Fractal Sets, Dimensions and Measures

This project is mostly about analysis and geometry, but it has almost no prerequisites. It is also possible to turn it into a computational project.

Introduction

The main goal of this project is to investigate the notion of *fractal dimension*. We are all familiar with integer dimensions, for example, an open set in $\mathbb{R}^n$ has dimension n and its 'size' can be measured by the corresponding n-dimensional Lebesgue measure. On the other hand, there are natural sets that are too large for one dimension but too small for the next integer dimension and to measure them in a meaningful way one has to introduce non-integer dimensions. A typical example is the Cantor set which in a certain sense is much smaller than the interval but much larger than a point. It is a set of dimension between 0 and 1 or a von Koch snowflake which has dimension between 1 and 2.

Such sets appear naturally as self-similar fractals (Cantor set, von Koch snowflake), attractors and other sets related to dynamical systems (Julia set, Mandelbrot set), stochastic processes (Brownian motion) and random models (Mandelbrot percolation, percolation, Ising model etc).

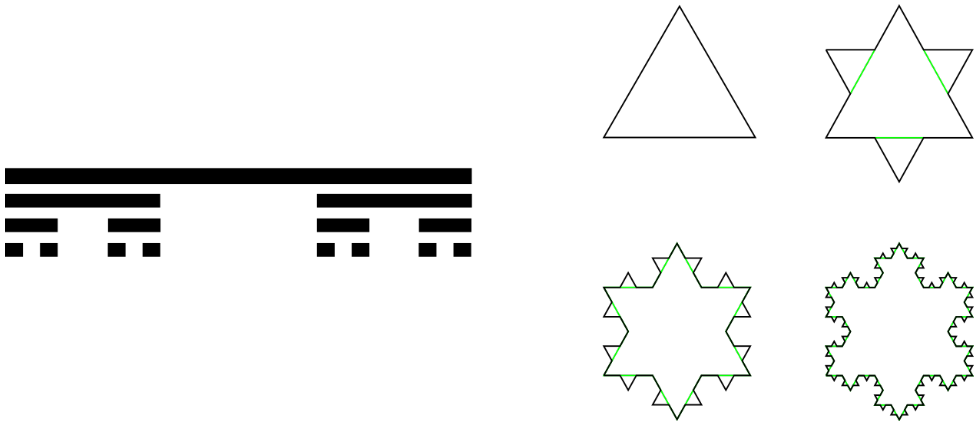

Figure 1: Left: The Cantor set is obtained by repeatedly removing the middle third. Right: von Koch snowflake is obtained by replacing the middle third by a triangle.

Project

This project can be developed in several directions. It could emphasise either geometry or measure theory or dynamics behind fractals. Either way, the project will start with the general study of non-integer dimensions and associated measures (non-integer-dimensional versions of the length or the volume) and how they could be described to study complicated sets. After that

The project is fairly open-ended but I can propose the following directions:

Option 1. The main task will be to compare different notions of dimension (Hausdorff dimension, box-counting dimension, packing dimension etc) for sets and measures. This will include introducing these notions and computing dimensions for a wide range of examples in order to show similarities and differences between different concepts.

Option 2. Many objects exhibit so-called 'multifractal' nature. Informally, this means that they are a mixture of different fractals. The goal here will be to define various notions describing multifractals and study their basic properties. It is also possible to study various multifractal decompositions. A typical example would be the following fractal decomposition of a unit interval. It is easy to show that the set of points such that the asymptotic frequency of every digit is $1/10$ has full Lebesgue measure (this is essentially the law of the large numbers) but given a set of frequencies there is a non-empty set of $x \in [0, 1]$ with given frequencies of digits. What is the dimension of such a set? How can such decompositions be used?

Option 3. There are many natural examples of *random* fractal sets (for example the trajectory of the Brownian motion or the shape of the lightning). The goal will be to study fractal techniques that are adapted to the random setting.

Option 4. It is also possible to do a computational project. Many fractals appear in dynamical systems. For example, one iterate $z \mapsto z^2 + c$. Julia set is defined as the set of points such that these iterations stay bounded. It is known that these sets are fractal, but there is no explicit formula for the dimension in terms of the parameter c. One possible direction is to study it numerically.

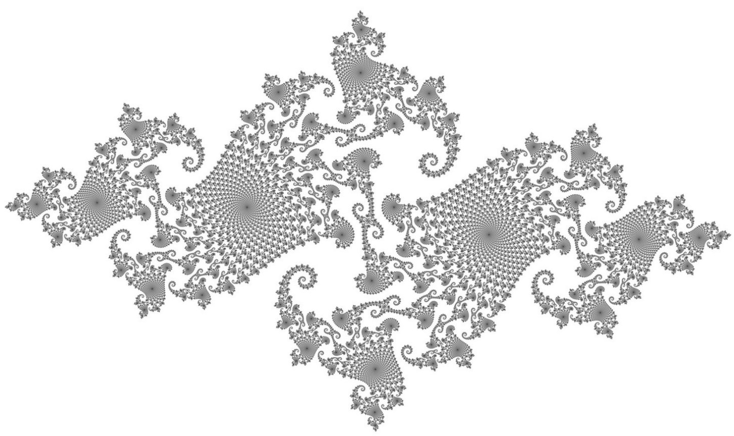

Figure 2: Julia set for $c = -0.75 + 0.11 * i$.

References

[1] K. Falconer. *Fractal geometry.* John Wiley & Sons, Ltd., Chichester, 1990. Mathematical foundations and applications.

[2] K. Falconer. *Techniques in fractal geometry.* John Wiley & Sons, Ltd., Chichester, 1997.