

# B8.5 Graph Theory

## Sheet 0 — MT24

*Not for classes*

The following problems (mostly mentioned in the lectures/notes) are primarily intended for students who did not do Part A Graph Theory, to help you get yourself up to speed. Other students may find them useful too. They will not be discussed in classes; solutions will be posted on the course website.

You don't need to do all the questions, but it may be helpful. I suggest comparing your answers to the model solutions one (or a few) at a time; having seen the model solution, try to write the next answer in a similar style/level of detail.

1. Let  $x$  and  $y$  be vertices of a graph  $G$ . Show that  $G$  contains an (i.e., at least one)  $x$ - $y$  walk if and only if  $G$  contains an  $x$ - $y$  path.
2. Let  $G = (V, E)$  be a graph, and define a relation  $\sim$  on  $V$  by  $x \sim y$  if  $x$  and  $y$  are connected in  $G$ , i.e., if there is an  $x$ - $y$  path/walk in  $G$  (possibly of length 0). Show, giving full details, that  $\sim$  is an equivalence relation.
3. [A little tedious; omit if you like.] Check that any graph  $G$  is the disjoint union of its components (maximal connected subgraphs). It may help to first show that the components correspond to equivalence classes of the relation  $\sim$  in the previous question.
4. Show that TFAE (The Following Are Equivalent): (a)  $T$  is a tree, (b)  $T$  is a minimal (w.r.t. edges) connected graph, (c)  $T$  is a maximal (w.r.t. edges) acyclic graph.
5. Modify the argument in lectures to show that any tree with at least two vertices has at least two leaves.
6. Let  $T$  be a tree with  $|T| \geq 2$ , and let  $P$  be a longest path in  $T$ . Prove, giving full details, that the ends of  $P$  are leaves. Deduce that  $T$  has at least two leaves.
7. Show that any two vertices of a tree  $T$  are joined by a *unique* path in  $T$ .
8. Let  $(d_1, \dots, d_n)$  be a sequence of integers with  $n \geq 2$ . Show that there is a tree on  $[n]$  with  $d(i) = d_i$  for each  $i$  if and only if  $d_i \geq 1$  for all  $i$  and  $\sum_{i=1}^n d_i = 2n - 2$ .
9. Show that deleting any edge from a tree  $T$  leaves a graph with exactly two components. Show that deleting a vertex  $v$  leaves  $d(v)$  components. [Hint: you could do this directly, or try a short cut using what we know about numbers of edges in trees.]