

Numerical Solution of Partial Differential Equations

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Lecture 4

Finite difference approximation of elliptic BVP's

In Lecture 3 we discussed the finite difference approximation of a two-point boundary-value problem. Here we shall carry out a similar analysis for the elliptic boundary-value problem

$$\begin{aligned} -\Delta u + c(x, y)u &= f(x, y) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega = (0, 1) \times (0, 1)$, c is a continuous function on $\overline{\Omega}$ and $c(x, y) \geq 0$. We shall consider two separate cases:

- First we shall assume that $f \in C(\overline{\Omega})$. In this case, the error analysis proceeds similarly as in Lecture 3.
- In Lecture 5 we shall consider the case when f is only in $L_2(\Omega)$. In that case the boundary-value problem (1) does not have a classical solution – only a weak solution exists; a different technique is then needed to prove the convergence of the scheme.

The case when $f \in C(\bar{\Omega})$

Definition of the mesh

Let N be an integer, $N \geq 2$, and let $h = 1/N$; the mesh-points are (x_i, y_j) , $i, j = 0, \dots, N$, where $x_i = ih$, $y_j = jh$. These mesh-points form the mesh

$$\bar{\Omega}_h := \{(x_i, y_j) : i, j = 0, \dots, N\}.$$

We consider the set of interior mesh-points

$$\Omega_h := \{(x_i, y_j) : i, j = 1, \dots, N - 1\},$$

and the set of boundary mesh-points $\Gamma_h := \bar{\Omega}_h \setminus \Omega_h$.

Definition of the finite difference scheme

$$\begin{aligned} -(D_x^+ D_x^- U_{i,j} + D_y^+ D_y^- U_{i,j}) + c(x_i, y_j) U_{i,j} &= f(x_i, y_j) && \text{for } (x_i, y_j) \in \Omega_h, \\ U &= 0 && \text{on } \Gamma_h. \end{aligned} \tag{2}$$

In an expanded form, this can be written as follows:

$$-\left\{ \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{h^2} \right\} + c(x_i, y_j)U_{i,j} = f(x_i, y_j), \quad (3)$$

for $i, j = 1, \dots, N - 1$,

$$U_{i,j} = 0 \quad \text{if } i = 0, i = N \text{ or if } j = 0, j = N. \quad (4)$$

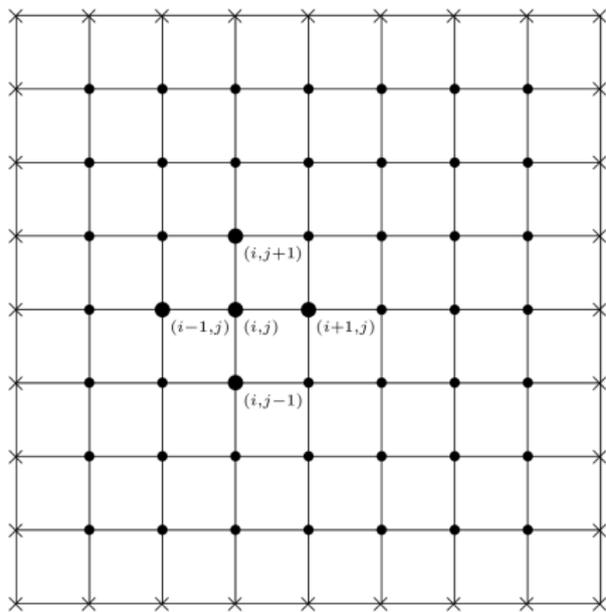


Figure 1: The mesh $\Omega_h(\cdot)$, the boundary mesh $\Gamma_h(\times)$, and a typical five-point difference stencil.

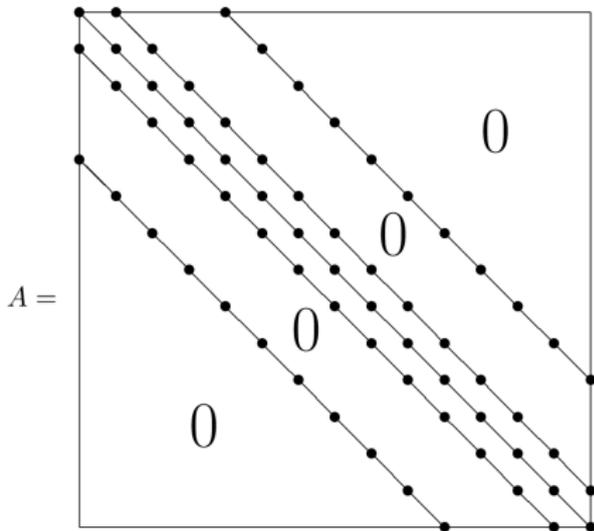


Figure 2: The sparsity structure of the banded matrix A .

A typical row of A has 5 non-zero entries, corresponding to the 5 values of U in the finite difference stencil shown in Figure. 1. The sparsity structure of A is shown in Figure 2.

Existence and uniqueness of solutions

Next we show that the finite difference scheme (2) has a unique solution.

For two functions, V and W , defined on Ω_h , we introduce the inner product

$$(V, W)_h = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} h^2 V_{i,j} W_{i,j},$$

which resembles the L_2 -inner product

$$(v, w) = \int_{\Omega} v(x, y) w(x, y) \, dx \, dy.$$

Lemma

Suppose that V is a function defined on $\bar{\Omega}_h$ and that $V = 0$ on Γ_h ; then,

$$\begin{aligned} & (-D_x^+ D_x^- V, V)_h + (-D_y^+ D_y^- V, V)_h \\ &= \sum_{i=1}^N \sum_{j=1}^{N-1} h^2 |D_x^- V_{i,j}|^2 + \sum_{i=1}^{N-1} \sum_{j=1}^N h^2 |D_y^- V_{i,j}|^2. \end{aligned} \tag{5}$$

PROOF. The identity (5) is a direct consequence of the corresponding univariate summation-by-parts result for $-D_x^+ D_x^-$ shown in Lecture 3, and the analogous identity for $-D_y^+ D_y^-$. \square

Returning to the analysis of the finite difference scheme (2), we shall now proceed in much the same way as in the univariate case in Lecture 3. As $c(x, y) \geq 0$ on $\bar{\Omega}$, by the summation-by-parts formula (5) we have that

$$\begin{aligned}
 (AV, V)_h &= (-D_x^+ D_x^- V - D_y^+ D_y^- V + cV, V)_h \\
 &= (-D_x^+ D_x^- V, V)_h + (-D_y^+ D_y^- V, V)_h + (cV, V)_h \\
 &\geq \sum_{i=1}^N \sum_{j=1}^{N-1} h^2 |D_x^- V_{i,j}|^2 + \sum_{i=1}^{N-1} \sum_{j=1}^N h^2 |D_y^- V_{i,j}|^2,
 \end{aligned} \tag{6}$$

for any V defined on $\bar{\Omega}_h$ such that $V = 0$ on Γ_h .

This implies, just as in the one-dimensional analysis presented in Section 3, that A is a non-singular matrix. Indeed if $AV = 0$, then (6) yields:

$$D_x^- V_{i,j} = \frac{V_{i,j} - V_{i-1,j}}{h} = 0, \quad \begin{array}{l} i = 1, \dots, N, \\ j = 1, \dots, N-1; \end{array}$$

$$D_y^- V_{i,j} = \frac{V_{i,j} - V_{i,j-1}}{h} = 0, \quad \begin{array}{l} i = 1, \dots, N-1, \\ j = 1, \dots, N. \end{array}$$

As $V = 0$ on Γ_h , these imply that $V \equiv 0$. Thus $AV = 0$ if and only if $V = 0$. Hence A is non-singular, and $U = A^{-1}F$ is the unique solution of (2). Thus the unique solution of the finite difference scheme (2) may be found by solving the system of linear algebraic equations $AU = F$.

Stability and convergence of the finite difference scheme

In order to prove the stability of the finite difference scheme (2), we introduce the mesh-dependent norms

$$\|U\|_h := (U, U)_h^{1/2},$$

and

$$\|U\|_{1,h} := (\|U\|_h^2 + \|D_x^- U\|_x^2 + \|D_y^- U\|_y^2)^{1/2},$$

where

$$\|D_x^- U\|_x := \left(\sum_{i=1}^N \sum_{j=1}^{N-1} h^2 |D_x^- U_{i,j}|^2 \right)^{1/2}$$

and

$$\|D_y^- U\|_y := \left(\sum_{i=1}^{N-1} \sum_{j=1}^N h^2 |D_y^- U_{i,j}|^2 \right)^{1/2}.$$

$\|\cdot\|_{1,h}$ is the discrete version of the Sobolev norm $\|\cdot\|_{H^1(\Omega)}$.

With this new notation, the inequality (6) can be rewritten in the following compact form:

$$(AV, V)_h \geq \|D_x^- V\|_x^2 + \|D_y^- V\|_y^2. \quad (7)$$

Using the discrete Poincaré–Friedrichs inequality stated in the next lemma, we shall be able to deduce that

$$(AV, V)_h \geq c_0 \|V\|_{1,h}^2,$$

where c_0 is a positive constant.

Lemma (Discrete Poincaré–Friedrichs inequality)

Suppose that V is a function defined on $\bar{\Omega}_h$ and such that $V = 0$ on Γ_h ; then, there exists a constant c_* , independent of V and h , such that

$$\|V\|_h^2 \leq c_* \left(\|D_x^- V\|_x^2 + \|D_y^- V\|_y^2 \right) \quad (8)$$

for all such V .

PROOF.

The inequality (8) is a straightforward consequence of its univariate counterpart proved in Lecture 3; indeed, for each fixed j , $1 \leq j \leq N - 1$,

$$\sum_{i=1}^{N-1} h |V_{i,j}|^2 \leq \frac{1}{2} \sum_{i=1}^N h |D_x^- V_{i,j}|^2. \quad (9)$$

Analogously, for each fixed i , $1 \leq i \leq N - 1$,

$$\sum_{j=1}^{N-1} h |V_{i,j}|^2 \leq \frac{1}{2} \sum_{j=1}^N h |D_y^- V_{i,j}|^2. \quad (10)$$

We first multiply (9) by h and sum through j , $1 \leq j \leq N - 1$, then multiply (10) by h and sum through i , $1 \leq i \leq N - 1$, and finally add these two inequalities to obtain

$$2 \|V\|_h^2 \leq \frac{1}{2} \left(\|D_x^- V\|_x^2 + \|D_y^- V\|_y^2 \right).$$

Hence we arrive at (8) with $c_* = \frac{1}{4}$. \square

Now the inequalities (7) and (8) imply that

$$(AV, V)_h \geq \frac{1}{c_*} \|V\|_h^2.$$

Finally, combining this inequality with (7) and recalling the definition of the norm $\|\cdot\|_{1,h}$, we obtain

$$(AV, V)_h \geq c_0 \|V\|_{1,h}^2, \tag{11}$$

where $c_0 = (1 + c_*)^{-1} = (1 + (1/4))^{-1} = \frac{4}{5}$.

Using the inequality (11) we can now prove the stability of the finite difference scheme (2).

Theorem

The finite difference scheme (2) is stable in the sense that

$$\|U\|_{1,h} \leq \frac{1}{c_0} \|f\|_h. \quad (12)$$

PROOF. The proof is identical to that of the analogous stability inequality from Lecture 3 in the univariate case. From (11) and (2) we have that

$$\begin{aligned} c_0 \|U\|_{1,h}^2 &\leq (AU, U)_h = (f, U)_h \leq |(f, U)_h| \\ &\leq \|f\|_h \|U\|_h \leq \|f\|_h \|U\|_{1,h}, \end{aligned}$$

and hence we arrive at the desired inequality (12). \square

Convergence in the class of classical solutions

Next, we turn to the study of accuracy of the finite difference scheme (2). We define the **global error**, e , by

$$e_{i,j} := u(x_i, y_j) - U_{i,j}, \quad 0 \leq i, j \leq N.$$

Assuming that $u \in C^4(\bar{\Omega})$, Taylor expansions with remainder terms in the x and y directions give:

$$\begin{aligned} Ae_{i,j} &= Au(x_i, y_j) - AU_{i,j} = Au(x_i, y_j) - f_{i,j} \\ &= \Delta u(x_i, y_j) - (D_x^+ D_x^- u(x_i, y_j) + D_y^+ D_y^- u(x_i, y_j)) \\ &= \left[\frac{\partial^2 u}{\partial x^2}(x_i, y_j) - D_x^+ D_x^- u(x_i, y_j) \right] + \left[\frac{\partial^2 u}{\partial y^2}(x_i, y_j) - D_y^+ D_y^- u(x_i, y_j) \right] \\ &= -\frac{h^2}{12} \frac{\partial^4 u}{\partial x^4}(\xi_i, y_j) - \frac{h^2}{12} \frac{\partial^4 u}{\partial y^4}(x_i, \eta_j), \quad 1 \leq i, j \leq N-1, \end{aligned}$$

where $\xi_i \in [x_{i-1}, x_{i+1}]$, $\eta_j \in [y_{j-1}, y_{j+1}]$.

We define the **consistency error** (or **truncation error**) of the finite difference scheme (2) by

$$\varphi_{i,j} := Au(x_i, y_j) - f_{i,j}.$$

Then, by the calculations above,

$$\varphi_{i,j} = -\frac{h^2}{12} \left(\frac{\partial^4 u}{\partial x^4}(\xi_i, y_j) + \frac{\partial^4 u}{\partial y^4}(x_i, \eta_j) \right), \quad 1 \leq i, j \leq N-1,$$

and

$$\begin{aligned} Ae_{i,j} &= \varphi_{i,j}, & 1 \leq i, j \leq N-1, \\ e &= 0 & \text{on } \Gamma_h. \end{aligned}$$

Thanks to the stability result (12), we therefore have that

$$\|u - U\|_{1,h} = \|e\|_{1,h} \leq \frac{1}{c_0} \|\varphi\|_h. \quad (13)$$

To arrive at a bound on the global error $e = u - U$ in the norm $\|\cdot\|_{1,h}$ it therefore remains to bound $\|\varphi\|_h$ and insert the resulting bound in the right-hand side of (13). Indeed, by noting that

$$|\varphi_{i,j}| \leq \frac{h^2}{12} \left(\left\| \frac{\partial^4 u}{\partial x^4} \right\|_{C(\bar{\Omega})} + \left\| \frac{\partial^4 u}{\partial y^4} \right\|_{C(\bar{\Omega})} \right),$$

we deduce that the consistency error, φ , satisfies

$$\|\varphi\|_h \leq \frac{h^2}{12} \left(\left\| \frac{\partial^4 u}{\partial x^4} \right\|_{C(\bar{\Omega})} + \left\| \frac{\partial^4 u}{\partial y^4} \right\|_{C(\bar{\Omega})} \right). \quad (14)$$

Finally (13) and (14) yield the following result.

Theorem

Let $f \in C(\bar{\Omega})$, $c \in C(\bar{\Omega})$, with $c(x, y) \geq 0$, $(x, y) \in \bar{\Omega}$, and suppose that the corresponding weak solution of the boundary-value problem (1) belongs to $C^4(\bar{\Omega})$; then

$$\|u - U\|_{1,h} \leq \frac{5h^2}{48} \left(\left\| \frac{\partial^4 u}{\partial x^4} \right\|_{C(\bar{\Omega})} + \left\| \frac{\partial^4 u}{\partial y^4} \right\|_{C(\bar{\Omega})} \right). \quad (15)$$

PROOF. Recall that $c_0 = (1 + c_*)^{-1}$, $c_* = \frac{1}{4}$, so that $1/c_0 = \frac{5}{4}$, and combine (13) and (14). \square

In other words,

stability + consistency \Rightarrow convergence.