lipschitz condition on a rectangle $R = [c, d] \times [b - k, b + k]$.

Let $a \in [c, d]$ and let $y(x)$ and $\tilde{y}(x)$ be solution of the same ODE $y'(x) = f(x, y(x))$ to (possibly different) initial values $y(a) = b$ and $\tilde{y}(a) = \tilde{b}$ whose graph is in R.

We want to understand how far from each other these solutions are so consider the function

$$v(x) = |\tilde{y}(x) - y(x)|$$

which is of course non-negative. As before we can see $y(x)$ and $\tilde{y}(x)$ as solutions of the corresponding integral equations

$$y(x) = b + \int_a^x f(t, y(t))dt \quad \text{resp.} \quad \tilde{y}(x) = \tilde{b} + \int_a^x f(t, \tilde{y}(t))dt.$$

We also note that the Lipschitz condition gives

$$|f(t, y(t)) - f(t, \tilde{y}(t))| \le L|y(t) - \tilde{y}(t)| = Lv(t).$$

Hence we get

$$\begin{aligned} v(x) &= \left| b - \tilde{b} + \int_a^x f(t, y(t)) - f(t, \tilde{y}(t)) dt \right| \\ &\leq \left| b - \tilde{b} \right| + \left| \int_a^x |f(t, y(t)) - f(t, \tilde{y}(t))| dt \right| \\ &\leq \left| b - \tilde{b} \right| + L \left| \int_a^x v(t) dt \right| \end{aligned} \tag{1.19}$$

for all $x \in [c, d]$. Gronwall's inequality (with $B = |b - \tilde{b}|$ and $A = L$) thus gives

$$|y(x) - \tilde{y}(x)| = v(x) \leq |b - \tilde{b}| e^{L|x-a|} \text{ for all } x \in [c, d]. \tag{1.20}$$

In the special case that $y(x)$ and $\tilde{y}(x)$ satisfy the same IVP for the same initial value $b = \tilde{b}$ they must hence agree on the whole interval so we get uniqueness of solutions.

In addition to proving the uniqueness part of Picard's theorem, this argument also shows that if b and $\tilde{b}$ are very close, then also the corresponding solutions are close.

To formulate this "continuous dependence on the data", we recall from the course A2 metric spaces that a good way of measuring the distance between two functions is the supremum norm

$$\|y - \tilde{y}\|_{\sup} := \sup_{x \in [c,d]} |y(x) - \tilde{y}(x)|.$$

Using the usual $\varepsilon - \delta$ characterisation of continuity we can then say that the solution depends continuously on the data since given any $\varepsilon > 0$ we can choose $\delta = \varepsilon e^{-Lh} > 0$ for $h := \max(a - c, d - a)$ and get that if $y(x)$ and $\tilde{y}(x)$ solve the same ODE with initial values b and $\tilde{b}$ which satisfy $|b - \tilde{b}| \leq \delta$ then

$$|y(x) - \tilde{y}(x)| \leq |b - \tilde{b}| e^{L|x-a|} \leq \delta e^{Lh} = \varepsilon$$

for any $x \in [c, d]$. Thus $|b - \tilde{b}| \leq \delta$ ensures that the supremum-distance $\|y - \tilde{y}\|_{sup}$ between the two functions is $\|y - \tilde{y}\|_{sup} \leq \varepsilon$.

1.7 Extension of solutions and characterisation of maximal existence interval

In most situations we are considering ODEs

$$y'(x) = f(x, y(x))$$

for functions f which are not only defined on a rectangle $[a - h, a + h] \times [b - k, b + k]$ but on a much larger set, often for all $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

This however does NOT mean that we can expect that the solution of our IVP itself also exists for all x, as we have already seen in the warning example of $y' = y^2, y(0) = 1$ where the solution $y(x) = \frac{1}{1-x}$ blows up as $x \nearrow 1$. At the same time, there are also simple examples of ODEs, such as $y'(x) = Cy(x)$ for which solutions exist for all x. While in these particular examples we can determine the maximal time until which a solution exists by determining an explicit solution, this is not the case for more general problems, so we need another way to determine whether or not a solution will exist for all x.

In the following we will consider this problem for IVPs for which the function f (of two variables) is defined and continuous on all of $\mathbb{R}^2$ and *locally Lipschitz with respect to* y, i.e. so that the

Lipschitz condition holds true for every compact rectangle $R \subset \mathbb{R}^2$ (for an L that is allowed to depend on R!). That is, we ask that

$$\begin{aligned} &\text{for all } c_1 < c_2 \text{ and } d_1 < d_2 \text{ there exists } L \text{ so that} \\ &\quad |f(x,y) - f(x,\tilde{y})| \leq L|y - \tilde{y}| \text{ for all } x \in [c_1, c_2] \text{ and } y, \tilde{y} \in [d_1, d_2] \end{aligned} \tag{1.21}$$

We note that the argument using the MVT seen above, and the fact that continuous functions on compact sets are bounded, ensures that this condition is in particular satisfied for all continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$ for which the partial derivative $\partial_y f$ exists and is continuous on all of $\mathbb{R}^2$. In contrast, the much more restrictive *global Lipschitz condition* (1.24) that we briefly discuss in below (and that guarantees that solutions exist for all x) only holds for very few functions $f : \mathbb{R}^2 \to \mathbb{R}$.

In this section we will see that for continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$ which satisfy the local Lipschitz condition on $\mathbb{R}^2$ solutions will exist (forwards and backwards in "time") for as long as there is no blow-up. In the following section we will then discuss how comparison with explicitly solvable ODEs can often be used to decide whether (and roughly when) such a finite time blow up actually happens and more generally to obtain "a priori estimates" on the behaviour of solutions for ODEs which are not explicitly solvable.

We first note that if $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and so that the local Lipschitz condition holds, then Picard's theorem guarantees that for every $(a, b) \in \mathbb{R}^2$ there exists a unique solution of the IVP

$$y'(x) = f(x, y(x)), \quad y(a) = b$$

on at least an interval $[a - h_0, a + h_0]$. Indeed, as the Lipschitz condition is satisfied on every bounded rectangle and as f is continuous on all of $\mathbb{R}^2$, so of course also on every rectangle, it suffices to find $h_0, k_0 > 0$ so that the condition $Mh_0 \leq k_0$ holds (which is always possible as we can e.g. fix any $k_0 > 0$ and then take $h_0 > 0$ small enough so that this is satisfied).

As already observed, we cannot expect that this interval $[a-h_0, a+h_0]$ is the "best", as in maximal possible, set on which the solution is (uniquely) defined. Indeed for functions f as considered above, it will actually never be the maximal such interval, since we can apply Picard's theorem on a suitable rectangle $[a_1 - h_1, a_1 + h_1] \times [b_1 - k_1, b_1 + k_1]$ around $(a_1, b_1) := (a + h_0, y(a + h_0))$ to obtain a solution y_1 of the ODE with this new initial data and then combine this solution with the original y to extend y a little bit beyond a_1. We note that the uniqueness aspect of Picard ensures that these two solutions agree on the interval where they are both defined, so can be glued together to give a (unique) solution on a larger interval.

It might be tempting to think that since we can continue to iterate this argument infinitely often this will eventually give us a solution for all x. This is however wrong since the values of h_i we can get depend on the new starting point (a_i, b_i) and so might get smaller and smaller, possibly getting small fast enough that the infinite sum of such h_i's obtained by Picard only gives us a finite number $h = \sum h_i$ and hence only gives us a solution on a finite interval.

Indeed, for equations such as $y' = y^2$ where we know that solutions blow up, we already know that this must happen (here you can e.g. check that if we always choose $k_i = 1$ then the condition of $Mh \leq k$ gives us values of $h_i = \frac{1}{(b_i+1)^2}$ which results in very small h_i as b_i becomes large.)

To discuss for how long a solution exists we hence need a different strategy. Given a function $f : \mathbb{R}^2 \to \mathbb{R}$ as above we define

$$T_+ := \sup\{t > a : \text{there exists a solution of IVP on } [a, t)\}$$

and

$$T_- := \inf\{t < a : \text{there exists a solution of IVP on } (t, a]\}$$

with the convention that $T_+ := +\infty$ if for every $t > a$ there is a solution on $[a, t)$ and $T_- = -\infty$ if there for every $t < a$ there is a solution on $(t, a]$.

We note that if y and $\tilde{y}$ are two such solutions that are defined on intervals $I, \tilde{I}$ then these solutions satisfy the same IVP on the interval $I \cap \tilde{I}$ where they are both defined. As f satisfies a Lipschitz condition on every bounded rectangle we can thus apply the uniqueness proof from section 1.6 to see that $y(x)$ and $\tilde{y}(x)$ must agree for all x where they are both defined.

We can hence combine all of these solutions together to obtain a (unique) solution y of the IVP that is defined on the *maximal existence interval* (T_-, T_+). We now want to show that the only reason that this interval might not be all of $\mathbb{R}$ is a potential blow-up.

Theorem 1.4. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous and locally Lipschitz with respect to y. Let $a, b \in \mathbb{R}$ and let (T_-, T_+) be the maximal existence interval of the solution $y(x)$ of the IVP*

$$y'(x) = f(x, y(x)), \quad y(a) = b$$

If $T_+ < \infty$ then the solution must blow up as we approach T_+, i.e. we must have $|y(x)| \to \infty$ as $t \nearrow T_+$. If $T_- > -\infty$ then we must have $|y(x)| \to \infty$ as $t \searrow T_-$.

Thinking of x as a time-parameter, we can hence distinguish between the following cases:

- we have global existence, i.e. the solution of the IVP exists for all $x \in \mathbb{R}$
- the solution exists for all times in the future, but blows up at some finite time $T_- < a$ in the past
- the solution exists for all times in the past, but blows up at some finite time $T_+ > a$ in the future
- the solution blows up both at some finite time in the past and at some finite time in the future.

We note that solutions which exists for all times (into future or past) can of course also tend to infinity as $x \to \pm\infty$, and that the above result gives no information on the behaviour of y as $x \to \pm\infty$. Indeed, unlike the behaviour near finite maximal existence times, solutions which are defined for all times (forwards and/or backwards in time) can show a variety of interesting asymptotic behaviour (including oscillations, blow up, convergence....), and we will see more on this also in the context of autonomous systems of ODEs in the following chapter.

Proof of Theorem 1.4. It suffices to prove the claim in the case where $T_+ < \infty$ as we can reverse the time direction by setting $\tilde{y}(x) = y(-x)$ and using that $\tilde{y}$ is a solution of the ODE $\tilde{y}' = \tilde{f}(x, \tilde{y})$ for $\tilde{f}(x, y) = -f(-x, y)$ to initial condition $\tilde{y}(-a) = b$.

So suppose that $T_+ < \infty$. We want to show that

$$|y(x)| \to \infty \text{ as } x \nearrow T_+,$$

i.e. that for every $K > 0$ there exists a $\delta > 0$ so that $|y(x)| > K$ for all $x \in (T_+ - \delta, T_+)$. We argue by contradiction so suppose that this was not true, i.e. that there is some $K > 0$ so that for every $\delta > 0$ there exists an $x_\delta \in (T_+ - \delta, T_+)$ with $|y(x_\delta)| \leq K$.

We now want to argue that there is a number $h > 0$ so that we can apply Picard's theorem with the *same* $h > 0$ *for all* initial data $(x_\delta, y(x_\delta))$ for which $\delta > 0$ is quite small, say $\delta < \frac{1}{2}$.

For this we fix some compact rectangle R_0 which contains the set $[-K, K] \times \{T_+\}$ in its interior, such as $R_0 = [T_+ - 1, T_+ + 1] \times [-K - 1, K + 1]$. As f is continuous and R_0 is compact we know that f is bounded on R_0, say $|f| \leq M_0$ on R_0. This same bound is of course also valid for every rectangle R which is contained in R_0.

This means that if we set $h := \min(\frac{1}{M_0}, \frac{1}{2})$ and $k := 1$ then $M_0 h \leq k$ holds and so the first assumption of Picard's theorem holds on $R = [\tilde{a} - h, \tilde{a} + h] \times [\tilde{b} - 1, \tilde{b} + 1]$ for every choice of $\tilde{a} \in [T_+ - \frac{1}{2}, T_+ + \frac{1}{2}]$ and $\tilde{b} \in [-K, K]$. Since f is locally Lipschitz wrt y on all of $\mathbb{R}^2$, we of course also get that the Lipschitz condition holds on every such rectangle, so can apply Picard's theorem with the same h for all initial conditions $y(\tilde{a}) = \tilde{b}$ for such $\tilde{a}$ and $\tilde{b}$.

Choosing a $\delta < h$, letting $x_\delta \in (T_+ - \delta, T_+)$ be so that $|y(x_\delta)| \leq K$ we can thus use $\tilde{a} = x_\delta$ and $\tilde{b} = y(x_\delta)$ as initial data to get a solution $\tilde{y}$ of the ODE which satisfies $\tilde{y}(x_\delta) = y(x_\delta)$ and which is defined on $[x_\delta - h, x_\delta + h]$. As $x_\delta + h > T_+ - \delta + h > T_+$ this allows us to extend the original solution y beyond the maximal existence time T_+, contradiction.

□

In order to understand whether a solution exists for all times (in forward and/or backwards) direction, we hence need to understand whether a solution can blow up. For this "a priori estimates", which provide upper or lower bounds on the unknown solution of our IVP which we can obtain without (or before) solving the equation, are a key tool. If we e.g. know that the solution $y(x)$ we are interested in is bounded from above by a function $z_+(x)$ which exists for all $x > a$ (and in particular does not tend to $+\infty$ in finite time), then this excludes the possibility that $y(x)$ tends to $+\infty$ in finite time. If we can obtain also a lower bound on y which remains valid for all $x \geq a$ this then excludes the possiblity of a finite time blow-up (forwards in time), which thanks to the above theorem ensures that the solution y exists for all times $x \geq a$.

Conversely, if we know that there is a function $z_-(x)$ which shoots off to $+\infty$ in finite time, say as $t \nearrow T_1$, and if we additionally know that $y(x) \geq z_-(x)$ for as long as both of these functions exist, then this forces $y(x)$ to tend to $+\infty$ in finite time, and tells us that a blow up must happen no later than at T_1 (though can of course happen before that as $y(x)$ can blow up before z_-).

The simplest example where we can argue this way is for ODEs for which the function f is globally bounded. Consider e.g. the IVP

$$y'(x) = \sin(xy^2(x)), \quad y(0) = 1.$$

In this case we can e.g. argue that since $-1 \leq y'(x) \leq 1$ we must have that $1 - x \leq y(x) \leq 1 + x$ for as long as the solution $y(x)$ exists. This excludes the possibility that $y(x)$ blows up in finite time (in either time direction) and hence ensures that this IVP has a global solution.

We now want to show that such arguments are applicable more generally and first consider the case where the function f that describes the right hand side of our ODE grows no more than linearly in the y direction. In such situations we can obtain a priori estimates on the growth of solutions from Gronwall's lemma:

Application: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous and locally Lipschitz wrt y and suppose that f is so that for some $c_1 < c_2$ there exist numbers $C_1, C_2 > 0$ so that

$$|f(x)| \leq C_1 + C_2|y| \text{ for all } x \in [c_1, c_2] \text{ and all } y \in \mathbb{R}. \tag{1.22}$$

Consider now the solution $y(x)$ of the usual IVP $y'(x) = f(x, y(x))$, $y(a) = b$ for some $a \in [c_1, c_2]$ and $b \in \mathbb{R}$. As $y(x)$ satisfies the integral equation $y(x) = y(a) + \int_a^x f(t, y(t))dt$ we see that the non-negative function $v(x) = |y(x)|$ is so that

$$\begin{aligned} v(x) = |y(x)| &\leq |y(a)| + |\int_a^x |f(t, y(t))|dt| \leq |b| + |\int_a^x C_1 + C_2 v(t)dt| \\ &\leq (|b| + C_1 h) + C_2 |\int_a^x v(t)dt| \end{aligned} \tag{1.23}$$

for all $x \in (T_-, T_+) \cap [c_1, c_2]$ and $h := \max(a - c_1, c_2 - a)$. Hence Gronwall's inequality ensures that

$$|y(x)| \leq (|b| + C_1 h)e^{C_2|x-a|} \text{ on } (T_-, T_+) \cap [c_1, c_2].$$

As the right-hand side does not blow-up, we thus cannot have a blow-up on the interval $[c_1, c_2]$ so, by the above theorem must have that the maximal existence times forwards and backwards in time are so that $T_- < c_1$ and $T_+ > c_2$.

If such a growth condition (1.22) is not only valid for some specific $c_{1,2}$'s, but if for all $c_1 < c_2$ there exist $C_{1,2}$ (allowed to depend on $c_{1,2}$) so that (1.22) holds, then we deduce that the solution cannot blow-up on any finite interval and hence must exist globally.

Example:
Consider the solution of the ODE $y'(x) = xy(x)\sin(xy(x)^2)$ to initial data $y(0) = 1$. As the corresponding function $f(x, y) = xy\sin(xy^2)$ is continuously differentiable on all of $\mathbb{R}^2$ it is continuous and satisfies the local Lipschitz condition on $\mathbb{R}^2$. Given any $h > 0$ we can furthermore bound $|f(x, y)| \leq h|y|$ on $[-h, h] \times \mathbb{R}$, so are dealing with a right hand side whose growth is no faster than linear. We can hence deduce that the solution exists globally and furthermore satisfies $|y(x)| \leq e^{h|x|}$ for all $|x| \leq h$, so (applying this bound for $h = |x|$ satisfies a priori bounds of

$$y(x) \leq e^{x^2} \text{ for all } x \in \mathbb{R}.$$

We will see in the next section that instead of arguing via Gronwall we could obtain such a priori bounds instead also by comparing y to solutions of simpler ODEs, here $y'(x) = \pm xy(x)$ that we can explicitly solve.

Application: A special case of functions with no more than linear growth are continuous functions which satisfy a *global (in y) Lipschitz condition*, i.e. which are so that for some $c_1 < a < c_2$ there exist a number L so that

$$|f(x, y) - f(x, \tilde{y})| \leq L|y - \tilde{y}| \text{ for all } x \in [c_1, c_2] \text{ and all } y, \tilde{y} \in \mathbb{R} \tag{1.24}$$

(rather than just for $y, \tilde{y}$ in a bounded interval). In this case the linear growth condition holds with $C_1 := \max_{[c_1,c_2]} |f(x, 0)|$, which is finite as f is continuous, and $C_2 = L$, so we again obtain that our solution needs to exist on the whole interval $[c_1, c_2]$ on which we have such a growth bound, respectively on all of $\mathbb{R}$ if for all $h > 0$ there exists an L (allowed to depend on h) so that (1.24) holds.

Remark. *For functions which satisfy such a global Lipschitz condition we could alternatively get the existence of a solution on the full interval* $[c_1, c_2]$ *directly by applying the proof of Picard via successive approximation carried out in section 1.4 on the full interval* $[c_1, c_2]$*. Indeed in this case the proof from section 1.4 can be simplified as there is no need to constrain the range of values of* y *to an interval* $[b-k, b+k]$*, which in practice means that the assumption* $Mh \le k$ *is no longer relevant and Claim 1 is no longer needed.*

1.8 Comparison Principle and a priori estimates

While we obtained the bounds $1 - x \le y(x) \le 1 + x$ for the solution of the IVP $y'(x) = \sin(xy^2(x)), \quad y(0) = 1$ above just by direct integration, we can think of these estimates also as a comparison of the solution of the original ODE that we cannot solve explicitly, with the solutions of the much simpler ODEs $y'(x) = \pm 1$.

More generally we can use solutions of ODEs $z'_\pm = g_\pm(x, z(x))$ which we can explicitly solve to get information on the solution of our original problem if the functions $g_\pm$ are chosen so that

$$g_-(x, y) \le f(x, y) \le g_+(x, y) \tag{1.25}$$

for all (x, y) which are relevant for our analysis.

We note that we cannot expect that $y'(x) = f(x, y(x))$ is bounded from above by the derivative $z'_+(x) = g_+(x, z_+(x))$ of such a solution z_+ at the corresponding point x, since we evaluate f and g_+ at different y-values and hence cannot expect that $f(x, y(x)) \le g(x, z_+(x))$.

We will however see that if we start with initial values $y(a) \le z_+(a)$ then the functions $y(x)$ and $z_+(x)$ remain in this order into the future, i.e. we have that $y(x) \le z_+(x)$ for all $x \ge a$.

The key point that makes this work is that while $y(x)$ does not solve the same ODE as $z_+(x)$, it satisfies the differential inequality

$$y'(x) \le g_+(x, y(x))$$

if f and g_+ are as in (1.25). This suffices to ensure that for $x \ge a$ the unknown solution $y(x)$ always remains bounded from above by $z_+(x)$ since we can prove the following more general statement:

Theorem 1.5 (Comparison Principle)**.** *Let* $g : \mathbb{R}^2 \to \mathbb{R}$ *be continuously differentiable and let* $u(x)$ *and* $v(x)$ *be differentiable functions which satisfy*

$$u'(x) \le g(x, u(x)) \text{ and } v'(x) \ge g(x, v(x)) \tag{1.26}$$

on some interval I*.*

If $u(x_0) \le v(x_0)$ *for some* $x_0 \in I$ *then we obtain that*

$$u(x) \le v(x) \text{ for all } x \in I \text{ with } x \ge x_0$$

while if $u(x_0) \ge v(x_0)$ *then we we obtain that*

$$u(x) \ge v(x) \text{ for all } x \in I \text{ with } x \le x_0$$

In most applications we will use this theorem for functions u and v which have the same value at x_0, in which case both of the estimates apply.

Proof of the comparison principle. Since g is continuously differentiable we can use the MVT to check that the function $h : \mathbb{R}^3 \to \mathbb{R}$ of 3 variables which is given by the difference quotient $h(x, y_1, y_2) := \frac{g(x,y_1)-g(x,y_2)}{y_1-y_2}$ if $y_1 \neq y_2$ respectively the partial derivative $h(x, y_1, y_2) := \partial_y g(x, y_1)$ if $y_1 = y_2$ is continuous. As both the functions $u(x)$ and $v(x)$ are continuous on the interval I this ensures that the composition $a(x) := h(x, u(x), v(x))$ is also continuous.

Crucially, this function is chosen so that we can write

$$g(x, v(x)) - g(x, u(x)) = a(x)(v(x) - u(x)).$$

From the assumptions of the theorem we thus get that $w(x) := v(x) - u(x)$ satisfies

$$w'(x) = v'(x) - u'(x) \geq g(x, v(x)) - g(x, u(x)) = a(x)w(x).$$

Setting $A(x) = \int_{x_0}^{x} a(t)dt$ to get a function with $A'(x) = a(x)$ we hence get that

$$(e^{-A(x)}w(x))' = e^{-A(x)}(w'(x) - A'(x)w(x)) \geq 0$$

and hence that $e^{-A(x)}w(x)$ is non-decreasing.

If $u(x_0) \leq v(x_0)$ and hence $e^{-A(x_0)}w(x_0) = w(x_0) \geq 0$ we thus get that for $x \geq x_0$ also $e^{-A(x)}w(x) \geq 0$, and hence $w(x) \geq 0$, i.e. $u(x) \leq v(x)$. On the other hand, if $u(x_0) \geq v(x_0)$ we can use that for $x \leq x_0$ we have $e^{-A(x)}w(x) \leq w(x_0) \leq 0$ so $u(x) \geq v(x)$. □

The above Theorem allows us to compare solutions of complicated ODEs which we cannot explicitly solve with solutions of simpler ODEs.

Example: Consider the solution of the IVP

$$y'(x) = y^2(x) + x, \qquad y(0) = 1.$$

Then for $x \geq 0$ we have that $y'(x) \geq y^2(x)$ so if we apply our result for $g_1(x, y) = y^2$ we can compare $y(x)$ with the solution $z_1(x) = \frac{1}{1-x}$ of the IVP $z_1'(x) = g_1(x, z_1(x))$, $z_1(0) = 1 = y(0)$ to deduce that

$$y(x) \geq \frac{1}{1-x} \text{ for all } x \geq 0 \text{ in the maximal existence interval } (T_-, T_+) \text{ of } y(x).$$

Hence $y(x)$ must blow up no later than at $x = 1$, i.e. we must have that $T_+ \leq 1$.

To get a lower bound on the maximal existence time, and an upper bound on $y(x)$ for some range of $x \geq 0$, we can then use that $y'(x) \leq y^2(x) + 1$ for $x \leq 1$. We hence instead compare with the solution z_2 of the ODE $z_2'(x) = z_2(x)^2 + 1$ with the same initial condition $z_2(0) = 1$. Separation of variables, using that $\arctan(x)' = \frac{1}{1+x^2}$, yields $z_2(x) = \tan(x + \frac{\pi}{4})$ so we deduce that $y(x) \leq \tan(x + \frac{\pi}{4})$ for as long as the right hand side does not blow up, i.e. obtain this estimate for $0 \leq x_0 < \pi/4$.

Thus the maximal existence time T_+ must be so that $\frac{\pi}{4} \leq T_+ \leq 1$ and we have furthermore a priori upper and lower bounds as described above.

Similarly we can obtain upper and lower bounds on $y(x)$ which remain valid for all $x \leq 0$, which ensures that the solution indeed exists for all $x \in (-\infty, T_+)$.

Remark. *While we have focused on the analysis of ODEs for functions f which are defined, continuous and locally Lipschitz w.r.t. y on all of $\mathbb{R}^2$, we could also consider the more general* *case where f has these properties on a set of the form $\hat{R} = (c_-, c_+) \times (d_-, d_+)$ where some of the $c_\pm, d_\pm$ might be $\pm\infty$ while others are finite numbers. Up to small modifications, all the statements and arguments also apply to these more general settings. In particular the comparison arguments still apply and we can modify the proof of Theorem 1.4 to see that the maximal existence interval (T_-, T_+) is given by the whole interval (c_-, c_+) of allowed x values, unless the solution $y(x)$ blows up or approaches the boundary of the allowed y range before reaching $c_\pm$.*

Example:
Consider the IVP

$$y'(x) = \frac{1}{x(y(x)-1)(x^2+y(x)^2)} \text{ with } y(1) = 2.$$

Then the corresponding function $f(x,y) = \frac{1}{x(y-1)(x^2+y^2)}$ is continuous on $\hat{R} = (0,\infty)\times(1,\infty)$ and satisfies a local Lipschitz condition on this set, i.e. is so that a Lipschitz condition holds on any compact rectangle that is contained in $\hat{R}$. So for the maximal existence interval $(T_-, T_+) \subset (0,\infty)$ we get that

- if $T_- > 0$ then we must either have that $y(x) \to \infty$ or that $y(x) \to 1$ as $x \searrow T_-$
- if $T_+ < \infty$ then we must either have that $y(x) \to \infty$ or that $y(x) \to 1$ as $x \nearrow T_+$

To determine which of the 4 possible types of maximal existence intervals, here $(0,\infty)$, (x_1,∞), (x_1,x_2) or $(0,x_2)$ for $x_1 > 0$ and $x_2 < \infty$, actually occurs we can then use comparison with ODEs which are separable and which we can explicitly solve:

As the right hand side is positive for $x > 0$ and $y > 1$ (which are the only allowed value for our problem) we know that $y(x)$ is increasing.

Analysis for $x \geq 1$: As $y(x) \geq y(1) = 2$ for all $x \geq 1$ we can immediately exclude the possibility that y approaches the value 1 where the ODE becomes undefined.

This bound of $y(x) \geq 2$ is also useful to get an upper bound on $y(x)$ as it means that for $x \geq 1$ we can bound $y(x) - 1 \geq \frac{1}{2}y(x)$ and $y^2(x) + x^2 \geq y^2(x)$, so get that $y'(x) \leq \frac{2}{xy(x)^3}$. We can thus bound $y(x) \leq z(x)$ for the solution $z(x)$ of ODE $z'(x) \leq \frac{2}{xz(x)^3}$ with $z(1) = 2$ which we can determine using separation of variables.

Analysis for $x \leq 1$: As $y(x) \leq y(0) = 2$ it will certainly not go to $+\infty$ and as the allowed x and y range are $x > 0$ and $y > 1$ we can bound $1 \leq x^2 + y(x)^2 \leq 1 + 4 = 5$. Thus $y(x)$ is sandwiched $z_1(x) \leq y(x) \leq z_2(x)$ between the solutions of $z_1'(x) = \frac{1}{x(z_1(x)-1)}$ and $z_2(x) = \frac{1}{5x(y-1)}$ with $z_{1,2}(1) = 2$. These satisfy

$$(z_1(x) - 1)^2 = 2\log x + 1 \text{ and } (z_2(x) - 1)^2 = \frac{2}{5}\log x + 1$$

these solutions are defined for $x \geq T_1 = e^{-\frac{1}{2}}$ respectively $x \geq T_2 = e^{-5/2}$ and tend to 1 when x approaches $T_{1,2}$ from above. Thus we know that the solution $y(x)$ of the original problem exists for $x \geq x_1$ for some $x_1 \in [e^{-5/2}, e^{-\frac{1}{2}}]$ which is so that $y(x) \nearrow 1$ as $x \nearrow x_1$.

1.9 Picard's Theorem for systems and higher order ODEs

We now want to look at existence and uniqueness of solutions of systems of ODEs. As well as being of interest in itself, this will be useful in particular for proving the existence of solutions of equations with higher order derivatives. We consider a pair of first order ODEs, for the functions y_1 and y_2.

$$\begin{aligned} y_1'(x) &= f_1(x, y_1(x), y_2(x)) && (1.27)\\ y_2'(x) &= f_2(x, y_1(x), y_2(x)) && (1.28)\\ \text{with initial data} \quad y_1(a) &= b_1, \quad y_2(a) = b_2. && (1.29) \end{aligned}$$

We can introduce vector notation

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \underline{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix};$$

So we can write equations (1.27)–(1.29) as

$$\begin{aligned} \underline{y}'(x) &= \underline{f}(x, \underline{y}(x)), && (1.30)\\ \underline{y}(a) &= \underline{b}, && (1.31) \end{aligned}$$

To adapt our previous arguments to such vector-valued ODEs we need a 'distance' in $\mathbb{R}^2$. In the Metric Spaces course the various norms l^1, l^2 (the Euclidean distance) and l^∞ on $\mathbb{R}^n$ were defined. We could use any of these (or any other norm on $\mathbb{R}^n$), but here it's easiest to work with the l^1 norm, $||\underline{y}||_1 = |y_1| + |y_2|$, as this allows us to obtain estimates on the norm of a vector by adding up estimates for the components.

Instead of considering scalar valued functions $y(x)$ that take values in an interval $[b-k, b+k]$ we now consider vector valued functions $\underline{y}(x)$ that take values in the ball with radius k around the given initial value $\underline{b}$, i.e. look at the set

$$B_k(\underline{b}) = \{\underline{y} \in \mathbb{R}^2 : ||\underline{y} - \underline{b}||_1 \leq k\} = \{(y_1, y_2) \in \mathbb{R}^2 : |y_1 - b_1| + |y_2 - b_2| \leq k\}$$

In place of the rectangle R we hence use the subset

$$S = [a-h, a+h] \times B_k(\underline{b}) \subset \mathbb{R}^3$$

and we replace the assumptions of Picard's theorem with the analogue assumptions on the behaviour of the function $\underline{f} : S \to \mathbb{R}^2$ which is now a function of 3 variables that takes values in $\mathbb{R}^2$.

We can then proceed exactly as in the proof of Picard's theorem carried out above (with absolute values replaced by norms for all vectors) and show

Theorem 1.6. (Picard's existence theorem for systems.) *Let $f : S \to \mathbb{R}^2$ be a function defined on the set $S = [a-h, a+h] \times B_k(\underline{b})$ which satisfies*

H(i) *$\underline{f}(x, \underline{y})$ is continuous on S, $||\underline{f}(x, \underline{y})||_1 \leq M$ and $Mh \leq k$*

H(ii) *$\underline{f}(x, \underline{y})$ is Lipschitz with respect to $\underline{y}$ on S. That is, there exists L such that for $x \in [a-h, a+h]$ and $\underline{y}, \underline{\tilde{y}} \in B_k(\underline{b})$,*

$$||\underline{f}(x, \underline{y}) - \underline{f}(x, \underline{\tilde{y}})||_1 \leq L||\underline{y} - \underline{\tilde{y}}||_1.$$

Then the IVP

$$\underline{y}'(x) = \underline{f}(x, \underline{y}(x))$$

has a unique solution on the interval $[a-h, a+h]$.

As above we stress that these are conditions on the functions $\underline{f} : S \to \mathbb{R}^2$ which is a function in three variables, and NOT on the composition $x \mapsto \underline{f}(x, \underline{y}(x))$ of $\underline{f}$ with a function $\underline{y} : [a-h, a+h] \to \mathbb{R}^2$. We also remark that the very same theorem (and proof) also apply for systems of n ODEs for any $n \in \mathbb{N}$, and that we only focus on the case of $n = 2$ to make the discussion below less messy.

We note that instead of checking the Lipschitz condition for the vector valued function $\underline{f}$ we can also check it for each of the components. Namely if there exist L_1 and L_2 such that for $x \in [a-h, a+h]$ and $\underline{y}, \underline{\tilde{y}} \in B_k(\underline{b})$,

$$\begin{aligned} |f_1(x, y_1, y_2) - f_1(x, \tilde{y}_1, \tilde{y}_2)| &\leq L_1(|y_1 - \tilde{y}_1| + |y_2 - \tilde{y}_2|) \text{ and} \\ |f_2(x, y_1, y_2) - f_2(x, \tilde{y}_1, \tilde{y}_2)| &\leq L_2(|y_1 - \tilde{y}_1| + |y_2 - \tilde{y}_2|). \end{aligned}$$

then we can add up these estimates to see that $\underline{f}$ satisfies the Lipschitz condition with $L = L_1 + L_2$.

We note that to obtain such Lipschitz estimates for functions of several variables it is often useful to "add in a smart 0", and e.g. to write

$$f_1(x, y_1, y_2) - f_1(x, \tilde{y}_1, \tilde{y}_2) = (f_1(x, y_1, y_2) - f_1(x, \tilde{y}_1, y_2)) + (f_1(x, \tilde{y}_1, y_2) - f_1(x, \tilde{y}_1, \tilde{y}_2))$$

and then estimate the two terms separately. Sometimes we can get such estimates just by hand, while at other times it is useful to note that if we only change one variable at a time as done in the above estimate (and if f is differentiable) then we can again apply the mean-value theorem: e.g. to estimate the first term in the above expression we can use that for any fixed x and y_2 the function $t \mapsto f_1(x, t, y_2)$ is a function of one variable as you have seen in prelims, so if f, and hence f_1, is differentiable then the mean-value theorem ensures that there is some ξ between y_1 and $\tilde{y}_1$ so that

$$f_1(x, y_1, y_2) - f_1(x, \tilde{y}_1, y_2) = \partial_{y_1} f_1(x, \xi, y_2)(y_1 - \tilde{y}_1).$$

Remark. *Also for systems the solutions will not only exist on the interval $[a-h, a+h]$ obtained above but on a maximal existence interval (T_-, T_+) which for continuous functions $\underline{f} : \mathbb{R}^3 \to \mathbb{R}^2$ which satisfy the Lipschitz condition on every compact subset S of $\mathbb{R}^3$ will again be given by all of $\mathbb{R}$ unless there is a blow-up.*

Unlike in the scalar case where $|y(x)| \to \infty$ as $x \nearrow T_+$ implies (by the IVT) that we must either have $y(x_n) \to +\infty$ for all sequences $x_n \nearrow T_+$ or $y(x_n) \to +\infty$ for all $x_n \nearrow T_+$, we cannot expect a similar statement for the components of vector valued functions for which $|\underline{y}(x)| \to \infty$ (such functions can e.g. spiral out to infinity like for $\underline{y}(x) = \frac{1}{1-x}(\cos(x), \sin(x))$ resulting in oscillatory behaviour of the components $y_{1,2}(x)$).

In contrast to the scalar case, the comparison principle seen in section 1.8 no longer applies for systems, which makes the analysis of potential blow-ups far more involved.

Picard for Higher Order ODEs

With Picard extended to first-order systems, it is a small step to extend it to a single, higher order ODE. For simplicity, we consider just the IVP for second-order ODEs (which will be considered in more detail in DEs2):

$$y'' = F(x, y(x), y'(x))$$

with initial data

$$y(a) = b \quad y'(a) = c.$$

We can reduce this to a first-order system by setting $y_1(x) = y(x)$ and $y_2(x) = y'(x)$ and note that $y(s)$ solves the above IVP if and only if $\underline{y}(x) = (y_1(x), y_2(x))$ satisfies the first order system of ODEs

$$\underline{y}'(x) = \underline{f}(x, \underline{y}(x))$$

for

$$\underline{f}(x, y_1, y_2) := (y_2, F(x, y_1, y_2))^T$$

The first component of this function $\underline{f}$ is just the function $f_1(x, y_1, y_2) = y_2$ which assigns to each triple $(x, y_1, y_2) \in \mathbb{R}^3$ the third number y_2, so obviously a continuous function which satisfies the Lipschitz condition with $L_1 = 1$.

So $\underline{f}$ is continuous if and only if the second component, which is just given by the original function $F(x, y_1, y_2)$ is continuous. Similarly, if F satisfies a Lipschitz condition (with a constant L_2) then we obtain that $\underline{f}$ also satisfies a Lipschitz condition (with $L = L_2 + 1$) so we can apply Picard's theorem for systems to deduce that for such F a unique solution of this higher order IVP exists at least on a small interval $[a - h, a + h]$ around $x = a$.

Clearly this method can be extended to the IVP for an n-th order linear ODE. In particular, this justifies our belief that an n-th order ODE needs n pieces of (initial) data to fix a unique solution.

2 Plane autonomous systems of ODEs

The definition: a *plane autonomous system of ODEs* is a pair of ODEs of the form;

$$\frac{dx}{dt} = X(x,y) \qquad (2.1)$$
$$\frac{dy}{dt} = Y(x,y)$$

Here "autonomous" means there is no t-dependence in X or Y, and "plane" means there are just two equations, so we can draw pictures in the (x, y) - plane, which will then be called the *phase plane*.

Given initial values $x(0) = a,\ y(0) = b$, expect that there exists a unique solution and this solution which will define *a trajectory* or *phase path* in the phase plane. It is convenient, though not necessary, to think of t as time, and the trajectory as the curve in the plane (including orientation) that is traced out by a moving particle. We put an arrow on the trajectory giving the direction of increasing t. We will denote $\dot{x} = \frac{dx}{dt}$ etc. We will assume throughout that X and Y Lipschitz continuous in x and y (on every bounded subset of $\mathbb{R}^2$) as this will allow us to apply Picard's theorem to obtain important properties of solutions for these plane autonomous system and of the corresponding trajectories.

Important observations

- If $(x(t), y(t))$ is a solution of (2.1) then for any fixed number $t_0 \in \mathbb{R}$ also

  $$\tilde{x}(t) := x(t + t_0), \tilde{y}(t) := y(t + t_0)$$

  solve (2.1) and they trace out the same trajectories.

- Through every point (x_0, y_0) there exists a UNIQUE trajectory. In particular, different trajectories can NEVER intersect, though they might asymptote to the same point (x^*, y^*) as $t \to \infty$ or as $t \to -\infty$ (and any such point must be a critical point, see below)

The first point immediately follows when we insert $(\tilde{x}(t), \tilde{y}(t))$ into the equations as this gives

$$\dot{\tilde{x}}(t) = \dot{x}(t + t_0) = X(x(t + t_0),\ y(t + t_0)) = X(\tilde{x}(t),\ \tilde{y}(t))$$

and

$$\dot{\tilde{y}}(t) = Y(\tilde{x}(t),\ \tilde{y}(t)).$$

Note that this does not work if the system is not autonomous (i.e. if either X or Y also depend on t).

The second point is an important consequence of Picard's theorem (and holds true as we assume X, Y Lipschitz): First of all Picard guarantees the existence of a solution $(x(t), y(t))$ with $x(0) = x_0$ and $y(0) = y_0$ and hence there is a trajectory through the point. If $(\tilde{x}(t), \tilde{y}(t))$ is any other solution that traces out a trajectory through (x_0, y_0) then there is a t_0 so that $(\tilde{x}(t_0), \tilde{y}(t_0)) = (x_0, y_0)$. Looking at $u(t) := \tilde{x}(t - t_0)$, $v(t) := y(t - t_0)$ we get a new solution of (2.1) which has the same initial values as the original $(x(t), y(t))$, namely $(u(0), v(0)) = (x_0, y_0) = (x(0), y(0))$. By the uniqueness part of Picard we thus know that these two solutions $(u(t), v(t))$ and $(x(t), y(t))$ must be the same, so $(\tilde{x}, \tilde{y})$ is nothing else than a time-shift of the original solution so must trace out the same trajectory.

2.1 Critical points and closed trajectories

A *critical point* is a point (x_0, y_0) in the phase plane where $X(x_0, y_0) = Y(x_0, y_0) = 0$. So a critical point is a particular (very special) trajectory corresponding to solutions $(x(t), y(t))$ of (2.1) that are constant in time.

There may be trajectories in the phase plane which are closed i.e. which return to the same point. Provided they don't just correspond to constant solutions and so are simply given by a single point, these correspond to **periodic solutions** of (2.1) as may be seen as follows:

Suppose the trajectory is closed so that for some finite value t_0 of t, $(x(t_0), y(t_0)) = (x(0), y(0))$, while $(x(t), y(t)) \neq (x(0), y(0))$ for $0 < t < t_0$. Define $\bar{x}(t) = x(t + t_0)$, $\bar{y}(t) = y(t + t_0)$. Then as before we see that $(\bar{x}(t), \bar{y}(t))$ is another solution of (2.1) with $\bar{x}(0) = x(t_0) = x(0)$; $\bar{y}(0) = y(t_0) = y(0)$. Now by uniqueness of solution (given Lipschitz again).

$$x(t + t_0) = \bar{x}(t) = x(t)$$

$$y(t + t_0) = \bar{y}(t) = y(t),$$

but this is now true for all t, so a **closed trajectory** corresponds to a **periodic solution** of (2.1) with period t_0. The converse is trivial. □

Note in particular that this means that a trajectory cannot intersect itself, but might close up to a closed curve (without self-intersections).

2.1.1 An example

Consider the harmonic oscillator equation

$$\ddot{x} = -\omega^2 x.$$

Turn this into a plane autonomous system by introducing y as follows:

$$\left.\begin{array}{rrl} & \dot{x} = y & = X(x,y) \\ \text{so} & \dot{y} = -\omega^2 x & = Y(x,y). \end{array}\right\} \tag{2.2}$$

(Clearly this trick often works for second-order ODEs arising from Newton's equations.) The only critical point is $(0, 0)$, but note that

$$\frac{d}{dt}(\omega^2 x^2 + y^2) = 2\omega^2 x\dot{x} + 2y\dot{y} = 0$$

so $\omega^2 x^2 + y^2 =$ constant. (which, from Prelims dynamics, we know to be proportional to the total energy). For a given value of the constant this is the equation of an ellipse, so we can draw all the trajectories in the phase plane as a set of nested (concentric) ellipses:

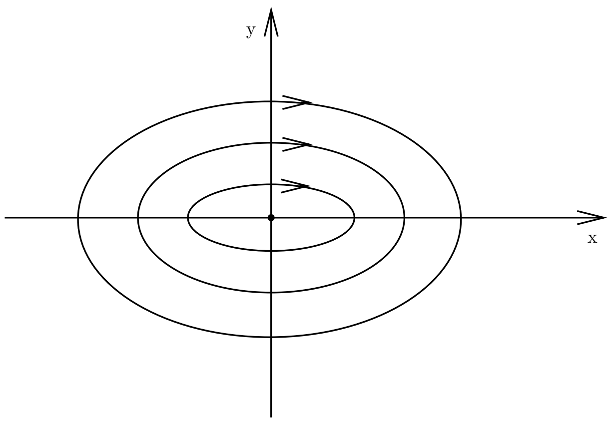


Figure 2.1: The phase diagram for the harmonic oscillator; to put the arrows on the trajectories, notice that $\dot{x} > 0$ if $y > 0$.

The picture in the phase plane is called the *phase diagram* (or *phase portrait*) and from that we see that all trajectories are closed, so all solutions are periodic (as we already know, from Prelims).

2.2 Stability and linearisation

We want to learn how to sketch the trajectories in the phase plane in general and to do this we first consider their *stability*. Intuitively we say a critical point (a, b) is *stable* if near (a, b) the trajectories have all their points close to (a, b) for all t greater than some t_0. We make the formal definition:

Definition A critical point (a, b) is *stable* if given $\epsilon > 0$ there exists $\delta > 0$ and t_0 such that for any solution $(x(t), y(t))$ of (2.1) for which $\sqrt{(x(t_0) - a)^2 + (y(t_0) - b)^2} < \delta$

$$\sqrt{(x(t) - a)^2 + (y(t) - b)^2} < \epsilon, \quad \forall t > t_0.$$

A critical point is *unstable* if it is not stable.

(Here we have used the Euclidean distance. We could use other norms such as l^1 or l^∞)

A common way to analyse the stability of a critical point is to linearise about the point and assume that the stability is the same as for the linearised equation. There are rigorous ways of showing when this is true. We will assume it is valid, pointing out the cases where it is likely fail. Linearising will also enable us to classify the critical points according to what the trajectories look like near the critical point .

So suppose $P = (a, b)$ is a critical point for (2.1), so

$$X(a, b) = 0 = Y(a, b). \tag{2.3}$$

Now $x = a$, $y = b$ is a solution of (2.1). We linearise by setting

$$x = a + \zeta(t); \; y = b + \eta(t)$$

where ζ and η are thought of as small. From (2.1), and Taylor's theorem

$$\dot{x} = \dot{\zeta} = X(a + \zeta, b + \eta) = X(a, b) + \zeta X_x|_p + \eta X_y|_p + \text{ h.o.}$$

$$\dot{y} = \dot{\eta} = Y(a,b) + \zeta Y_x|_p + \eta Y_y|_p + \text{ h.o.}$$

where 'h.o.' means quadratic and higher order terms in ζ and η. Now use (2.3) and neglect higher order terms to find

$$\left.\begin{aligned} \begin{pmatrix} \dot{\zeta} \\ \dot{\eta} \end{pmatrix}^{\cdot} &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \zeta \\ \eta \end{pmatrix} \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} X_x|_p & X_y|_p \\ Y_x|_p & Y_y|_p \end{pmatrix} \end{aligned}\right\} \tag{2.4}$$

Call this (constant) matrix M and set $\underline{Z}(t) = \begin{pmatrix} \zeta \\ \eta \end{pmatrix}$ then (2.4) becomes

$$\dot{\underline{Z}} = M\,\underline{Z}. \tag{2.5}$$

We can solve (2.5) with eigen-vectors and eigen-values as follows: $\underline{Z}_0 e^{\lambda t}$ is a solution, with constant vector $\underline{Z}_0$ and constant scalar λ if

$$\lambda \underline{Z}_0 = M\ \underline{Z}_0,$$

i.e. $\underline{Z}_0$ is an eigen-vector of M with eigen-value λ. We are considering just 2×2-matrices, with eigen-values say λ_1 and λ_2 so the general solution if $\lambda_1 \neq \lambda_2$ is

$$\underline{Z}(t) = c_1 \underline{Z}_1 e^{\lambda_1 t} + c_2 \underline{Z}_2 e^{\lambda_2 t}, \tag{2.6}$$

for constants c_i. Recall λ_1, λ_2 may be real, in which case the c_i and the $\underline{Z}_i$ are real, or a complex conjugate pair, in which case the c_i and the $\underline{Z}_i$ are too.

If $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$ say, we need to take more care. The Cayley-Hamilton Theorem (see Algebra I) implies that $(M - \lambda I)^2 = 0$ since the characteristic polynomial is $c_M(x) = (x - \lambda)^2$, so either $M - \lambda I = 0$ or $M - \lambda I \neq 0$. We have a dichotomy:

(i) if $M - \lambda I = 0$ then $M = \lambda I$ and the solution is

$$\underline{Z}(t) = \underline{C} e^{\lambda t} \tag{2.7}$$

for any constant vector $\underline{C}$.

(ii) if $M - \lambda I \neq 0$ then there exists a constant vector $\underline{Z}_1$ with

$$\underline{Z}_0 := (M - \lambda I)\underline{Z}_1 \neq 0$$

but

$$(M - \lambda I)\underline{Z}_0 = (M - \lambda I)^2 \underline{Z}_1 = 0.$$

(So $\underline{Z}_0$ is the one linearly independent eigenvector of M.) One now checks that the solution of (2.5) is

$$(c_1 \underline{Z}_1 + (c_0 + c_1 t)\underline{Z}_0)e^{\lambda t}. \tag{2.8}$$

Now we can use (2.6) and (2.8) to classify critical points.

2.3 Classification of critical points

We shall assume that neither eigenvalue of the matrix M is zero, which is the requirement that the critical point be *non-degenerate*. A proper discussion of this point would take us outside the course but roughly speaking if a critical point is *degenerate* then we need to keep more terms in the Taylor expansion leading to (2.4), and the problem is much harder.

Case 1. $0 < \lambda_1 < \lambda_2$ (both real of course)

From (2.6), as $t \to -\infty$, $\underline{Z}(t) \to 0$, and $\underline{Z}(t) \sim c_1 \underline{Z}_1 e^{\lambda_1 t}$ unless $c_1 = 0$ in which case $\underline{Z}(t) \sim c_2 \underline{Z}_2 e^{\lambda_2 t}$, while as $t \to +\infty$, $\underline{Z}(t) \sim$ a large multiple of $\underline{Z}_2$, unless $c_2 = 0$ when $\underline{Z}(t) \sim$ a large multiple of $\underline{Z}_1$

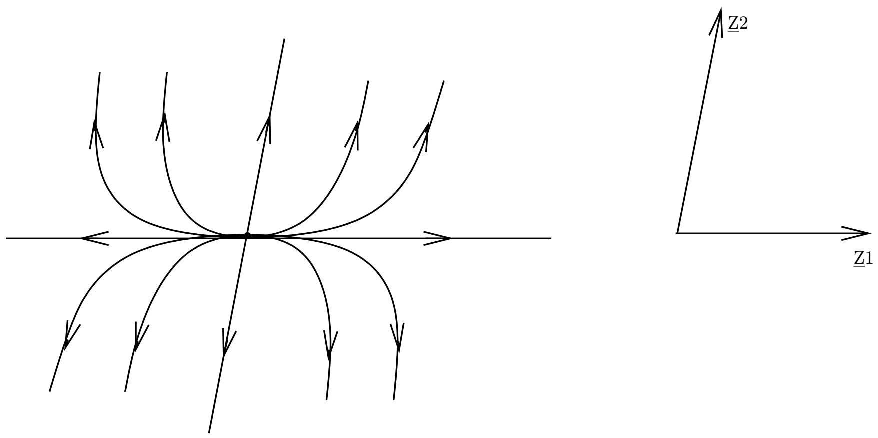


Figure 2.2: An unstable node.

These trajectories converge on the critical point into the past, but go off to infinity in the future. A critical point with these properties is called an **unstable node**.

Case 2: $\lambda_1 < \lambda_2 < 0$ (both real)

This is as above but with $t \to -t$ and the roles of $\underline{Z}_1$, $\underline{Z}_2$ switched. The trajectories converge on the critical point into the future and come in from infinity in the past.

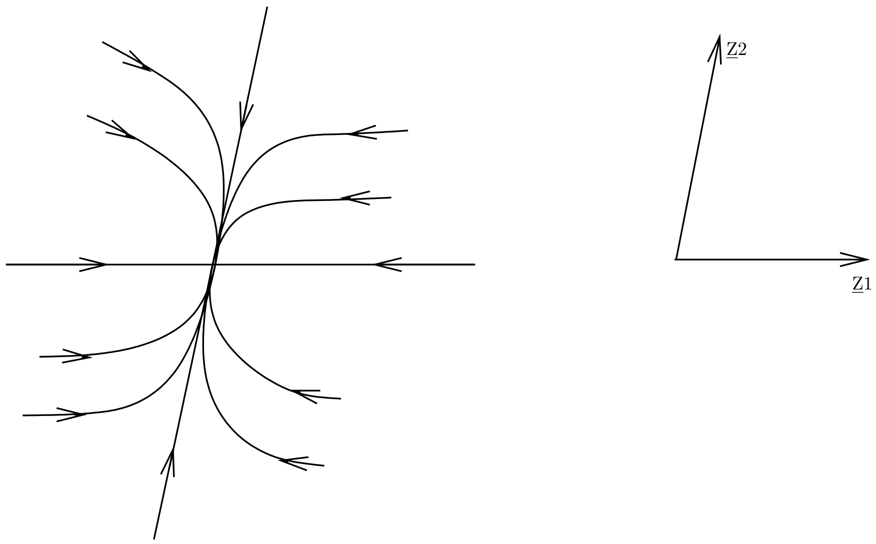


Figure 2.3: A stable node.

This is a **stable node**.

Case 3: $\lambda_1 = \lambda_2 = \lambda$. If the solution of the linearised equation is given by (2.7) (case (i)) we have a **star** , while if the solution is given by (2.8) (case (ii)) there is an **inflected node** . In both cases the critical point is stable if $\lambda < 0$ and unstable if $\lambda > 0$.

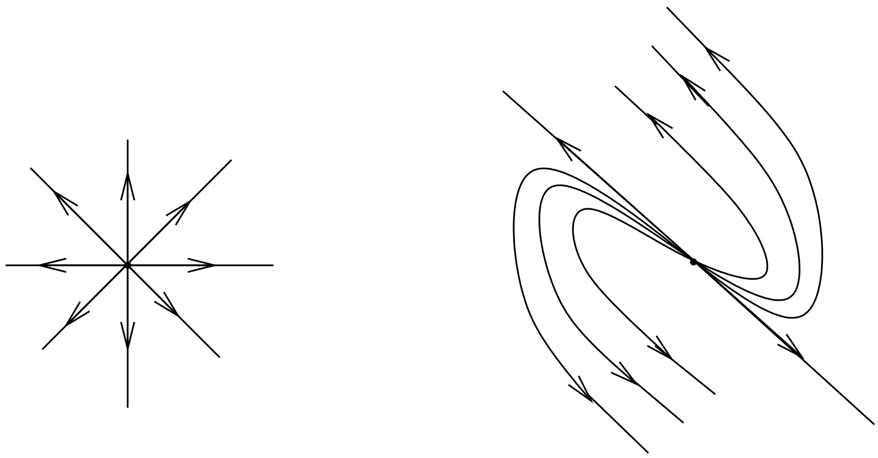


Figure 2.4: Unstable star case (i) and unstable inflected node case (ii)

Case 4: $\lambda_1 < 0 < \lambda_2$ (both real)

$$\begin{array}{llrlr} \text{If } c_1 = 0 \text{ then} & & \underline{Z}(t) \to \infty & \text{along } \underline{Z}_2 & \text{as } t \to \infty \\ & & \to 0 & \text{along } \underline{Z}_2 & \text{as } t \to -\infty. \\ \text{If } c_2 = 0 \text{ then} & & \underline{Z}(t) \to 0 & \text{along } \underline{Z}_1 & \text{as } t \to \infty \\ & & \to \infty & \text{along } \underline{Z}_1 & \text{as } t \to -\infty. \end{array}$$

If $c_1, c_2 \neq 0$ then $\underline{Z}(t) \to \infty$ along $\underline{Z}_2$ as $t \to \infty$ and along $\underline{Z}_1$ as $t \to -\infty$.

Most trajectories come in approximately parallel to $\pm\underline{Z}_1$ and go out becoming asymptotic to $\pm\underline{Z}_2$.

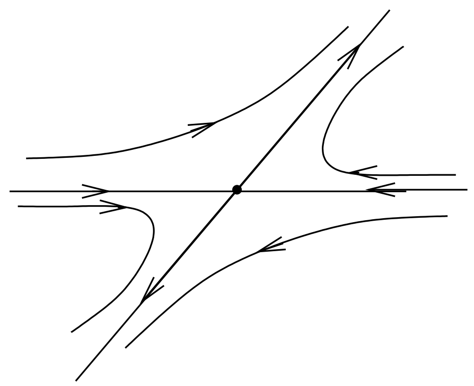


Figure 2.5: A saddle.

This is a **saddle** (to motivate the name, think of the trajectories as contour lines on a map; then two opposite directions from the critical point are uphill and the two orthogonal directions are downhill).

If the eigen-values are a complex conjugate pair we may write

$$\lambda_1 = \mu - i\nu, \quad \lambda_2 = \mu + i\nu \quad \mu, \nu \in \mathbb{R},$$

and the classification continues in terms of μ and ν.

In (2.6) the $c_i Z_i$ are a conjugate pair so if we put $c_1 = re^{i\theta}$, $\underline{Z}_1 = (1, ke^{i\phi})^T$, then

$$c_1\underline{Z}_1 = \begin{pmatrix} re^{i\theta} \\ rke^{i(\phi+\theta)} \end{pmatrix}$$

so that

$$\underline{Z}(t) = e^{\mu t}\begin{pmatrix} 2r\cos(\nu t - \theta) \\ 2rk\cos(\nu t - (\phi + \theta)) \end{pmatrix}.$$

Case 5: $\mu = 0$

Then $\underline{Z}(t)$ is periodic.

This case is called a **centre**, and is stable. The sense of the trajectories, clockwise or anticlockwise, depends on the sign of B; $B > 0$ is clockwise (take $\zeta = 0$ and $\eta > 0$, then $\dot{\zeta} = B\eta > 0$).
To see that this centre is stable: Take $t_0 = 0$. Consider the path whose maxmum distance from the critical point, (a, b), is $\epsilon > 0$. Let $\delta > 0$ be the minimum distance of this path from (a, b). Then
$\sqrt{(x(0) - a)^2 + (y(0) - b)^2} \leq \delta$ implies $\sqrt{(x(t) - a)^2 + (y(t) - b)^2} \leq \epsilon$, for all $t \geq 0$.

Case 6: $\mu \neq 0$

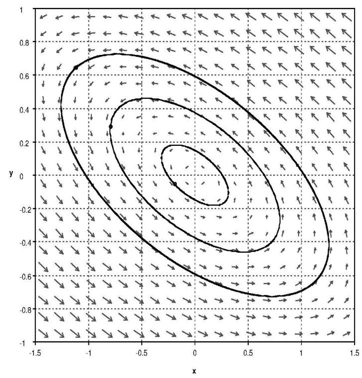


Figure 2.6: An anticlockwise centre ($B < 0$); $X = -x - 3y,\ Y = x + y$

This is just like case 5, but with the extra factor $e^{\mu t}$, which is monotonic in time. We have another dichotomy:

(i) $\mu > 0$ then $|\underline{Z}(t)| \to \infty$ as $t \to \infty$ so the trajectory spirals out, into the future. This is called an **unstable spiral**.

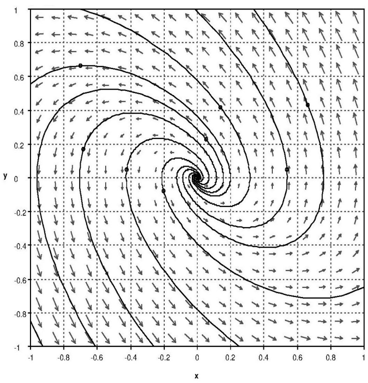


Figure 2.7: A antilockwise unstable spiral; X=-y, Y=x+y. Reverse the arrows for a stable spiral .

(ii) $\mu < 0$ this is the previous with time reversed so it spirals in, and is called a **stable spiral**.

In case 6, as in case 5, the sense of the spiral is dictated by the sign of B.

[*An alternative method of looking at case 5 and 6:*

Case 5: $\mu = 0$

so $\lambda_1 = -i\nu$ *and* $\lambda_i^2 = -\nu^2 < 0$*; but, as both the trace and determinant of a matrix are invariant under* $P^{-1}M\,P$ *transformations, in terms of the matrix* M *of (2.4),* trace M $=$ A$+$D $= \lambda_1 + \lambda_2 =$ i$\nu -$ i$\nu = 0$ *so* det M $=$ AD $-$ BC $= -\mathrm{A}^2 - \mathrm{BC} = \lambda_1\lambda_2 = \nu^2 > 0$

Equation (2.4) becomes

$$\begin{pmatrix} \dot{\zeta} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \begin{pmatrix} \zeta \\ \eta \end{pmatrix}. \qquad (2.9)$$

As an exercise, show that now $-C\zeta^2 + 2A\zeta\eta + B\eta^2$ *is constant in time. We know that* B, C *have opposite signs with* $(-BC) > A^2$ *so this is the equation of an ellipse.*

This case is called a **centre**.

Case 6: $\mu \neq 0$

So, in (2.6), we must have $\underline{Z}_1 = \bar{\underline{Z}}_2$ *and* $c_1 = \bar{c}_2$ *and*

$$\underline{Z}(t) = e^{\mu t}\left[c_1 \underline{Z}_1 e^{-i\nu t} + \bar{c}_1 \bar{\underline{Z}}_1 e^{i\nu t}\right],$$

which is just like case 5, but with the extra factor $e^{\mu t}$*, which is monotonic in time. So:*

(i) $\mu > 0$ *then* $|\underline{Z}(t)| \to \infty$ *as* $t \to \infty$ *so the trajectory spirals out, into the future. An* **unstable spiral**.

(ii) $\mu < 0$ *this is the previous with time reversed so it spirals in, a* **stable spiral**.]

Important observation:

Both the trace and determinant of a matrix are invariant under $P^{-1}M\,P$ transformations, so in terms of the matrix M of (2.4), trace M = A + D = $\lambda_1 + \lambda_2$ and det M = AD − BC = $\lambda_1\lambda_2$

Thus: if $A + D > 0$ then we have one of the cases 1, 4 or 6(i), all of which are **unstable** (but if $A + D < 0$ the critical point can be stable or unstable). Further det $M = \lambda_1\lambda_2$. So when the eigenvalues are real the sign of det M tells us whether the signs of the eigenvalues are the same or different. The determinant is always positive in the case of complex eigenvalues.

Relationship to non-linear problem: One hopes that the linearistion will have the same type of critical point as the original system. In general if the linearisation has a node, saddle point or spiral, then so does the original system, but proving this is beyond the scope of this course. However, a centre in the linearisation does not imply a centre in the nonlinear system. This is not surprising when one reflects that a centre in the linear system arises when $\operatorname{Re}\lambda = 0$ so the perturbation involved when one returns to the nonlinear system, however small, can change this property.

Analysing the critical points and their local behaviour is important in determining the general behaviour of trajectories of an ODE system. Connecting the various critical points together requires care. It helps to remember that trajectories can never intersect and that while different trajectories can asymptote to the same point (x_0, y_0) this can only be the case if (x_0, y_0) is a critical point. Also that the signs of $X(x_0, y_0)$ and $Y(x_0, y_0)$ give the signs of $dx/dt(t)$ and $dy/dt(t)$ respectively of solutions of (2.1) at the time where they pass through this point.

We note that a trajectory can only become horizontal in a point (x_0, y_0) if $Y(x_0, x_0) = 0$ as this means that the corresponding solution of (2.1) has velocity $dy/dt = 0$ in the moment where it passes through that point.

Similarly, the only points (x_0, y_0) in the plane where trajectories can become vertical are points where $X(x_0, y_0) = 0$.

To draw a phase diagram it hence helps to draw the "nullclines", which are the curves in the plane on which $X(x, y) = 0$ respectively $Y(x, y) = 0$.

Such nullclines obviously cross at critical points. To find the nullclines sketch the curves $X(x, y) = 0$ and the curves $Y(x, y) = 0$. In particular in any region bounded by nullclines the trajectories must have a single sign for dx/dt and for dy/dt. Hence a simple examination of the expressions for X and Y in any region will determine if all the arrows in that region are “up and to the left”, “up and to the right”, “down and to the left” or “down and to the right”.

2.3.1 An example

Find and classify the critical points for the system

$$\begin{aligned} \dot{x} &= x - y = X(x, y) \\ \dot{y} &= 1 - xy = Y(x, y) \end{aligned} \tag{2.10}$$

Solution: for the critical points, from $X = 0$ deduce $x = y$, therefore from $Y = 0$ deduce $x^2 = 1$, and we have either $(1, 1)$ or $(-1, -1)$.

For the classification, calculate

$$M = \begin{pmatrix} X_x & X_y \\ Y_x & Y_y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -y & -x \end{pmatrix},$$

and evaluate at the critical points:

$$\text{at } (1, 1) : M = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} : \ \lambda^2 - 2 = 0 : \ \lambda = \pm\sqrt{2}$$

this is a **saddle**. The corresponding eigenvectors are:

$$\lambda_1 = -\sqrt{2} \quad \underline{Z}_1 = \begin{pmatrix} 1 \\ 1 + \sqrt{2} \end{pmatrix} \text{ direction in}$$

$$\lambda_2 = \sqrt{2} \quad \underline{Z}_2 = \begin{pmatrix} 1 \\ 1 - \sqrt{2} \end{pmatrix} \text{ direction out}$$

$$\text{at } (-1, 1) : \ M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} : \lambda^2 - 2\lambda + 2 = 0 : \lambda = 1 \pm i.$$

this is an **unstable spiral**; $B < 0$, so its described anticlockwise.

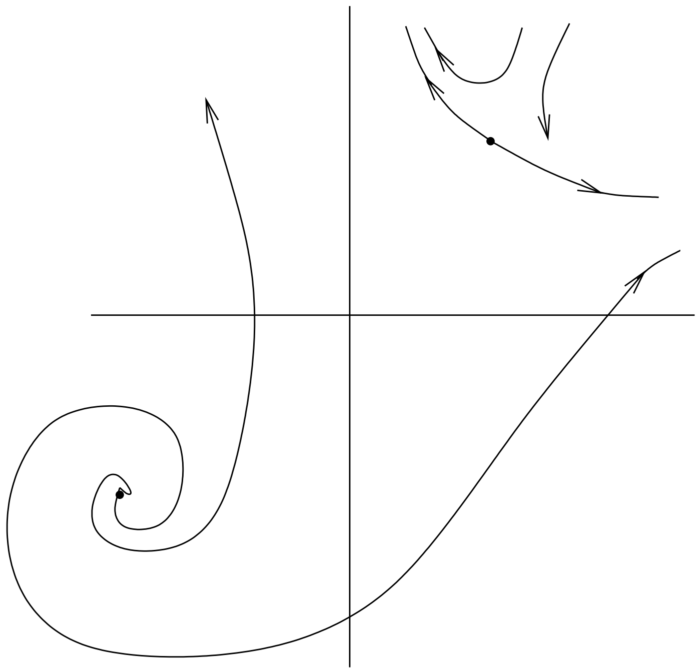


Figure 2.8: The phase diagram of (2.10)

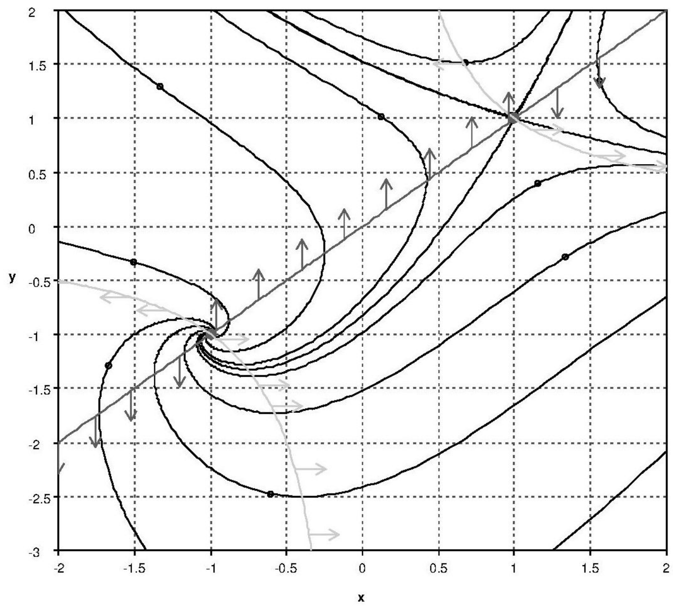


Figure 2.9: The phase plane diagram of (2.10) showing the nullclines $y = x$ and $xy = 1$.

2.3.2 Further example: the damped pendulum

Another example from mechanics: a simple plane pendulum with a damping force proportional to the angular velocity. We shall use the analysis of plane autonomous systems to understand the motion.

Take θ to be the angle with the downward vertical, then Newton's equation is

$$ml\ddot{\theta} = -mg\sin\theta - mkl\dot{\theta},$$

where m is the mass of the bob, l is the length of the string, g is the acceleration due to gravity and k is a (real, positive) constant determining the friction. We cast this as a plane autonomous system in the usual way: set $x = \theta$ and $y = \dot{x} = \dot{\theta}$ so

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{g}{l}\sin x - ky \end{aligned}$$

For simplicity below, we'll also assume that $k^2 < \frac{4g}{l}$, so that the damping isn't too large.

To sketch the phase diagram, we first find and classify the critical points. The critical points satisfy $y = 0 = \sin x$, so are located at $(x, y) = (N\pi, 0)$. Then

$$M = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos x & -k \end{pmatrix}$$

The classification depends on whether N is even or odd:

$$\text{for } x = 2n\pi \quad M = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & -k \end{pmatrix}$$

which gives a stable spiral (clockwise);

$$\text{for } x = (2n+1)\pi \quad M = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} & -k \end{pmatrix}$$

which gives a saddle.
We now have enough information to sketch the phase diagram (note that $\dot{x}$ is positive or negative according as y is).

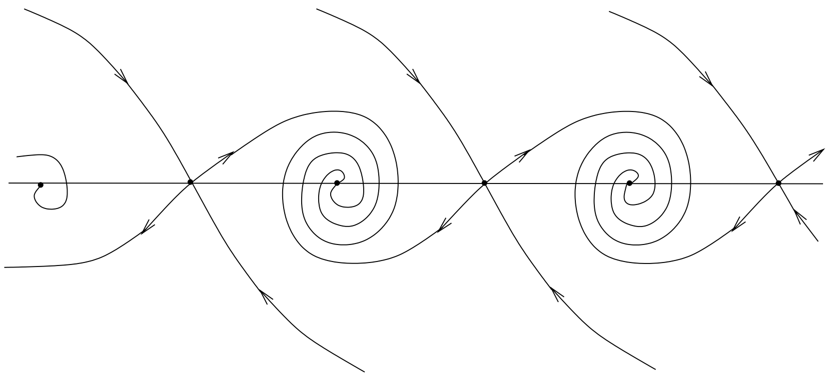


Figure 2.10: The phase diagram of the damped pendulum

2.3.3 An important example: The Lotka–Volterra predator-prey equations

This is a simplified mathematical model of a *predator-prey* system. Think of variables x standing for the population of prey, and y for the population of predators, both functions of t for time. As time passes, x increases as the prey breed, but decreases as the predators predate; likewise y increases by predation but decreases if too many predators compete. We assume that x and y are governed by the following plane autonomous system:

$$\begin{aligned} \dot{x} &= \alpha x - \gamma xy \\ \dot{y} &= -\beta y + \delta xy, \end{aligned} \tag{2.11}$$

where $\alpha,\ \beta,\ \gamma,\ \delta$ are positive real constants. Because of the interpretation as populations, we only care about $x \geq 0,\ y \geq 0$ but we shall consider the whole plane for simplicity. Again, the aim is to use the analysis of plane autonomous systems to lead us to the phase diagram and an understanding of the dynamics.

For the critical points first, set

$$X := x(\alpha - \gamma y) = 0$$

$$Y := y(-\beta + \delta x) = 0.$$

There are two solutions, $(0,0)$ and $(\frac{\beta}{\delta},\ \frac{\alpha}{\gamma})$. For the matrix:

$$M = \begin{pmatrix} X_x & X_y \\ Y_x & Y_y \end{pmatrix} = \begin{pmatrix} \alpha - \gamma y & -\gamma x \\ \delta y & -\beta + \delta x \end{pmatrix}$$

so first

$$\text{at } (0,0): \ M = \begin{pmatrix} \alpha & 0 \\ 0 & -\beta \end{pmatrix}$$

which gives a saddle, where, it is easy to see, the out-direction is the x-axis and the in-direction is the y-axis. Next

$$\text{at } \left(\frac{\beta}{\delta},\ \frac{\alpha}{\gamma}\right): \ M = \begin{pmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\alpha\delta}{\gamma} & 0 \end{pmatrix} : \lambda^2 + \alpha\beta = 0$$

which gives a centre, described anticlockwise since $B < 0$.

We have found and classified the critical points. Before sketching the phase diagram, it is worth noting, from (2.11), that the axes are particular trajectories, and trajectories can only cross at critical points (as noted before).

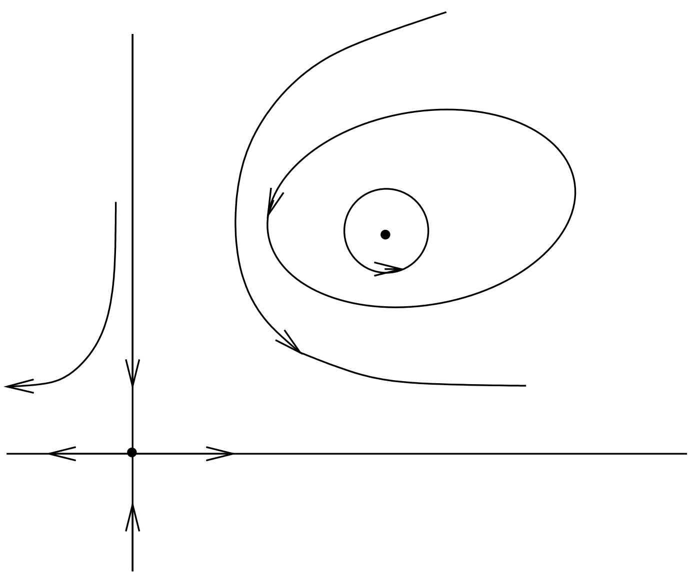


Figure 2.11: The phase diagram for the Lotka–Volterra system

Therefore any trajectory which is ever in the first quadrant is confined to the first quadrant, and no trajectory can enter the first quadrant from outside. Since there is a centre in the first quadrant, it looks as though all trajectories in the first quadrant may be **periodic**. This is true, and can be seen by the following argument: form the ratio

$$\frac{\dot{y}}{\dot{x}} = \frac{y(-\beta + \alpha x)}{x(\alpha - \gamma y)} = \frac{dy}{dx}.$$

and separate

$$\frac{(\alpha - \gamma y)}{y} dy - \frac{(-\beta + \delta x)}{x} dx = 0;$$

now integrate

$$\beta \log x - \delta x + \alpha \log y - \gamma y = C. \tag{2.12}$$

for a constant C. For different values of C, (2.12) is the equation of the trajectory or equivalently the trajectories are the level sets of the function $f(x,y) = \beta \log x - \delta x + \alpha \log y - \gamma y$.

One can see graphically that these level sets $\{(x,y) : f(x,y) = C\}$ cannot spiral to a closed curve or to the critical point $(\frac{\beta}{\delta}, \frac{\alpha}{\gamma})$ (the latter can be seen as $(\frac{\beta}{\delta}, \frac{\alpha}{\gamma})$ is the unique maximum of f so cannot be approached by another level set, to rigorously prove that level sets cannot spiral towards closed curves one could argue using the analyticity of f but this goes beyond the remit of this course).

This only leaves the possibilities that the level sets are either closed curves or that they are curves that either escape to infinity or approach one of the axes. The second possibility is excluded as f tends to minus infinity on the axes and at infinity while the function f is given by a fixed number C on the level set.

Having excluded all other possibilities we hence deduce that the level sets of f, and hence the trajectories, are all closed curves and hence that all solutions of (2.11) are periodic.

This useful technique can be applied to other examples.

2.3.4 Another example from population dynamics.

This is a simple model for two species in competition. Suppose that, when suitably scaled, the population on an island of rabbits ($x \geq 0$) and sheep ($y \geq 0$) satisfies the plane autonomous system:

$$\dot{x} = x(3 - x - 2y), \quad \dot{y} = y(2 - x - y). \tag{2.13}$$

(The populations are in competition for resources so each has a negative effect on the other)

If we analyse this system we find that the critical points are $(0,0)$, $(3,0)$, $(0,2)$, $(1,1)$.

Then at $(0,0)$:

$$M = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

which has eigenvalues 3 and 2, with eigenvectors are (1,0), and (0,1) and is an unstable node.

At $(3,0)$:

$$M = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$$

which has eigenvalues -3 and -1, with eigenvectors (1,0), and (-3,1) and is a stable node.

At $(0,2)$:

$$M = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$$

which has eigenvalues -1 and -2, with eigenvectors (-1,2), and (0,1) and is a stable node.

At $(1,1)$:

$$M = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

which has eigenvalues $-1 - \sqrt{2}$ and $-1 + \sqrt{2}$, with eigenvectors $(\sqrt{2}, 1)$ and $(-\sqrt{2}, 1)$. and is a saddle point.

Again, as x and y represent populations we require that any trajectory which starts out in the first quadrant will remain there. As in the previous example this is indeed the case as the axes are particular trajectories.

Looking at the phase diagram we can see that, in the long term, depending on the initial data, either the rabbits or the sheep will survive.

Other values of the coefficients will give different outcomes - see problem sheet 2.

2.3.5 Another important example: limit cycles

Consider the plane autonomous system:

$$\dot{x} = (1 - (x^2 + y^2)^{\frac{1}{2}})x - y$$

$$\dot{y} = (1 - (x^2 + y^2)^{\frac{1}{2}})y + x.$$

Put $x^2 + y^2 = r^2$ then

$$X = x(1 - r) - y$$

$$Y = y(1 - r) + x$$

and one sees that only critical point is $(0, 0)$. One could go through the classification for this to find that it is an unstable spiral (exercise!).

Alternatively, in this case, we can analyse the full nonlinear system. We shall transform to polar coordinates. The simplest way to do this is as follows: first

$$r\dot{r} = x\dot{x} + y\dot{y} = x[x(1 - r) - y] + y[y(1 - r) + x]$$

$$= r^2(1 - r)$$

or

$$\dot{r} = r(1 - r).$$

Then, with

$$y = r \sin\theta,$$

we find

$$\dot{y} = \dot{r} \sin\theta + r \cos\theta\dot{\theta} = y(1 - r) + x,$$

which gives $\dot{\theta}$, so the system becomes

$$\left.\begin{aligned} \dot{\theta} &= 1 \\ \dot{r} &= r(1 - r). \end{aligned}\right\}$$

Unlike the system in its previous form, we can solve this. First

$$\theta = t + \text{ const},$$

and then

$$\int dt = \int \frac{dr}{r(1 - r)} = \int \left(\frac{1}{r} + \frac{1}{1 - r}\right) dr$$

so

$$\log \frac{r}{|1 - r|} = t + \text{ const}$$

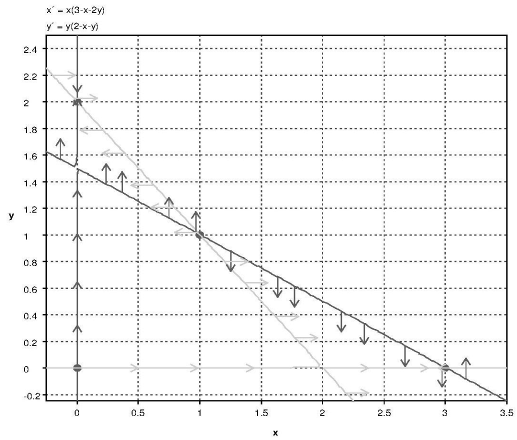


Figure 2.12: The nullclines for the equations (2.13).

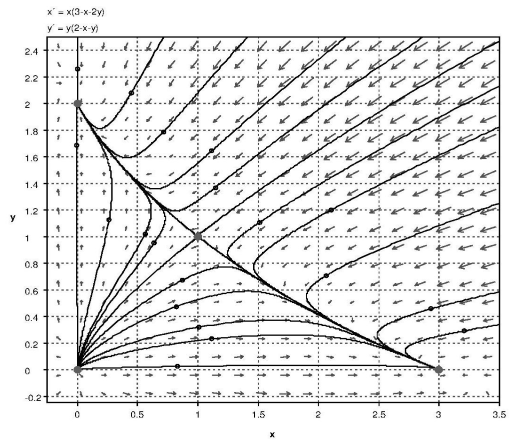


Figure 2.13: Phase diagram for the equations (2.13) for competitive species - no nullclines.

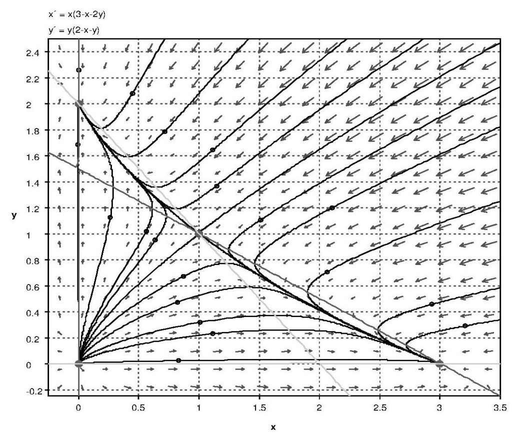


Figure 2.14: The phase diagram for the equations (2.13) for competitive species - with the nullclines Rabbits or sheep survive, depending on the initial data.

i.e.

$$\frac{r}{1-r} = Ae^t.$$

Solve for r and change the constant:

$$r = \frac{1}{1+Be^{-t}} = \frac{1}{1+(\frac{1}{r_0}-1)e^{-t}}$$

where $r(0) = r_0$.

Note that as $t \to \infty,\ r \to 1$, while as $t \to -\infty$ either $r \to 0$ if $r_0 < 1$ or $r \to \infty$ at some finite t if $r_0 > 1$.

Now it is clear that the origin is an unstable spiral, and that the trajectories spiral out of it anticlockwise. We can also see that $r = 1$ is a closed trajectory and that all other trajectories (except the fixed point at the origin) tend to it; we call such a closed trajectory a **limit cycle**. It is **stable** because the other trajectories converge on it. (For an example of an unstable limit cycle we could consider the same system but with t changed to $-t$.)

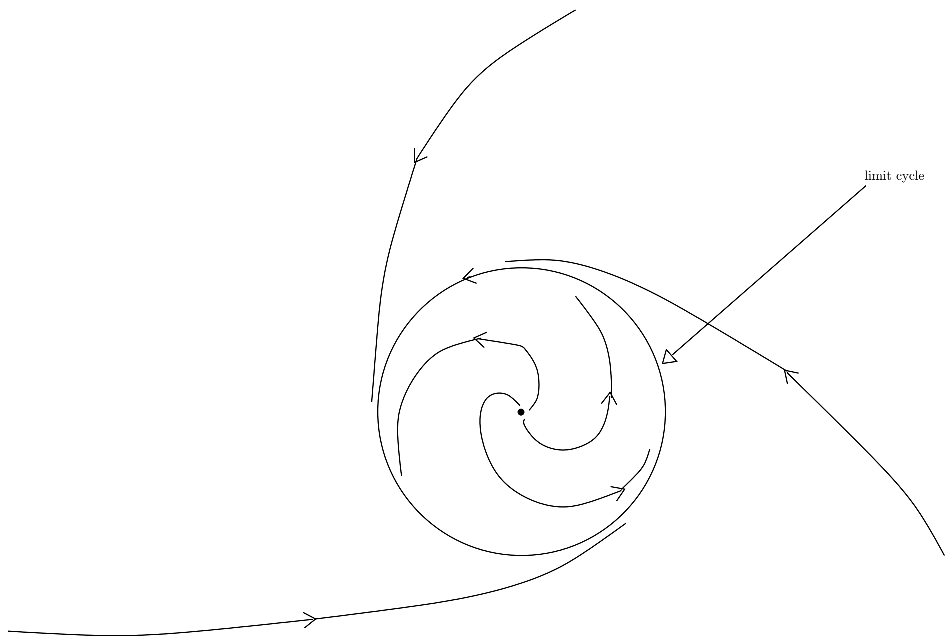


Figure 2.15: Phase diagram with a limit cycle

Another system with a limit cycle arises from the *Van der Pol equation*:

$$\ddot{x} + \epsilon(x^2-1)\dot{x} + x = 0$$

where ϵ is a positive real constant. If $\epsilon = 0$ this is the harmonic oscillator again. If $\epsilon \neq 0$ then the usual trick produces a plane autonomous system:

$$\dot{x} = y$$

$$\dot{y} = -\epsilon(x^2-1)y - x.$$

The only critical point is $(0, 0)$ and it's an unstable spiral for $\epsilon > 0$ (exercise!).

Claim: Its beyond us to show this, but this system has a unique limit cycle, which is stable. There are some good illustrations for this in e.g. Boyce and di Prima (pp 496–500 of the 5th edition).

2.4 The Bendixson–Dulac Theorem

It's important to be able to detect periodic solutions, but it can be tricky. We end this section with a discussion of a test that can rule them out.

Theorem 2.1. (Bendixson–Dulac) *Consider the system $\dot{x} = X(x, y)$, $\dot{y} = Y(x, y)$, with $X, Y \in C^1$. If there exists a function $\varphi(x, y) \in C^1$ with*

$$\rho := \frac{\partial}{\partial x}(\varphi X) + \frac{\partial}{\partial y}(\varphi Y) > 0$$

in a simply connected region R then there can be no nontrivial closed trajectories lying entirely in R.

Proof. (By nontrivial, I mean I want the trajectory must have an inside i.e. it isn't just a fixed point.) So suppose C is a closed trajectory lying entirely in R and let D be the disc (which also lies entirely in R, as R is simply connected) whose boundary is C. We apply Green's theorem in the plane. Consider the integral

$$\begin{aligned} \int\int_D \rho \, dxdy &= \int\int_D \left[\frac{\partial}{\partial x}(\varphi X) + \frac{\partial}{\partial y}(\varphi Y)\right] \, dxdy \\ &= \oint_C -\varphi Y \, dx + \varphi X \, dy \\ &= \oint_C -\varphi \left(-\dot{y} dx + \dot{x} dy\right). \end{aligned}$$

But on C, $dx = \dot{x}dt$, $dy = \dot{y}dt$ so this is zero, which contradicts positivity of ρ, so there can be no such C.

2.4.1 Corollary.

If

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}$$

has fixed sign in a simply connected region R, then there are no nontrivial closed trajectories lying entirely in R.

This is just the previous but with φ const — in an example, always try this first!

2.4.2 Examples

(i) the damped pendulum (section 2.3.2)

$$\dot{x} = y$$
$$\dot{y} = -\frac{g}{l}\sin x - ky$$

has no periodic solutions.
Here

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = -k \ < 0;$$

now use the corollary.

(ii)

$$\ddot{x} + f(x)\dot{x} + x = 0$$

has no periodic solutions in a simply connected region where f has a fixed sign.

By the usual trick we get the system

$$\dot{x} = y$$
$$\dot{y} = -yf(x) - x$$

then

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = -f(x)$$

and we use the corollary.

(iii) The system

$$\dot{x} = y$$
$$\dot{y} = -x - y + x^2 + y^2$$

has no periodic solutions.

The corollary doesn't help so try the general case:

$$\rho := (\varphi X)_x + (\varphi Y)_y = \varphi(-1 + 2y) + X\varphi_x + Y\varphi_y.$$

Now guess: $\varphi_y = 0$ then

$$\rho = \varphi(-1 + 2y) + y\varphi_x = -\varphi + y(\varphi_x + 2\varphi)$$

so if we take $\varphi = -e^{-2x}$ the coefficient of y (which can take either sign) is zero and $\rho = 2e^{-2x} > 0$ and we are done.

PART II Partial Differential Equations.

3 First order semi-linear PDEs: method of characteristics

3.1 The problem

In this chapter, we consider *first-order* PDEs

$$P(x,y)\frac{\partial z}{\partial x} + Q(x,y)\frac{\partial z}{\partial y} = R(x,y,z(x,y)) \tag{3.1}$$

for an unknown function $z = z(x,y)$.

The PDE is said to be *semi-linear* as it is linear in the highest order partial derivatives, with the coefficients of the highest order partial derivatives depending only on x and y. If P and Q depend also on z the PDE is said to be *quasi-linear*. Everything that we discuss in this chapter is valid for semi-linear equations, though some parts can also be applied to quasilinear PDEs (see Remarks 3.1 and 3.2 below).

We will assume throughout this section that, in the region specified, $P(x,y)$ and $Q(x,y)$ are Lipschitz continuous in x and y and $R(x,y,z)$ is continuous and Lipschitz continuous in z. This will be enough to apply Picard's theorem to ensure that the characteristic equations have a unique solution through each point (see Proposition 3.1 below).

We want to find a unique solution to (3.1) given suitable data and determine its *domain of definition*. This is the region in the (x,y)-plane in which the solution is uniquely determined by the data. It turns out to depend on both the equation and the data.

The solution of (3.1) will be a function

$$z = f(x,y)$$

and to construct f it will be useful to consider the graph of this function, i.e. the surface $\Sigma := \{(x,y,z) : z = f(x,y)\}$. We shall refer to this as the *solution surface.*

The idea of the method of characteristics is to try to generate this solution surface, initially as a parametrised surface, and then from it obtain the desired solution $z = f(x,y)$ of (3.1).

Recall that if a surface Σ is given by the graph of a function f then the vectors $(1,0,\partial_x f)$ and $(0,1,\partial_y f)$ are tangent to Σ. We can thus generate a vector $\mathbf{n}$ that is normal to Σ via

$$\mathbf{n} = (1,0,\partial_x f) \wedge (0,1,\partial_y f) = (-\partial_x f, -\partial_y f, 1).$$

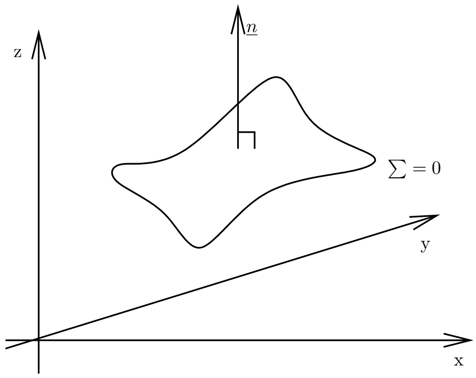


Figure 3.1: The solution surface

Hence we see that f is a solution of the PDE

$$P(x,y)\partial_x f(x,y) + Q(x,y)\partial_y f(x,y) = R(x,y,f(x,y)) \tag{3.2}$$

if and only if the the vector $\mathbf{t} = (P,Q,R)$ satisfies

$$\mathbf{t} \cdot \mathbf{n} = -P\partial_x f - Q\partial_y f + R = 0,$$

for $\mathbf{n}$ as above. I.e. f is a solution if $\mathbf{t}$ is perpendicular to the normal vector $\mathbf{n}$ of $\Sigma = \text{graph}(f)$ or equivalently if $\mathbf{t}$ is tangential to Σ.

We can hence reformulate our PDE for the function f as the geometric condition on the solution surface $\Sigma = \text{graph}(f)$ that $\mathbf{t}(x,y,z) = (P,Q,R)(x,y,z)$ is tangential to Σ at every point $(x,y,z) \in \Sigma$.

To solve the PDE (3.1) we hence want to construct such a surface Σ, initially as a parametrised surface, and then determine the function f by writing this surface as a graph, i.e. by solving for $z = z(x,y)$ (if possible).

We usually consider the PDE (3.1) together with data, which asks that the unknown function $z(x,y)$ is given by a prescribed function $g(x,y)$ on a curve $\gamma_0 = (\gamma_1, \gamma_2)$ in the xy-plane. This is equivalent to asking that the solution surface Σ contains the *initial curve* $\gamma(s) = (\gamma_1(s), \gamma_2(s), g(\gamma_1(s), \gamma_2(s)))$.

To construct our solution surface, we will start with the given initial curve, then determine curves, so called characteristics, that start on the initial curve and that move in direction of the vector (P,Q,R). The solution surface Σ will then be generated by all of these curves.

3.2 The big idea: characteristics

We look for a curve Γ whose tangent is $\mathbf{t}$. If $\Gamma = (x(t), y(t), z(t))$ in terms of a parameter t this means

$$\begin{aligned} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y) \\ \dot{z} &= R(x,y,z) \end{aligned} \tag{3.3}$$

These are the *characteristic equations* and the curve Γ is a *characteristic curve* or just *a characteristic*. Given a characteristic $(x(t), y(t), z(t))$, we call the curve $(x(t), y(t))$, which lies below it in the (x,y)-plane, the *characteristic projection* or *characteristic trace*.

The next result shows that characteristics exist, and gives the crucial property of them:

Proposition 3.1. *Suppose that $P(x, y)$ and $Q(x, y)$ are Lipschitz continuous in x and y and $R(x, y, z)$ is continuous and satisfies a Lipschitz condition in z. Then*

(a) Through every point $(x_0, y_0) \in \mathbb{R}^2$ there is a unique characteristic projection.

(b) Through every $(x_0, y_0, z_0) \in \mathbb{R}^3$ there is a unique characteristic.

(c) If f is a solution of the PDE (3.1) and Γ is a characteristic through a point p that is contained in the solution surface $\Sigma = graph(f)$ then the whole characteristic is contained in Σ.

We note in particular that characteristic projections can never intersect (this is very much a feature of semilinear equations and does not hold true for quasilinear PDEs, compare remark 3.1 below.).

Proof. (a) Since P and Q do not depend on z the first two equations $\dot{x} = P(x, y), \dot{y} = Q(x, y)$ of (4.15) are a plane autonomous system (with Lipschitz functions $X = P, Y = Q$). From the previous chapter we know that Picard guarantees a unique trajectory through every point and as these trajectories are simply the characteristics projections we obtain (a).

(b) Part (a) already provides a unique solution $(x(t), y(t))$, t in some interval I_1, to the first two equations of (4.15) with $x(0) = y(0) = (x_0, y_0)$.

Given this $(x(t), y(t))$ we can set $F(t, z) := R(x(t), y(t), z)$ and view the third equation as an ODE $\dot{z}(t) = F(t, z(t))$ for z. Since R satisfies a Lipschitz condition with respect to z we obtain from Picard's theorem that there is a unique solution $z(t)$ with $z(0) = z_0$. Hence we find a unique characteristic through (x_0, y_0, z_0).

(c) Let f be a solution of the PDE and let $\Sigma = \text{graph}(f) = \{(x, y, f(x, y))\}$ be the corresponding solution surfaces. Suppose that $(x(t), y(t), z(t))$ is a characteristic through a point $p \in \Sigma$. Shifting time we can assume that $p = (x, y, z)(0)$ and hence $z(0) = f(x(0), y(0))$. We now want to prove that the whole curve stays in Σ, i.e. that

$$w(t) := z(t) - f(x(t), y(t))$$

remains equal to zero.

To do this, we want to show that $|w(t)|$ satisfies the conditions of Gronwall's inequality (1.27) with $b = 0$. Namely we want to show that $|w(t)| \leq L|\int_0^t |w(s)|ds|$, L the constant from the Lipschitz condition $|R(x, y, z) - R(x, y, \tilde{z})| \leq L|z - \tilde{z}|$.

To see this we differentiate $w(t) = z(t) - f(x(t), y(t))$ and use first the characteristic equations and then that f solves the PDE (3.2) to get

$$\begin{aligned} \dot{w} &= \dot{z} - (\partial_x f(x, y)\dot{x} + \partial_y f(x, y)\dot{y}) = R(x, y, z) - (\partial_x f(x, y)P(x, y)\partial_y f(x, y)Q(x, y)) \\ &= R(x, y, z) - R(x, y, f(x, y)). \end{aligned} \tag{3.4}$$

As $w(0) = 0$ we hence get

$$|w(t)| = |\int_0^t \dot{w}(s)ds| \leq |\int_0^t |R(x, y, z) - R(x, y, f(x, y))|ds| \leq L|\int_0^t |w(s)|ds| \tag{3.5}$$

and we can apply Gronwall to see that $|w(t)| \leq 0 \cdot e^{L|t|}$, i.e. that $w(t) = z(t) - f(x(t), y(t))$ remains zero for all t.

$\square$

Remark 3.1. *If we consider instead quasilinear PDEs*

$$P(x, y, z(x, y))\partial_x z(x, y) + Q(x, y, z(x, y))\partial_y z(x, y) = R(x, y, z(x, y)) \tag{3.6}$$

then statement (a) is false. If two characteristics $\Gamma(t) = (x, y, z)(t)$ *and* $\tilde{\Gamma}(t) = (\tilde{x}, \tilde{y}, \tilde{z})(t)$ *pass through points* (x_0, y_0, z_0) *and* $(x_0, y_0, \tilde{z}_0)$ *with the same* x *and* y *coordinates, but with* $z_0 \neq \tilde{z}_0$ *then we cannot expect that the projections of these characteristics agree, as the ODEs satisfied by* (x, y) *and* $(\tilde{x}, \tilde{y})$ *also contain a dependence on the functions* $z(t)$ *and* $\tilde{z}(t)$*. For quasilinear PDEs it is hence possible that characteristic projections intersect, while the above Proposition excludes this possibility for semilinear PDEs.*

Conversely, statements (b) and (c) of the Proposition remain valid also for quasilinear PDEs (provided P, Q, R *are Lipschitz wrt all variables* x, y, z*) and the proofs can be easily adapted to this setting.*

It is hence possible to solve also quasilinear PDEs with the method of characteristics, but one needs to be more careful, in particular when determining the domain of definition.

3.2.1 Examples of characteristics

We need to gain proficiency in calculating characteristics and calculate the characteristics for the followig PDEs:

Example (a):

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z. \tag{3.7}$$

From (3.3) we write down the characteristic equations and solve them giving

$$\begin{aligned} \dot{x} &= P = x; \qquad & x &= Ae^t \\ \dot{y} &= Q = y; \qquad & y &= Be^t \\ \dot{z} &= R = z; \qquad & z &= Ce^t \end{aligned}$$

with $A,\ B,\ C$ constants (trivial to solve). The characteristics are hence half-lines from the origin (not including the origin) in $\mathbb{R}^3$ and the characteristic projections are half-lines from the origin (not including the origin) in the xy plane.

Example (b):

$$y\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z. \tag{3.8}$$

The characteristic equations and their solutions are

$$\begin{aligned} \dot{x} &= y; \qquad & x(t) &= Bt + \frac{t^2}{2} + A \\ \dot{y} &= 1; \qquad & y(t) &= B + t \\ \dot{z} &= z; \qquad & z(t) &= Ce^t \end{aligned} \tag{3.9}$$

with A, B, C constants. To solve this system, pass over the first, solve the second, then come back to the first and third. (I am adopting a convention to introduce the constants A, B, C in the first, second and third of the characteristic equations respectively.)

In general solving the characteristic equations needs experience and luck; there isn't a general algorithm.

3.3 The Cauchy problem

A Cauchy Problem for a PDE is the combination of the PDE together with boundary data that, in principle, will give a unique solution, at least locally. We will look for suitable data and determine the domain on which the solution is uniquely determined.

Suppose we are prescribing the solution z of (3.1) along a curve $\gamma_0 = (\gamma_1, \gamma_2)$ (called the *data curve*) in the (x, y)-plane, i.e. ask that $z(x, y) = g(x, y)$ for a given function g for all points on the data curve. Setting $\gamma_3 = g(\gamma_1, \gamma_2)$ this produces a curve $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ in space (called the *initial curve*) which needs to be in our solution surface Σ.

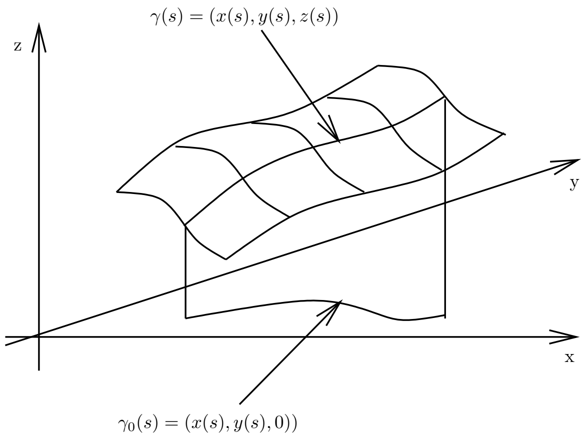


Figure 3.2: Geometry of the Cauchy problem.

To determine Σ we first parametrise the curve γ over some interval I, so consider $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ for $s \in I$. Here we will always assume that the components of γ are continuously differentiable (though will discuss the possibility that
$gamma_3'$ is discontinuous later in section 3.7). Then, to solve (3.1), we construct the solution surface Σ by taking the characteristics through the points of $\gamma(s)$ (because Proposition 3.1(b) tells us that the solution surface is generated by these characteristics) . Thus the method of solution, the *method of characteristics*, is

(i) Parametrise γ over some interval I.

(ii) Determine the solutions $(x(t, s), y(t, s), z(t, s))$ of the characteristic equations

$$\begin{aligned} \dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y) \\ \dot{z} &= R(x, y, z) \end{aligned} \tag{3.10}$$

with initial data $x(0,s) = \gamma_1(s);\ y(0,s) = \gamma_2(s);\ z(0,s) = \gamma_3(s),\ s \in I$.

(iii) This yields the solution surface Σ in parametric form, i.e. $\Sigma = \{(x(t,s), y(t,s), z(t,s))\}$ where s ranges over the interval I over which we parametrised the initial curve γ, while for each s we consider the maximal set of t's for which we can solve all three characteristic equations.

(iv) Having a parametric form of Σ we then want to eliminate the parameters s, t and write Σ as a graph to read off the solution. This is a question we will explore below, and there is a restriction on the data for the method to work, also to be found later.

3.4 Examples

(a) Solve

$$y\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z,$$

with $z(x,0) = x$ for $1 \le x \le 2$.

We introduce a parameter s for the data, say $\gamma(s) = (s, 0, s)$, for $1 \le s \le 2$, and then solve the characteristic equations (done in section 3.2.1) with this as data at $t = 0$

$$x = Bt + \frac{t^2}{2} + A; \quad x(0,s) = A = s$$

$$y = B + t; \quad y(0,s) = B = 0$$

$$z = Ce^t; \quad z(0,s) = C = s$$

So, $C = s$, $B = 0$, $A = s$ and the parametric form of the solution is

$$\left.\begin{array}{l} x = s + \frac{1}{2}t^2 \\ y = t \\ z = se^t \end{array}\right\} \tag{3.11}$$

for $1 \le s \le 2$ and $t \in \mathbb{R}$.

(b) (From Ockendon et al) Solve

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = (x+y)z \tag{3.12}$$

with $z = 1$ on the segment of the circle $(x-2)^2 + y^2 = 2$, $y \ge 0$.

So we can take $\gamma(s) = (2 - \sqrt{2}\cos s, \sqrt{2}\sin s, 1)$, $s \in [0, \pi]$, and solve the characteristic equations:

$$\frac{\partial x}{\partial t} = P = x; \quad x = Ae^t; \quad A = (2 - \sqrt{2}\cos s)$$

$$\frac{\partial y}{\partial t} = Q = y; \quad y = Be^t; \quad B = \sqrt{2}\sin s$$

$$\frac{\partial z}{\partial t} = R = (x+y)z = ((2 - \sqrt{2}\cos s + \sqrt{2}\sin s)e^t)z.$$

We can integrate the final equation to get

$$\log|z| = ((2 - \sqrt{2}\cos s + \sqrt{2}\sin s)e^t) + C; \quad C = -(2 - \sqrt{2}\cos s + \sqrt{2}\sin s).$$

So the parametric form of the solution is

$$\left.\begin{aligned} x &= (2-\sqrt{2}\cos s)e^t \\ y &= (\sqrt{2}\sin s)e^t \\ z &= \exp[((2-\sqrt{2}\cos s+\sqrt{2}\sin s)(e^t-1)] \end{aligned}\right\} \tag{3.13}$$

3.5 Domain of definition

Where is the solution determined uniquely by the data? This is the domain of definition and is the region in the (x, y)-plane where the solution surface is uniquely determined and is given explicitly as $z = f(x, y)$.

The solution surface is swept out by the characteristics through the initial curve, so the solution will be defined in the region swept out by the projections of the characteristics through the initial curve provided these characteristic projections only intersect the data curve once (otherwise the problem can be overdetermined, compare section 3.6 below).

In particular if the initial curve is bounded, and if we don't have to deal with the problem of characteristic projections intersecting the data curve multiple times, or with having characteristic projections that are closed curves, and obtain a then the domain of definition will usually be bounded by the projections of the characteristics through the end points of the initial curve, see also below.

Example (a) from section 3.4 Here the solution surface is swept out by the characteristics through γ, so has edges given by the characteristics through the ends of γ, which are at $s = 1$ and $s = 2$.

The characteristics are $(x, y, z)(s, t) = (s + \frac{1}{2}t^2, t, se^t)$ for $s \in [1, 2]$ and $t \in \mathbb{R}$ and we can solve for z to get

$$z(x, y) = (x - \frac{1}{2}y^2)e^y.$$

The characteristic projections are given by $x = s + \frac{1}{2}y^2$ and none of them intersect the data curve more than once.

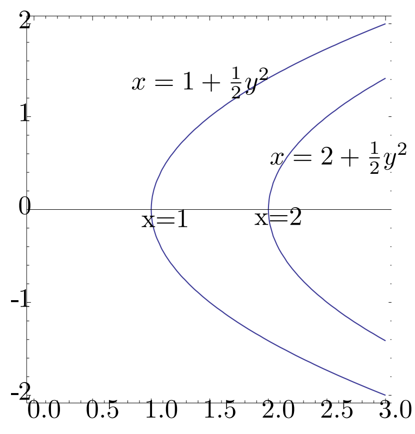


Figure 3.3: The domain of definition for this problem

Hence the domain of definition Ω is the region between the characteristic projections $x = 1 + \frac{1}{2}y^2$ at $s = 1$ and $x = 2 + \frac{1}{2}y^2$ at $s = 2$, i.e.

$$\Omega = \{(x, y) : 1 + \frac{1}{2}y^2 \leq x \leq 2 + \frac{1}{2}y^2\} \tag{3.14}$$

Blow up:

The method of characteristics reduces the PDE (3.1) to a system of ODEs. As we have already seen nonlinear ODEs can give rise to solutions which blow up, so the same must be true of non linear PDEs, even if those that are semi-linear.

If we have characteristics $t \mapsto (x, y, z)(s, t)$ for which the z component tends to $\pm\infty$ as t approaches a finite time $T_{max}(s)$ (or $T_{min}(s)$), while the (x, y) components of the characteristics remain regular beyond $T_{max}(s)$, then the corresponding solution $z = f(x, y)$ must become singular as (x, y) approach $(x(T_{max}), y(T_{max})$.

In situations like this the domain of definition Ω is still generated by the characteristic projections, but we need to be aware that we are only allowed to consider $(x, y)(s, t)$ for t so that $z(s, t)$ is well defined, i.e. only for $t < T_{max}(s)$. One part of the boundary of the domain of definition Ω is then given by the curve $\{(x, y)(s, T_{max}(s))\}$, $s \in I$, at which the solution f blows-up.

As a simple example which illustrates this behaviour you can consider the equation

$$x\partial_x z + y\partial_y z = -z^2$$

with prescribed data of $z(x, y) = \alpha \in \mathbb{R}$ for on $\{(x, y) : x + y = 1, \quad x \in [0, 1]\}$.

3.6 Cauchy data:

Once we are given the surface Σ that is spanned by the characteristics we then want to solve for z as a function of x and y. To do so we need to be able to eliminate t and s in favour of x and y, at least in principle. For this, recall from Prelims the definition of the Jacobian:

$$J(s, t) = \frac{\partial(x, y)}{\partial(t, s)} = \det \begin{pmatrix} x_t & y_t \\ x_s & y_s \end{pmatrix}. \tag{3.15}$$

Now if

$$x = x(t, s), \text{ and } y = y(t, s)$$

are continuously differentiable functions of t and s in a neighbourhood of a point, then a sufficient condition to be able to find unique continuously differentiable functions

$$t = t(x, y) \text{ and } s = s(x, y)$$

in some neighbourhood of the point, is that J be non-zero at the point. We can then substitute into $z = z(t, s)$ to get

$$z = z(t(x, y), s(x, y)) = f(x, y),$$

a continuously differentiable function of x and y as required. This comes from a result known as the Inverse Function Theorem that you can see in AOS Multidimensional Analysis and Geometry which says that if the matrix $\begin{pmatrix} x_t & y_t \\ x_s & y_s \end{pmatrix}$ is invertible at a point (s,t), i.e. if the Jacobian is not zero, then we can invert the function $(x,y)(s,t)$ at least near this point, and the inverse function $(s,t)(x,y)$ are again differentiable. Here we will have to take it on trust, but it is the two dimensional equivalent of the one dimensional result you saw in Analysis in Prelims - where you saw that a function $g:\mathbb{R}\to\mathbb{R}$ has a differentiable inverse near $x=a$ if $g'(a)\neq 0$.

If we require that $J(s,0)\neq 0$ on the initial curve γ_0 then we hence get that the problem has a unique solution at least close to the initial curve. As

$$J(s,0) = \det\begin{pmatrix} x_t & y_t \\ x_s & y_s \end{pmatrix} = \det\begin{pmatrix} P & Q \\ x_s & y_s \end{pmatrix} = Py_s - Qx_s \tag{3.16}$$

for P,Q evaluated at points $x(s,0)=y(s,0)$ on the data curve $\gamma_0=(\gamma_1(s),\gamma_2(s))$ we hence say that the data is *Cauchy data* if

$$P(x,y)y_s - Q(x,y)x_s \neq 0 \text{ on the data curve} \tag{3.17}$$

i.e. if

$$P(\gamma_1(s),\gamma_2(s))\gamma_2'(s) - Q(\gamma_1(s),\gamma_2(s)))\gamma_1'(s)\neq 0$$

for all s in the interval over which we parameterise the data curve γ_0.

Geometrically the condition that the data is Cauchy corresponds to asking that the tangent vector $\gamma_0'(s)=(x_s(s,0),y_s(s,0))$ along the data curve and the tangent vector $(P,Q)=(\dot{x},\dot{y})=(x_t(s,0),y_t(s,0))$ along the characteristic projection through the same point are never parallel. I.e. the data is Cauchy if there are no characteristic projections that meet the data curve tangentially.

As characteristic projections of semilinear PDEs cannot intersect we can use this to detect whether there are any characteristic projections which intersect the data curve more than once and if so, at what points we need to split the data curve to obtain well posed problems.

Remark. *Often we can see graphically for what (x,y) and (s,t) we are able to invert $(x(s,t),y(s,t))$ and hence solve for z, as we simply need to know whether for a point (x,y) there is a unique (s,t) with $(x,y)=(x(s,t),y(s,t))$. If characteristic projections are not closed (and hence reach the same point for different values of t and the same s) and if they do not intersect the data curve multiple times, then we can solve for (s,t) for all points (x,y) in the set that is generated by the characteristic curves that intersect the data curve as different characteristic projections cannot intersect. In any case we still need to make sure that the resulting function z is still differentiable, either by computing $z(x,y)$ explicitly, or by checking that the Jacobian $J(s,t)$ is non-zero for all s and t that we consider, and then using that the inverse function theorem ensures that $(s(x,y),t(x,y))$ is differentiable.*

Remark 3.2. *For the more general quasilinear PDEs, we can still obtain a solution in a small neighbourhood of the data curve if the data is Cauchy using this method. However, for quasilinear PDEs characteristic projections can intersect, so to determined the domain of definition we need to determine how to restrict the range of (s,t) so that characteristic projections don't intersect. To detect whether this can happen it is again useful to consider whether $J(s,t)\neq 0$, not only for $t=0$ but more more generally.*

Example (b) from section 3.4

Here

$$J = \det\begin{pmatrix} (2-\sqrt{2}\cos s)e^t & (\sqrt{2}\sin s)e^t \\ (\sqrt{2}\sin s)e^t & (\sqrt{2}\cos s)e^t \end{pmatrix} = 2e^{2t}(1-\sqrt{2}\cos s),$$

so $J(s, 0) = (1 - \sqrt{2} \cos s)$ vanishes when $s = \frac{\pi}{4}$. Note that this corresponds to the characteristic projection $y = x$, which touches the data curve at $(1, 1)$.

If we were hence to consider the PDE with data prescribed on the full data curve $\gamma(s)$, $s \in [0, \pi]$ then there will be problems as each characteristic projection meets the data curve in two points, so the initial data is likely to be inconsistent.

So we will restrict the data curve to either $s \in [0, \pi/4)$ or to $s \in (\pi/4, \pi]$. This means that we end up with data that is Cauchy and so with a data curve that is so that characteristic projections only intersect once. We hence get a well defined (rather than an overdetermined) problem and the domain of definition will again be traced out by the characteristic projections that intersect (the corresponding part of) the data curve.

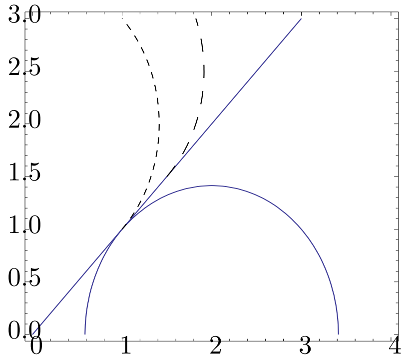


Figure 3.4: Different data curves

In the first case the data curve starts at $s = 0$, so the solution surface will have an edge given by the characteristic through the end of γ at $s = 0$ and the corresponding characteristic projection is $y = 0$, which forms an edge of the domain of definition. The domain of definition is then swept out by the projections of the characteristics through $\gamma(s)$ for $s \in [0, \pi/4)$ so the other boundary curve is the characteristic projection $x = y$ at $s = \pi/4$ and the domain of definition is $0 \leq y < x$. Similarly, we can also determine the domain of definition if we use the other part of the data curve, and in this situation we obtain the same set (this is not always the case and here comes from the original data curve being so the every characteristic projection that intersect the first part $s \in [0, \pi/4)$ of the data curve also intersects the second part and visa versa).

Note that if we instead had a data curve which turned to the left of $y = x$ after following the circle upto $s = \pi/4$ such as one of the the dashed curves in the diagram then we would still detect that $J(s, 0) = 0$ at $s = \pi/4$ but we would not have the problem that characteristic projections intersect the data curve in multiple points, hence it is likely there would only be problems on the characteristic projection $y = x$.

Remark: The extreme case of the data failing to be Cauchy is if the data curve is a characteristic projection, i.e. if we can parametrise the initial curve as $\gamma(s) = (\gamma_1, \gamma_2, \gamma_3)(s)$ so that $\gamma_{1,2}$ satisfy the characteristic equations $\gamma_1' = P$ and $\gamma_2' = Q$.

Then, if the initial curve is a characteristic, i.e. if also the 3rd charachteristic equation $\gamma_3' = R$ is satisfied, then there will be an infinity of solutions through γ, while otherwise there will be no solution.

For, if γ is a characteristic, then let C be any curve through γ whose projection is nowhere tangent to a characteristic projection. Then there is a solution surface through C. But this was any C so there is an infinity of solutions. On the other hand, if the data curve is a characteristic projection but the initial curve isn't a characteristic then we can have no solution. Indeed if there was a function f that solves the PDE for which γ is in the solution surface $\Sigma = \text{graph}(f)$ then we'd need to have $\gamma_3 = f(\gamma_1, \gamma_2)$ so by chain rule

$$\gamma_3' = f_x \gamma_1' + f_y \gamma_2' = P f_x + Q f_y = R$$

where the second equality holds as we assumed that (γ_1, γ_2) is a characteristic projection. But this would mean that γ is indeed a characteristic, contradiction.

3.7 Discontinuities in the first derivatives

The characteristic projections have another property. They are the only curves across which the solution surface can have discontinuities in the first derivatives.

For, suppose that $z(x, y)$ is a solution of the PDE (3.1) which is continuously differentiable away from a curve $\alpha_0 = (\alpha_1, \alpha_2)$ in the xy-plane, continuous across α_0 but for which there are discontinuities in the first order partial derivatives as we cross α_0.

We use the superscript $\pm$ to denote the solution on either side of γ and denote the jumps in the partial derivative by $[z_x]_-^+ = z_x^+ - z_x^-$ and $[z_y]_-^+ = z_y^+ - z_y^-$.

As z is continuous across α_0 we know that $z^+(\alpha_1(s), \alpha_2(s)) - z^-(\alpha_1(s), \alpha_2(s)) = 0$. Differentiating this gives gives that

$$[z_x]_-^+ \alpha_1' + [z_y]_-^+ \alpha_2' = 0 \tag{3.18}$$

At the same time, both z^+ and z^- are solutions of the PDE so

$$P z_x^+ + Q z_y^+ = R(x, y, z^+) \text{ and } P z_x^+ + Q z_y^+ = R(x, y, z^-). \tag{3.19}$$

As $z^+ = z^-$ on α the right hand sides agree, so subtracting these equations gives

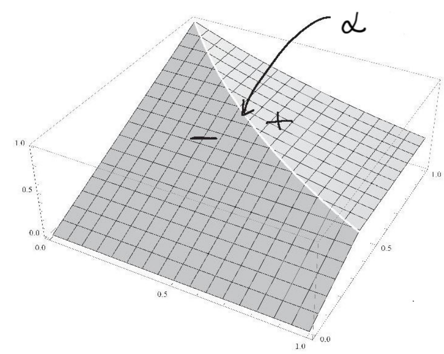
$$0 = P[z_x]_-^+ + Q[z_y]_-^+ \text{ on } \alpha_0. \tag{3.20}$$

The vector $j = ([z_x]_-^+, [z_y]_-^+)$ of the jumps in first derivatives hence solves the homogeneous system of linear equations $\begin{pmatrix} P & Q \\ \alpha_1' & \alpha_2' \end{pmatrix} \cdot j = 0$. So for there to be a non-zero jump, this matrix must be singular, i.e. the rows must be linearly dependent.

As the first row gives the tangent to a characteristic projection, while the second row is the tangent to the curve α_0 across which the derivatives jump, we must have that α_0 is a characteristic projection. So the only curves in the xy-plane across which the first order derivatives of a solution can jump are characteristic projections. In the picture of the solution surface we see this jump in derivative along the curve $\alpha(s) = (\alpha_1(s), \alpha_2(s), z(\alpha_1(s), \alpha_2(s))$, see figure below.

3.8 General Solution

Another problem we could have considered, is what is the most general solution of (3.1)? Just as we expect the most general solution of an ODE to have n arbitrary constants, so we expect the most general solution of a PDE of order n to have n arbitrary functions.



For example: The first order PDE $\frac{\partial z}{\partial x}(x, y) = 0$, has the most general solution $z = f(y)$ where f is an arbitrary function.

4 Second order semi-linear PDEs

4.1 Classification

In this section, we are interested in second-order PDEs of the following form:

$$\underbrace{a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy}}_{\text{principal part}} = f(x,y,u,u_x,u_y). \tag{4.1}$$

This PDE is said to be *linear* if f is linear in u, u_x, u_y, otherwise it is said to *semi-linear*. (If the coefficients a, b, c also depend on u, u_x, u_y it is said to be *quasi-linear*. We will consider only semi-linear equations.) You have seen the following examples in Prelims:

$$\begin{aligned} u_{xx} + u_{yy} = 0 &\quad \text{Laplace's equation} \\ u_{xx} - u_{yy} = 0 &\quad \text{wave equation if } y = ct \\ u_{xx} - u_y = 0 &\quad \text{heat equation if } y = t/\kappa. \end{aligned}$$

Equations that are linear (in the dependent variable) have solutions that can be combined by linear superposition (taking linear combinations). In general PDEs that are nonlinear (for example where f, above, depends nonlinearly on u or its derivatives) do not have solutions that are superposable.

We will assume throughout that functions are suitably differentiable.

4.1.1 The idea:

In this section, the key idea is to change coordinates so as to simplify the principal part. So we make the change of variables

$$(x,y) \to (\varphi(x,y), \psi(x,y));$$

with non vanishing Jacobian (basically this ensures that the map is locally invertible):

$$\frac{\partial(\varphi,\psi)}{\partial(x,y)} = \varphi_x\psi_y - \varphi_y\psi_x \neq 0.$$

We will abuse the notation a little and write (the solution) u as either a function of (x,y) or (φ,ψ) as required.

For the change in the partials, we calculate

$$u_x = u_\varphi\varphi_x + u_\psi\psi_x; \quad u_y = u_\varphi\varphi_y + u_\psi\psi_y$$

then

$$\left.\begin{aligned} u_{xx} &= u_{\varphi\varphi}\varphi_x^2 + 2u_{\varphi\psi}\varphi_x\psi_x + u_{\psi\psi}\psi_x^2 + u_\varphi\varphi_{xx} + u_\psi\psi_{xx} \\ u_{xy} &= u_{\varphi\varphi}\varphi_x\varphi_y + u_{\varphi\psi}(\varphi_x\psi_y + \psi_x\varphi_y) + u_{\psi\psi}\psi_x\psi_y + u_\varphi\varphi_{xy} + u_\psi\psi_{xy} \\ u_{yy} &= u_{\varphi\varphi}\varphi_y^2 + 2u_{\varphi\psi}\varphi_y\psi_y + u_{\psi\psi}\psi_y^2 + u_\varphi\varphi_{yy} + u_\psi\psi_{yy} \end{aligned}\right\} \tag{4.2}$$

so that (4.1) becomes

$$A(\varphi,\psi)u_{\varphi\varphi} + 2B(\varphi,\psi)u_{\varphi\psi} + C(\varphi,\psi)u_{\psi\psi} = F(\varphi,\psi,u,u_\varphi u_\psi) \tag{4.3}$$

with

$$\left.\begin{array}{l} A = a\varphi_x^2 + 2b\varphi_x\varphi_y + c\varphi_y^2 \\ B = a\varphi_x\psi_x + b(\varphi_x\psi_y + \varphi_y\psi_x) + c\varphi_y\psi_y \\ C = a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2. \end{array}\right\} \tag{4.4}$$

(Beware, F will include lower order derivatives from (4.2).) In a matrix notation (4.4) is (check!)

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \varphi_x & \psi_x \\ \varphi_y & \psi_y \end{pmatrix}$$

so that, taking determinants,

$$(AC - B^2) = (ac - b^2)(\varphi_x\psi_y - \psi_x\varphi_y)^2 = (ac - b^2)\left(\frac{\partial(\varphi, \psi)}{\partial(x, y)}\right)^2. \tag{4.5}$$

(We could obtain (4.5) directly from (4.4) but the matrix notation makes the computation simpler.) Now (4.5) leads to a classification of second-order linear PDEs:

4.1.2 The Classification

Second-order linear PDEs are classified into three types as follows:

1. $ac < b^2$ hyperbolic: e.g. wave equation;
2. $ac > b^2$ elliptic: e.g. Laplace equation;
3. $ac = b^2$ parabolic: e.g. heat equation.

So, by (4.5) the class of the equation is invariant under transformations with non-vanishing Jacobian.

We shall look at the classification in terms of the quadratic polynomial

$$a(x, y)\lambda^2 - 2b(x, y)\lambda + c(x, y) = 0. \tag{4.6}$$

Note: We will assume that $a \neq 0$, in the domain under consideration. If $a = 0$ but $c \neq 0$, we can swap the roles of x and y.

Case 1: hyperbolic type

So $ac < b^2$ and the quadratic has distinct real roots λ_1, λ_2, say. So

$$a(x, y)\left(\frac{dy}{dx}\right)^2 - 2b(x, y)\frac{dy}{dx} + c(x, y) = 0. \tag{4.7}$$

is equivalent to

$$\frac{dy}{dx} = \lambda_1(x, y), \qquad \frac{dy}{dx} = \lambda_2(x, y). \tag{4.8}$$

Suppose these equations have solutions $\varphi(x, y)$=constant, $\psi(x, y)$ =constant, respectively. Set as change of variables

$$\varphi = \varphi(x, y), \quad \psi = \psi(x, y).$$

Then, on $\varphi(x, y)$=constant,

$$\varphi_x + \varphi_y \frac{dy}{dx} = 0$$

so that

$$\lambda_1 \varphi_y = -\varphi_x$$

and thus

$$A(\varphi, \psi) = a(x, y)(\varphi_x)^2 + 2b(x, y)\varphi_x \varphi_y + c(x, y)(\varphi_y)^2 = 0,$$

and analogously $C(\varphi, \psi) = 0$. But $\lambda_1 \neq \lambda_2$, so $\frac{\varphi_x}{\varphi_y} \neq \frac{\psi_x}{\psi_y}$, and from (4.5) $B \neq 0$. Divide (4.3) by B to obtain the equation in the form

$$u_{\varphi\psi} = G(\varphi, \psi, u, u_\varphi, u_\psi). \tag{4.9}$$

This is the *normal form* (or *canonical form*) for a hyperbolic equation; the equation (4.7) is the *characteristic equation*; φ, ψ are *characteristic variables*; curves on which φ or ψ are constant are *characteristic curves*. We can often solve (4.9) explicitly.

Examples:

(a)

$$u_{xx} - u_{yy} = 0.$$

We already know how to solve this, but let us apply the method. So

$$a = 1,\ b = 0,\ c = -1, \text{and } \lambda^2 - 1 = 0.$$

We can take

$$\lambda_1 = 1, \quad \lambda_2 = -1$$

and solve (4.8)

$$y'(x) = 1 \quad y'(x) = -1$$

to get

$$\varphi = x - y \quad \psi = x + y.$$

(There is clearly lots of choice at this stage.) The equation has become

$$u_{\varphi\psi} = 0,$$

which we solve at once by

$$u = f(\varphi) + g(\psi),$$

a solution known from Prelims. So the characteristic curves of the wave equation are $x + ct$=const and $x - ct$ =const.

(b) An example with data: solve

$$x u_{xx} - (x + y)u_{xy} + y u_{yy} + \frac{(x + y)}{(y - x)}(u_x - u_y) = 0, \text{ for } y \neq x$$

with

$$u = \frac{1}{2}(x - 1)^2, \quad u_y = 0 \quad \text{on} \quad y = 1.$$

Problem is hyperbolic provided $x \neq y$ (check). The quadratic (4.6) is

$$x\lambda^2 + (x+y)\lambda + y = 0$$
$$= (\lambda+1)(x\lambda + y);$$

so choose

$$\lambda_1 = -1 \quad \lambda_2 = -\frac{y}{x}$$

and solve

$$y'(x) = -1; \quad y'(x) = -\frac{y}{x},$$

by $x + y =$ const; $xy =$const, so put

$$\varphi = x + y; \quad \psi = xy.$$

Calculate

$$u_x = u_\varphi + yu_\psi$$
$$u_y = u_\varphi + xu_\psi$$

so that

$$u_{xx} = u_{\varphi\varphi} + 2yu_{\varphi\psi} + y^2 u_{\psi\psi}$$
$$u_{xy} = u_{\varphi\varphi} + xu_{\varphi\psi} + yu_{\varphi\psi} + xyu_{\psi\psi} + u_\psi$$
$$u_{yy} = u_{\varphi\varphi} + 2xu_{\varphi\psi} + x^2 u_{\psi\psi}.$$

(It is always better to calculate the derivatives directly, rather than trying to remember formulae.)

Now the PDE becomes

$$0 = x[u_{\varphi\varphi} + 2yu_{\varphi\psi} + y^2 u_{\psi\psi}]$$
$$-(x+y)[u_{\varphi\varphi} + (x+y)u_{\varphi\psi}xyu_{\psi\psi} + u_\psi]$$
$$+y[u_{\varphi\varphi} + 2xu_{\varphi\psi} + x^2 u_{\psi\psi}]$$
$$+(x+y)u_\psi$$
$$= (4xy - (x+y)^2)u_{\varphi\psi}$$

so

$$u_{\varphi\psi} = 0$$

and the solution is

$$u = f(\varphi) + g(\psi) = f(x+y) + g(xy).$$

To impose the data, calculate

$$u_y = f'(x+y) + xg'(xy)$$

so on $y = 1$,

$$u = f(x+1) + g(x) = \frac{1}{2}(x-1)^2$$
$$u_y = f'(x+1) + xg'(x) = 0.$$

Differentiate the first:

$$f'(x+1) + g'(x) = x - 1$$

and solve simultaneously with the second:

$$g'(x) = -1,$$

and integrate to find

$$g(x) = -x + c.$$

Substitute back in $u(x, 1)$:

$$f(x+1) = \frac{1}{2}(x-1)^2 + x - c = \frac{1}{2}(x+1)^2 - x - c,$$

so

$$f(x) = \frac{1}{2}x^2 - x + 1 - c.$$

Finally

$$u = f(x+y) + g(xy) = \frac{1}{2}(x+y)^2 - (x+y) + 1 - xy.$$

Sketch of analysis for Cases 2 and 3: Elliptic and parabolic type

In these situations we can similarly change variables to get to the normal form, but as this rarely results in equations that we can explicitly solve we only sketch the corresponding argument:

In the elliptic case where $ac > b^2$ the equation (4.6) has a complex conjugate pair of roots, and the integral curves of

$$y'(x) = \lambda(x, y); \quad y'(x) = \bar{\lambda}(x, y)$$

are in complex conjugate pairs, $\varphi(x, y)$ =const; $\psi(x, y) = \bar{\varphi}(x, y)$ =const. In these complex coordinates we get $A = C = 0$, $B \neq 0$ as above so the equation becomes

$$u_{\varphi\bar{\varphi}} = G(\varphi, \bar{\varphi}, u, u_\varphi, u_{\bar{\varphi}}).$$

Introduce new variables, ζ, η, given by $\varphi = \zeta + i\eta$, $\bar{\varphi} = \zeta - i\eta$, to obtain the normal form for an elliptic equation (check):

$$u_{\zeta\zeta} + u_{\eta\eta} = H(\zeta, \eta, u, u_\zeta, u_\eta), \tag{4.10}$$

which closely resembles Laplace's equation.

In the parabolic case $ac = b^2$ the equation (4.6) has a repeated root $\lambda(x, y)$. Solving $y'(x) = \lambda(x, y)$ gives one new coordinate φ, and one can then pick any ψ with

$$\varphi_x \psi_y - \varphi_y \psi_x \neq 0 \tag{4.11}$$

as the other. This gives then $A = 0$ so $B^2 = AC = 0$, while $C \neq 0$, as ψ =const is not a characteristic curve by (4.11), so we get the normal form for a parabolic equation:

$$u_{\psi\psi} = G(u, \varphi, \psi, u_\varphi, u_\psi).$$

A typical example of a parabolic equation is the heat equation $u_{\psi\psi} = u_\varphi$.

Example:

Classify and reduce to normal form the equation

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0 \text{ for } x > 0. \tag{4.12}$$

The relevant quadratic is

$$x^2\lambda^2 - 2xy\lambda + y^2 = 0 = (x\lambda - y)^2$$

which has equal roots, so this equation is *parabolic*; $\lambda = \frac{y}{x}$ so solve

$$\frac{dy}{dx} = \frac{y}{x}, \text{ to get, for example, } \varphi = \frac{y}{x}$$

and take, for example, $\psi = x$. Calculate

$$u_x = -\frac{y}{x^2}u_\varphi + u_\psi$$

$$u_y = \frac{1}{x}u_\varphi$$

so that

$$u_{xx} = \frac{y^2}{x^4}u_{\varphi\varphi} - 2\frac{y}{x^2}u_{\varphi\psi} + u_{\psi\psi} + 2\frac{y}{x^3}u_\varphi$$

$$u_{xy} = -\frac{y}{x^3}u_{\varphi\varphi} + \frac{1}{x}u_{\varphi\psi} - \frac{1}{x^2}u_\varphi$$

$$u_{yy} = \frac{1}{x^2}u_{\varphi\varphi}.$$

The equation becomes

$$x^2\left[\frac{y^2}{x^4}u_{\varphi\varphi} + \frac{2y}{x^2}u_{\varphi\psi} + u_{\psi\psi} + \frac{2y}{x^3}u_\varphi\right] + \\ + 2xy\left[-\frac{y}{x^3}u_{\varphi\varphi} + \frac{1}{x}u_{\varphi\psi} - \frac{1}{x^2}u_\varphi\right] + y^2\left[\frac{1}{x^2}u_{\varphi\varphi}\right] = x^2u_{\psi\psi} = 0 \tag{4.13}$$

so the normal form is

$$u_{\psi\psi} = 0$$

with general solution $u = F(\varphi) + \psi G(\varphi)$. In terms of the original variables this is:

$$u = F\left(\frac{y}{x}\right) + xG\left(\frac{y}{x}\right). \tag{4.14}$$

NB Very often, a question like this will be phrased in the form 'Classify and reduce to normal form the equation (4.12) and show that the general solution can be written as (4.14)'. Therefore candidates for φ and ψ are proposed by the question itself.

A warning example:

The type can change e.g. classify the equation

$$u_{xx} + yu_{yy} = 0.$$

Then

$$\lambda^2 + y = 0, \quad \lambda^2 = -y,$$

and this is:

- elliptic in $y > 0$,
- parabolic at $y = 0$,
- hyperbolic in $y < 0$.

4.2 Characteristics:

The characteristics of second order semi-linear PDEs have analogous properties to the characteristic *projections* of first order semi-linear PDEs. (Note the difference in terminology.)

Firstly, if there are discontinuities in the second derivatives of a solution across a given curve, then that curve must be a characteristic curve. To see this, suppose that the curve Γ, given parametrically by $(x(s), y(s))$, is a curve across which there are discontinuities in the second derivatives of the solution. Let u^+_{xx} etc denote values on one side of Γ and u^-_{xx} denote values on the other side of Γ. Then differentiating $u_x(x(s), y(s))$ and $u_y(x(s), y(s))$ along Γ, and noting that u, u_x, u_y are continuous across the curve,

$$\begin{aligned} \frac{du_x}{ds} &= \frac{dx}{ds}u^{\pm}_{xx} + \frac{dy}{ds}u^{\pm}_{xy} \\ \frac{du_y}{ds} &= \frac{dx}{ds}u^{\pm}_{yx} + \frac{dy}{ds}u^{\pm}_{yy} \\ \text{and also } f(x, y, u, u_x, u_y) &= a(x, y)u^{\pm}_{xx} + 2b(x, y)u^{\pm}_{xy} + c(x, y)u^{\pm}_{yy}. \end{aligned}$$

Subtracting the 'minus' equation from the 'plus' equation

$$\begin{aligned} 0 &= \frac{dx}{ds}[u_{xx}]^+_- + \frac{dy}{ds}[u_{xy}]^+_- \\ 0 &= \frac{dx}{ds}[u_{yx}]^+_- + \frac{dy}{ds}[u_{yy}]^+_- \\ 0 &= a(x, y)[u_{xx}]^+_- + 2b(x, y)[u_{xy}]^+_- + c(x, y)[u_{yy}]^+_-, \end{aligned}$$

where $[u_{xx}]^+_- = u^+_{xx} - u^-_{xx}$ denotes the jump in u_{xx} across Γ, etc. If there are to be discontinuities in the second derivatives, then this set of equations in the jumps must have a nonzero solution, so that the determinant of the coefficients must be zero. Thus

$$a(x, y)\left(\frac{dy}{ds}\right)^2 - 2b(x, y)\frac{dx}{ds}\frac{dy}{ds} + c(x, y)\left(\frac{dx}{ds}\right)^2 = 0, \tag{4.15}$$

so Γ is a characteristic.

Furthermore, under suitable smoothness conditions, the Cauchy problem for a second order semi-linear PDE, where u and $\frac{\partial u}{\partial n}$ are given along a curve Γ, will have a unique local solution provided Γ is nowhere tangent to a characteristic curve. This result is beyond the scope of this course and will be investigated further in the Part B course, Applied PDEs. It can be seen that it is necessary that Γ is not a characteristic curve, as if u exists then it must have unique second order partial derivatives along Γ and exactly as above this can only be true when Γ is not a characteristic curve.

Remark: Our previous work carried the implicit assumption that $a \neq 0$. Note that (4.15) gives a method of calculating the characteristic curves if $a = 0$. In particular if $a = 0$ and $c = 0$ then the characteristic curves are $x =$const, $y =$const.

4.3 Type and data: well posed problems

We want to say something about the notion of *well posedness* and its connection with type. Our examples are mostly based on knowledge acquired in Prelims.

A problem, consisting of a PDE with data, is said to be *well posed* if the solution:

- exists
- is unique
- depends *continuously* on the data.

Recall that, in Section 1.6, we said that a solution of a DE is *continuously dependent on the data* if the error in the solution is small provided the error in the initial data is small enough.. We then gave a precise definition for ODEs. We now want to extend this definition to PDEs. Data can be given in different ways, so to be precise we will consider a problem where $u(x, y)$ is the solution of a certain PDE in a bounded subset of the plane D, with u given on some curve Γ. Then we will say that the solution depends continuously on the data if :

$\forall \epsilon > 0\ \exists \delta > 0$ such that if u_i, $i = 1, 2$, are solutions with $u_i = f_i$ on Γ then

$$\sup_{\Gamma} |f_1 - f_2| < \delta \quad \Rightarrow \quad \sup_{D} |u_1 - u_2| < \epsilon.$$

The definition extends in a fairly obvious way to other types of data. (Note that there are plenty of other 'distances ' we could use in place of taking the sup, but that is what we will use here.)

Of the three requirements for well posedness it is existence which is the hardest to obtain. In Prelims solutions were found for a number of linear problems, either by making a change of variables and then integrating, or by using separation of variables and Fourier series. Anything more than this is beyond the scope of this course. Uniqueness of solution was also proved for a number of linear problems (even when you didn't know if the solution existed). Proving uniqueness and continuous dependence on the data is much easier for linear problems as we can then start out by looking at the difference between two solutions, which will then be the solution of some suitable problem. Later we will look at the linear equations Poisson's equation and the heat equation and state and prove the maximum principle, which will enable us to prove uniqueness and continuous dependence for suitable boundary data.

But first we need to consider what data might be appropriate. In Prelims you considered three particular PDEs, each with a different type of data which arose from the particular physical problem they modelled. These are summarised in the table below. It turns out that there are mathematical as well as physical reasons why each problem had a different type of data. So first we will look at some of these problems and consider which may be well posed.

Some Models from Prelims:

	PDE	Models	Boundary conditions	
Wave Equation: Hyperbolic	$c^2 u_{xx} - u_{tt} = 0$	Waves on a string; u gives displacement	u, $\frac{\partial u}{\partial t}$ given $t = 0$ $\frac{\partial u}{\partial t}$ given $t = 0$; plus end point condition - say $u = 0$ at ends	(IBVP) (Infinite string) (IVP) (Finite string)
Laplace's Equation: Elliptic	$u_{xx} + u_{yy} = 0$, $(x, y) \in D$	Potential Theory or Steady Heat; u gives potential or temperature	u or $\frac{\partial u}{\partial n}$ given on boundary of D	(BVP)
Heat Equation: Parabolic	$u_t = \kappa u_{xx}$	Heat; u gives temperature	u given at $t = 0$; plus endpoint conditions - u or $\frac{\partial u}{\partial x}$ given at ends	(IBVP) (Temperature in finite bar)

Some examples from Prelims: (We will assume that the data is smooth enough for the following to hold.)

(a) Hyperbolic equation: The IVP and IBVP (initial-boundary-value problem) for the wave equation

$$u_{xx} - u_{yy} = 0 \quad (ct = y).$$

For the IVP (modelling an infinite string, where u is the displacement), we know the solution is

$$u = \frac{1}{2}\left[f(x+y) + f(x-y)\right] + \frac{1}{2}\int_{x-y}^{x+y} g(s)ds, \tag{4.16}$$

where the data are $u(x,0) = f(x)$ and $u_y(x,0) = g(x)$, $-\infty < x < \infty$. This is d'Alembert's solution of the IVP: it exists, and is unique and, intuitively at least, a small change in φ, ψ gives a small change in u (see problem sheet for proof). So this problem is well-posed.

For the IBVP (modelling a finite string length L, fixed at each end) consider the data:

$$u(x,0) = f(x), \quad u_y(x,0) = g(x) \quad 0 < x < L$$

$$u(0,y) = 0 = u(L,y).$$

So the boundaries are at $x = 0, L$. This IBVP is solved using separation of variables and Fourier series to get a solution

$$u = \sum_n \sin\frac{n\pi x}{L}\left(a_n \cos\frac{n\pi y}{L} + b_n \sin\frac{n\pi y}{L}\right)$$

with $a_n,\ b_n$ given in terms of f and g. Uniqueness was shown in Prelims, so this is the unique solution. If we now appeal to intuition for continuous dependence on the data this problem is well-posed.

(b) Elliptic equation:The BVP for the Laplace/Poisson's equation (modelling steady state heat, for example, where u is the temperature)

$$u_{xx} + u_{yy} = 0.$$

Do this first with data at the sides of a square, so $0 \le x, y \le a$ with

$$u(0,y) = u(a,y) = u(x,0) = 0;\ u(x,a) = f(x).$$

Consider separable solutions $u_n = \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$, then

$$u = \sum_n a_n \frac{\sinh\frac{n\pi y}{a}}{\sinh(n\pi)}\sin\frac{n\pi x}{a}$$

and

$$f(x) = \sum a_n \ \sin\frac{n\pi x}{a}$$

which determines the solution as a Fourier series.

Now a different BVP, with data at the circumference of the unit circle:

$$\text{on } r = 1,\ u = f(\theta)$$

and in polars

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

The separable solutions are

$$\begin{cases} \left(Ar^n + \frac{B}{r^n}\right)(C\cos n\theta + D\sin n\theta) \\ A + B\log r, \quad n = 0 \end{cases}$$

Regularity at $r = 0$ implies

$$u = \frac{1}{2}a_0 + \sum_1^\infty r^n(a_n \cos n\theta + b_n \sin n\theta)$$

and the boundary value at $r = 1$ requires

$$\frac{1}{2}a_0 + \sum(a_n \cos n\theta + b_n \sin n\theta) = f(\theta)$$

which again is solved by Fourier methods.

Uniqueness was proved in Prelims so in each of these cases we have the unique solution. It is plausible, but beyond our scope, to show that there is existence of solution in general. Later we will prove that there is continuous dependence on the data. So this problem is well posed

(c) Parabolic equation: The IBVP for the heat equation (modelling heat flow in a bar length L, with the ends held at zero temperature, u is temperature.)

$$u_{xx} = u_y$$

on the semi-infinite strip where $y = t > 0$ and $0 < x < L$, and data

$$u(x, 0) = f(x);\ u(0, y) = 0 = u(L, y).$$

The relevant separable solutions are

$$u_n = \sin\frac{n\pi x}{L}e^{-\frac{n^2\pi^2 t}{L^2}}$$

so that

$$u = \sum_n a_n \sin\frac{n\pi x}{L}e^{-\frac{n^2\pi^2}{L^2}t}$$

and the initial value requires

$$f(x) = \sum a_n \sin\frac{n\pi x}{L}$$

which is solved by Fourier methods. The solution exists *provided the series for u converges* which it will do for positive t. However, note that for negative t the exponentials grow rapidly with n and there is no reason to expect existence.

Uniqueness for this problem was done in Prelims, so again this is the unique solution for positive t. Later we will prove continuous dependence on the data. Thus the problem is well posed *forward in time.*

(d) What then is not well-posed? We give a few examples:

- BVPs for hyperbolic
  e.g. $u_{xx} - u_{yy} = 0$ on the unit square with data

$$u(0, y) = u(1, y) = u(x, 0) = 0;\ u(x, 1) = f(x).$$

  Recall this data gave a well-posed problem for the Laplace equation, but here if $f = 0$, then $\sin n\pi x \sin n\pi y$ will do, for any n, while it can be proved that there is no solution at all if $f \neq 0$ (try the Fourier series to see what goes wrong).

- IVPs for elliptic
  e.g. $u_{xx} + u_{yy} = 0$ in the horizontal strip $0 \leq y \leq Y$, $-\infty < x < \infty$, with data

$$u(x, 0) = 0,\ u_y(x, 0) = f(x).$$

  We know (from the problem sheet) that this data gives continuous dependence on the data for the wave equation. But not for Laplace's equation. For if $f(x) = \frac{1}{n} \sin nx$ it can be seen that $u(x, y) = \frac{1}{n^2} \sinh ny \sin nx$. But $\sup |\frac{1}{n^2} \sinh ny \sin nx| \to \infty$ as $n \to \infty$, whereas $1/n \sin nx \to 0$. Thus small changes in the initial data can lead to large changes in the solution. [More precisely: Suppose that there is continuous dependence on the initial data about the zero solution. That is: $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$\sup_{x \in \mathbb{R}} |f(x) - 0| < \delta \Rightarrow \sup_{x \in \mathbb{R}, 0 \leq y \leq Y} |u(x, y) - 0| < \epsilon.$$

  But taking $\epsilon = 1$, say, there exists N such that for all $n > N$, $\sup |\frac{1}{n^2} \sinh ny \sin nx| > 1$, but for any $\delta > 0$ we can choose $n > N$ such that $|\frac{1}{n} \sin nx| < \delta$, giving a contradiction.

- IBVP for elliptic
  e.g. $u_{xx} + u_{yy} = 0$ on the semi-infinite strip $0 \leq x \leq 1$, $y \geq 0$, with data

$$u(0, y) = u(1, y) = 0,\ u(x, 0) = 1,\ u_y(x, 0) = 0.$$

  This data gives a well-posed problem for the wave equation. If we try for separable solutions here, we have $u_n = \sin n\pi x \cosh n\pi y$ so

$$u = \sum_n a_n \sin n\pi x \cosh n\pi y.$$

  Initial conditions need

$$1 = \sum a_n \sin n\pi x$$

  whence

$$\begin{aligned} a_n &= 0 \quad n \text{ even} \\ &= \frac{4}{n\pi} \quad n \text{ odd}, \end{aligned}$$

  and then

$$u\left(\frac{1}{2}, y\right) = \sum_n \frac{4}{(2n+1)\pi} (-1)^n \cosh(2n+1)\pi y,$$

  which does not converge for any $y > 0$ (because the "cosh" terms grow rapidly with n) - there is no solution (strictly speaking, we've only shown that there is no solution of the form considered; we need more).

- The BVP for the heat equation is not well-posed, but we won't show that.

Again, it is beyond our scope to prove it in this course, but these different behaviours are universal for the different types of second-order, linear PDEs. In tabulated form, which problems are well-posed?

	IVP	IBVP	BVP
Hyperbolic	yes	yes	no
Elliptic	no	no	yes
Parabolic	yes	yes	no

where the 'yes' for parabolic equations are only valid forward in time.

4.4 The Maximum Principle

4.4.1 Poisson's equation

The normal form for second-order elliptic PDEs is

$$u_{xx} + u_{yy} = f(x, y, u, u_x, u_y) \tag{4.17}$$

The operator on the left-hand side is referred to as the Laplacian, for which the symbols $\nabla^2 u$ and Δu are often used as shorthand. Poisson's equation is a special case of (4.17), in which f depends only on x and y. We have already seen that appropriate boundary data for (4.17) is to give just one boundary condition on u everywhere on a closed curve.

We will consider the Dirichlet problem where u is given on the boundary of D:

$$u_{xx} + u_{yy} = f(x, y) \quad \text{in } D \tag{4.18}$$

$$u = g(x, y) \quad \text{on } \partial D. \tag{4.19}$$

It was shown in Prelims, using the divergence theorem, that if a solution exists, then it is unique. Using the maximum principle we will give another proof of this and also show that there is continuous dependence on the data. The solution does exist, but apart from the particular cases considered in Prelims that is beyond the scope of this course. Existence of solutions of general elliptic problems will be considered in the Part C courses Functional Analytic Methods for PDEs and Fixed point methods for nonlinear PDEs.

Theorem 4.1. *(The Maximum principle for the Laplacian) Suppose u satisfies*

$$\Delta u := u_{xx} + u_{yy} \geq 0 \quad (x, y) \in D, \tag{4.20}$$

everywhere within a bounded domain D. Then u attains its maximum value on ∂D.

Remark: This Theorem of course applies to the Poisson equation where we ask that $u_{xx} + u_{yy}$ is given by a prescribed function f, but is equally applicable to get information for non-linear problems, such as solutions of the equation $u_{xx} + u_{yy} = u^2$ for which we know that the right hand side has a given sign.

Remark: As u is a continuous function on the set $\bar{D}$ which is a closed and bounded subset of $\mathbb{R}^2$ and thus compact, we know that u achieves its maximum in some point $p \in \bar{D}$. The above theorem now tells us that this maximum value will indeed always be achieved on the boundary, though does not exclude that the maximum is also achieved at further points which might be in the interior.

In fact however the so called *strong maximum principle* (which is off syllabus) asserts that a function u with $\Delta u \geq 0$ cannot have an interior maximum unless it is constant.

Proof:

If we denote the boundary of D by ∂D , then as $D \cup \partial D$ is a closed bounded set and thus compact, u must attain its maximum somewhere in D or on its boundary. The proof now proceeds in two parts.

Suppose first that $u_{xx} + u_{yy} > 0$ in D.

If u has an interior maximum at some point (x_0, y_0) inside D, then the following conditions must be satisfied at (x_0, y_0):

$$u_x = u_y = 0, \quad u_{xx} \leq 0, \quad u_{yy} \leq 0.$$

But, as we assumed that $u_{xx} + u_{yy} > 0$ in all of D it is impossible for both u_{xx} and u_{yy} to be non positive . Hence u cannot have an interior maximum within D. so it must attain its maximum value on the boundary ∂D.

Suppose now that we only have $u_{xx} + u_{yy} \geq 0$ in D. We perturb u to get a function v which satisfies $v_{xx} + v_{yy} \geq 0$, so we can apply the first part of the proof.

Consider the function

$$v(x, y) = u(x, y) + \frac{\epsilon}{4}(x^2 + y^2),$$

where ϵ is a positive constant.

Then

$$v_{xx} + v_{yy} = u_{xx} + u_{yy} + \epsilon > 0$$

in D. So using the result just proved, v attains its maximum value on ∂D.

Now, suppose that the maximum value of u on ∂D is M and the maximum value of $(x^2 + y^2)$ on ∂D is R^2 , then the maximum value of v on ∂D (and thus throughout D) is $M + (\epsilon/4)R^2$. In other words, the inequality

$$u + \frac{\epsilon}{4}(x^2 + y^2) = v \leq M + \frac{\epsilon}{4}R^2$$

holds for all $(x, y) \in D$. Letting $\epsilon \to 0$, we see that $u \leq M$ throughout D, i.e. that u attains its maximum value on ∂D. □

It obviously follows (by using the above result with u replaced by $-u$) that, if $\Delta u \leq 0$ in D, then u attains its minimum value on ∂D.

In the case $\Delta u = 0$, u therefore attains both its maximum and minimum values on ∂D. This is an important property of *Laplace's equation.*

Corollary 4.2. *(a) Consider the Dirichlet problem* (4.18), (4.19). *Then if the solution exists, it is unique.*

(b) The Dirichlet problem (4.18), (4.19) *has continuous dependence on the data.*

Proof: (a) Suppose that u_1, u_2 are two solutions, so $u = u_1 - u_2$ satisfies

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \quad \text{in } D & (4.21) \\ u &= 0 \quad \text{on } \partial D. & (4.22) \end{aligned}$$

By (4.21) the maximum and minimum of u occur on ∂D, so by (4.22) $u \leq 0$ and $u \geq 0$ in D. Thus $u = 0$ in D as required.

(b) We have to prove that for all $\epsilon > 0$ there exists $\delta > 0$ such that if u_i, $i = 1, 2$ are solutions with boundary data g_i, then

$$\sup_{(x,y)\in\partial D} |g_1(x,y) - g_2(x,y)| < \delta \quad \Rightarrow \quad \sup_{(x,y)\in D} |u_1(x,y) - u_2(x,y)| < \epsilon.$$

By linearity $u = u_1 - u_2$ satisfies

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \text{ in } D & (4.23) \\ u &= g_1 - g_2, \text{ on } \partial D. & (4.24) \end{aligned}$$

We now apply the maximum principle to see that $u \leq \max_{(x,y)\in\partial D}(g_1 - g_2)$, and applying the same result to $-u$, $-u \leq \max_{(x,y)\in\partial D} -(g_1 - g_2)$.

Hence in D

$$|u_1 - u_2| \leq \max_{(x,y)\in\partial D} |g_1 - g_2|, \tag{4.25}$$

so we may take $\delta = \epsilon$. □

4.4.2 The heat equation

In parabolic PDEs it is usually the case that one independent variable represents time, so we now use x and t as independent variables instead of x and y. The normal form for second-order parabolic equations is

$$u_{xx} = F(x, t, u, u_t, u_x)$$

and specific examples include the inhomogeneous heat equation, often called the diffusion equation:

$$u_t = u_{xx} + f(x,t).$$

and the reaction-diffusion equation

$$u_t = u_{xx} + f(x,t,u),$$

Well posed boundary data: Typical boundary data for a diffusion equation are to give an initial condition for u at $t = 0$ and one boundary condition on each of two curves C_1 and C_2 in the (x,t)-plane that do not meet and are nowhere parallel to the x-axis.

For example: The inhomogeneous heat equation

$$u_t = u_{xx} + f(x,t)$$

is a simple model for the temperature $u(x,t)$ in a uniform bar of conductive material, with heat source $f(x,t)$, where x is position and t is time. Suppose the bar is of length L, its initial temperature is given via $u_0(x)$, and its ends are kept at zero temperature. Then the initial and boundary conditions are

$$u = u_0(x) \text{ at } t = 0, \quad u = 0 \text{ at } x = 0; \; u = 0 \text{ at } x = L.$$

If, instead of being held at constant temperature, an end is insulated, then the Dirichlet boundary condition, $u = 0$, there is replaced by the Neumann boundary condition, $u_x = 0$. Alternatively , the boundary conditions at $x = 0$ and $x = L$ may, in general, be replaced by conditions at moving boundaries, say $x = x_1(t)$ and $x = x_2(t)$.

Theorem 4.3. *(The Maximum principle for the heat equation) Suppose that $u(x, t)$ satisfies*

$$u_t - u_{xx} \leq 0 \tag{4.26}$$

in a region D_τ bounded by the lines $t = 0$, $t = \tau > 0$, and two non-intersecting smooth curves C_1 and C_2 that are nowhere parallel to the x-axis. Suppose also that $f \leq 0$ in D_τ. Then u takes its maximum value either on $t = 0$ or on one of the curves C_1 or C_2 .

Proof:

The proof is similar to that for Poisson's equation.

We first observe that since u is a continuous function on a compact set $\bar{D}_\tau$ it will achieve its maximum on $\bar{D}_\tau$.

Suppose first that $u_t - u_{xx} < 0$ in D_τ. At an internal maximum inside D_τ, u must satisfy

$$u_x = u_t = 0 \; u_{xx} \leq 0, \; (u_{tt} \leq 0).$$

On the other hand, if u has a maximum at a point on $t = \tau$, then there it must satisfy

$$u_x = 0, \; u_t \geq 0, u_{xx} \leq 0.$$

With $u_t - u_{xx}$ assumed to be strictly negative, both of these lead to contradictions, and it follows that u must take its maximum value somewhere on ∂D_τ but not on $t = \tau$. We are done.

Suppose now that $u_t - u_{xx} \leq 0$, then define

$$v(x, t) = u(x, t) + \frac{\epsilon}{2} x^2,$$

where ϵ is a positive constant. Then v satisfies

$$v_t - v_{xx} = u_t - u_{xx} - \epsilon < 0$$

in D_τ. So by the earlier step v takes its maximum value on ∂D_τ but not on $t = \tau$.

Now if the maximum value of u over these three portions of ∂D_τ is M, and the maximum value of $|x|$ on C_1 and C_2 is L, then

$$u \leq v \leq \frac{L^2 \epsilon}{2} + M.$$

Now we let $\epsilon \to 0$ and conclude that $u \leq M$, i.e. u takes its maximum value on ∂D_τ but not on $t = \tau$ □

If $u_t - u_{xx} \geq 0$ in D_τ, then a similar argument shows that u attains its minimum value on ∂D_τ but not on $t = \tau$. Thus, for the homogeneous equation (the heat equation) u attains both its maximum and its minimum values on ∂D_τ but not on $t = \tau$.

Remark: Physical interpretation: In a rod with no heat sources the hottest and the coldest spot will occur either initially or at an end. (Because heat flows from a hotter area to a colder area.)

Corollary 4.4. *Consider the IBVP consisting of* (4.26) *in* D_τ *with* u *given on* $\partial D_\tau \backslash \{t = \tau\}$*. Then if the solution exists, it is unique and depends continuously on the initial data.*

The proof works like he one for the Poisson's equation and is part of problem sheet 4.

5 Where does this course lead?

The course leads to DEs2, where, among other topics, boundary value problems for ODEs are discussed. Further discussion of differential equations comes in the Part B courses 'Non-linear Systems' and 'Applied Partial Differential Equations'. The use of abstract methods such as the Contraction Mapping Theorem to investigate the solutions of differential equations is taken further in various C4 courses, which require some knowledge of Banach and Hilbert spaces.

The techniques taught in DEs1 and DEs2 will be useful in various applied maths courses such as the Part A short course 'Modelling in Mathematical Biology' and the Part B courses 'Mathematical Ecology and Biology', 'Viscous Flow' and 'Waves and Compressible Flow'.

5.1 Section 1

Fixed point results such as the CMT provide very powerful methods for proving existence of solution for ODEs and PDEs. For PDEs we have to work in Banach spaces of functions rather than $\mathbb{R}^n$. For example for parabolic equations such as the reaction-diffusion equation:

$$u_t = u_{xx} + f(t, u),$$

with suitable boundary data, our proof of Picard's theorem can be extended to prove local existence (in t), provided f is continuous and is Lipschitz in u, where for each time t the solution u lives within a Banach space of x-dependent functions (which is infinite dimensional space) rather than the Euclidean n-dimensional space that we considered previously for ODEs. The technical details of this require methods from Functional Analysis as covered in the courses B4,1, B4.2 and C4.1 courses, but the basic ideas are just the same - there is just more to do, because more can go wrong!

An example of a reaction diffusion equations which occurs in applications is Fisher's equation:

$$u_t = u_{xx} + u(1 - u),$$

Another example of a second order parabolic equation is the Black-Scholes equation in mathematical finance.

Existence theorems play an important role in the theory of PDEs, which is a large and active field of current research, and you will be able to learn more about this in the Part C courses on Functional Analytic methods of PDEs and on Fixed Point Methods for Nonlinear PDEs.

5.2 Section 2

Phase plane analysis is a very important tool in mathematical modelling. It will be used, for example, in the Part A short course 'Modelling in Mathematical Biology' and the Part B course, 'Mathematical Ecology and Biology'.

The theory will be taken further in the Part B course, 'Nonlinear Systems'.

5.3 Section 3

We have considered only semi-linear first order PDEs. Similar methods can be extended to quasi-linear and fully non-linear equations. In these cases the characteristic equations are generally more difficult to solve. Such equations allow for the formation of shocks - see Part B 'Applied PDEs'.

These first order PDEs model many physical processes, including particularly conservation laws, and will appear in many of the modelling courses in Part B and beyond.

There are other methods for producing explicit solutions of PDEs:

Transform methods are very useful for linear PDEs- see the Part A short course in HT.

Similarity solutions can be used for both linear or non-linear PDEs - see Part B course 'Applied PDEs' - and involve reducing the PDE to an ODE in a 'similarity variable' involving both independent variables in the PDE.

5.4 Section 4

As we have already observed, proving the existence of solution, even of semi-linear second order PDEs is challenging. The different types of equation demand different approaches. For example:

Semi-linear hyperbolic equations with the solution u and its normal derivative prescribed on a given initial curve: One method to prove existence of solutions proceeds by showing that on initial curves other than characteristic curves we can find all the derivatives of u and that if all coefficients etc in the equation are very smooth (analytic) the solution is given by a power series near the initial curve. This is a version of the Cauchy-Kowalevski theorem.

Elliptic equations are treated in the Part C course 'Functional analytic methods for PDEs'.

A Appendix(off syllabus): An alternative proof of Picard

A.1 Picard's Theorem via the contraction mapping theorem

We can prove Picard's theorem in a more efficient way by using the contraction mapping theorem (CMT). This is a very useful method of proving existence and uniqueness of solutions of nonlinear differential equations and many, many other things besides. The results we need will be discussed in the course on Metric Spaces and Complex Analysis. We will assume the results proved there.

Define $\mathcal{C}_{h,k} = \mathcal{C}([a-h, a+h]; [b-k, b+k])$, the space of continuous functions $y : [a-h, a+h] \to [b-k, b+k]$. As is shown in the Metric Spaces course, for $y, z \in \mathcal{C}_{h,k}$ if we define

$$d(y, z) := ||y - z||_{\text{sup}} := \sup_{x \in [a-h, a+h]} |y(x) - z(x)|$$

then $(\mathcal{C}_{h,k}, d)$ is a complete metric space (we call $|| \cdot ||_{\text{sup}}$ the "sup norm").

Also we say that a map $T : \mathcal{C}_{h,k} \to \mathcal{C}_{h,k}$ is a contraction if there exists $K < 1$ such that

$$||T(y) - T(z)||_{\text{sup}} \leq K||y - z||_{sup},$$

and then we have the CMT, which says:

Theorem A.1. (Contraction Mapping Theorem) *(Banach) Let X be a complete metric space and let $T : X \to X$ be a contraction. Then there is a unique fixed point $y \in X$, i.e. a unique y such that $Ty = y$.*

To prove Picard's Theorem via the CMT we will first apply this theorem for $X = \mathcal{C}_{\eta,k} = \mathcal{C}([a-\eta, a+\eta]; [b-k, b+k])$ for a small enough $0 < \eta \leq h$ that we chose below, which will give that there exists a unique solution for $|x-a| \leq \eta$. In a second step we will then discuss how this solution can be extended to all of $[a-h, a+h]$ if $Mh \leq k$ by repeating the argument with a new choice of the space X.

We again consider the IVP (1.1)

Theorem A.2. (Picard's existence theorem.) *Let $f : R \to \mathbb{R}$ be a function defined on the rectangle $R := \{(x, y) : |x-a| \leq h,\ |y-b| \leq k\}$ which satisfies conditions* **P(i)***(a) and* **P(ii)** *and let $\eta > 0$ be so that $L\eta < 1$ and $M\eta \leq k$.*

Then the initial value problem (1.1) has a unique solution for $x \in [a-\eta, a+\eta]$.

Proof.

The strategy is to express (1.1) as a fixed point problem and use the CMT.

As before, we can write the initial value problem as an integral equation

$$y(x) = b + \int_a^x f(s, y(s))ds \tag{A.1}$$

If we define

$$(Ty)(x) = b + \int_a^x f(s, y(s))ds$$

then we can write (A.1) as a fixed point problem

$$y = Ty.$$

We will work in the complete metric space $\mathcal{C}_{\eta,k} = \mathcal{C}([a-\eta, a+\eta]; [b-k, b+k])$, where we will choose $\eta \leq h$ so that $T : \mathcal{C}_{\eta,k} \to \mathcal{C}_{\eta,k}$ and so that T is a contraction. We begin by proving

Claim 1: If $\eta > 0$ is so that $M\eta \leq k$ then $T : \mathcal{C}_{\eta,k} \to \mathcal{C}_{\eta,k}$

Proof. First we note that from the properties of integration, $(Ty)(x) \in \mathcal{C}([a-\eta, a+\eta]; \mathbb{R})$. All that we require is thus to show that $||Ty - b||_{\sup} \leq k$ if $||y - b||_{\sup} \leq k$.

But

$$\begin{aligned} ||Ty - b||_{\sup} &= \sup_{x \in [a-\eta, a+\eta]} \left| \int_a^x f(s, y(s))ds \right| && \text{(A.2)} \\ &\leq \sup_{x \in [a-\eta, a+\eta]} \left| \int_a^x |f(s, y(s))|ds \right| && \text{(A.3)} \\ &\leq M\eta \leq k, && \text{(A.4)} \end{aligned}$$

provided $M\eta \leq k$. □

Claim 2: If $L\eta < 1$ then T is a contraction (with $K = L\eta$):

Proof. Given $y, z \in C_{\eta,k}$ we can bound

$$\begin{aligned} ||Ty - Tz||_{\sup} &= \sup_{x\in[a-\eta,a+\eta]} \left| \int_a^x f(s, y(s)) - f(s, z(s)) ds \right| \\ &\leq \sup_{x\in[a-\eta,a+\eta]} \left| \int_a^x |f(s, y(s)) - f(s, z(s))| ds \right| \\ &\leq \sup_{x\in[a-\eta,a+\eta]} \left| \int_a^x L|y(s) - z(s)| ds \right| \leq L\eta ||y - z||_{\sup} \leq K||y - z||_{\sup} \end{aligned}$$

where $K := \eta L < 1$ provided $\eta < 1/L$. □

If we hence choose $\eta < \min\{h, k/M, 1/L\}$ then T satisfies the conditions of the CMT and has a unique fixed point, $y(x)$. As explained before, a (continuous) function y solves the integral equation $Ty = y$ if and only if it is continuously differentiable and a solution of the initial value problem, so we have established that the initial value problem has a unique solution on the interval $[a - \eta, a + \eta]$. □

Note that our proof using CMT produces a more restricted range of x values than did our proof on one dimension. The range of η depends on L as well as M and k. However, if $Mh \leq k$, actually we only need $\eta \leq h$, and we can now extend the range of the solution to all $x \in [a - h, a + h]$, by iteration.