Random functions and fields

This project does not have formal prerequisites, but Part A Integration and Probability will be useful. This project can go in different directions, in particular, it could be a project in pure mathematics but it could be a numerical project as well.

Introduction

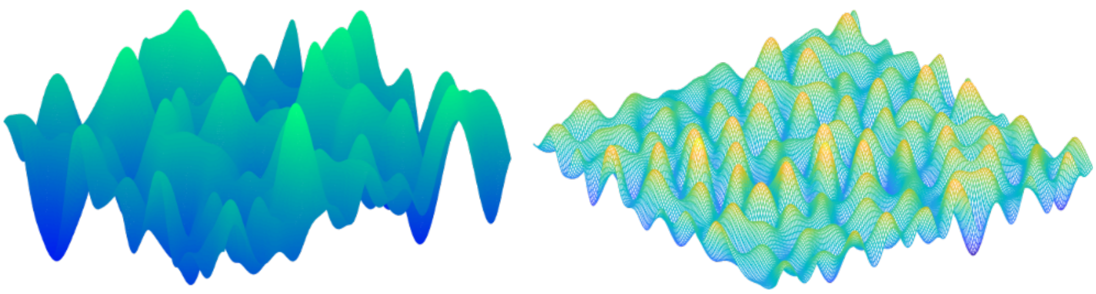
Let us start with a simple question: what is the number of real roots of a random real polynomial of degree n? First of all, there are many ways to define what is a 'random polynomial'. There are many different models and the answer depends on the model. Probably the simplest model is

$$p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_i \text{ are independent standard normal variables.}$$

Kac and Rice developed techniques that allow to obtain an explicit formula for the expected number of real roots between a and b and in particular to show that for large n is behaves as $\frac{2}{\pi}\log n + o(\log n)$.

It turns out that there is nothing special about polynomials of the fact that they depend on just one variable. Kac-Rice formula can be generalised to give the expectation and all higher moments of the volume of the zero set of *any* smooth function f such that its values at any points have jointly normal distribution. Such functions are called *Gaussian fields* and they play a very important role in mathematics, physics and many other areas of science. They are used to describe a wide range of natural phenomena and they naturally appear in several areas of mathematics. For example, there is a beautiful conjecture of M. Berry which states that high-energy eigenfunctions of the Laplacian behave like a particular Gaussian field. Gaussian fields are also used to model the Cosmic Microwave Background Radiation which plays an important role in cosmology. They are naturally appear in statistics and data analysis.

One can think of Gaussian fields as *collections of Gaussian random variables* indexed by points in some reference domain (a Euclidean space, a sphere, a manifold or even more abstract spaces). Alternatively, one can think that they are *random functions*. Such fields can be uniquely described by their mean value



There are two examples that we are particularly interested in: Random Plane Wave and Bargmann-Fock field. The first one appears as a universal model for Laplace eigenfunctions and the second is relevant for quantum physics and algebraic geometry. These fields could be defined by their covariances. In the case of the Random Plane Wave it is

$$\mathbb{E}f(x)f(y) = J_0(|x-y|)$$

where $x, y \in \mathbb{R}^2$ and J_0 is the 0th Bessel function. For the Bargmann-Fock the covariance is

$$\mathbb{E}f(x)f(y) = e^{-|x-y|^2}.$$

Alternatively, they could be written as convergent series. For the Random Plane Wave it is

$$f_{RPW}(x) = \sum_{-\infty}^{\infty} C_n J_{|n|}(|x|) e^{in\theta},$$

where θ is the argument of x, J_n are Bessel functions and C_n are independent complex normal variables subject to $C_{-n} = \bar{C}_n$. For the Bargmann-Fock it is

$$f_{BF}(x) = e^{-|x|^2/2} \sum_{m,n=0}^{\infty} a_{n,m} \frac{1}{\sqrt{n!m!}} x_1^n x_2^m,$$

where $a_{n,m}$ are independent standard normal random variables. Beyond these main examples there are many other fields of interest.

There is a beautiful conjecture which connects the behaviour of the level sets of Gaussian fields and *percolation models*. There are many versions of percolation, one of the simplest is the *bond percolation on* $\mathbb{Z}^2$. The idea is very simple, we take the square lattice $\mathbb{Z}^2$ and remove each edge independently with probability p. It is conjectured that the large-scale behaviour of level sets is the same as the large-scale behaviour of the connected components in percolation models. One of the main goals of this project is to explore this connection.

Task 1. Read some background about level lines of Gaussian fields. One possible starting point is the survey [2].

Read something about percolation. You don't have to learn too much, most of this project is not about percolation, but you have to know some basic notions. You can read introduction chapter from books by Grimmett [4] or Bollobás and Riordan [3].

Task 2. Write a code to simulate random plane wave, Bargmann-Fock field or any other smooth Gaussian field. Generate a few pictures of the level lines and excursion sets.

Task 3. Study large-scale behaviour of level lines or excursion sets. In particular, percolation conjecture suggests that for levels $\ell < 0$ there is a unique giant component of the excursion set and for $\ell > 0$ all connected components are 'small' (and there is a way to quantify this). There is an interesting phase transition happening at $\ell = 0$. There are many predictions one can make based on the percolation conjecture and the aim is to verify some of them numerically.

References

[1] R. J. Adler and J. E. Taylor. *Random fields and geometry.* Springer Monographs in Mathematics. Springer, New York, 2007.

[2] D. Beliaev. Smooth Gaussian fields and percolation. *Probab. Surv.*, 20:897–937, 2023.

[3] B. Bollobás and O. Riordan. *Percolation.* Cambridge University Press, New York, 2006.

[4] G. Grimmett. *Percolation*, volume 321 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, second edition, 1999.