Numerical Solution of Partial Differential Equations

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Lecture 6

Nonaxiparallel domains and nonuniform meshes

When Ω has a curved boundary, a non-uniform mesh has to be used near $\partial\Omega$ to avoid loss of accuracy. To be more precise, let us introduce the following notation: let $h_{i+1} := x_{i+1} - x_i$, $h_i := x_i - x_{i-1}$, and let

$$\hbar_i := \frac{1}{2}(h_{i+1} + h_i).$$

We define

$$D_x^+ U_i := \frac{U_{i+1} - U_i}{\hbar_i}, \quad D_x^- U_i := \frac{U_i - U_{i-1}}{h_i},$$

$$D_x^+ D_x^- U_i := \frac{1}{\hbar_i}\left(\frac{U_{i+1} - U_i}{h_{i+1}} - \frac{U_i - U_{i-1}}{h_i}\right).$$

Similarly, let $k_{j+1} := y_{j+1} - y_j$, $k_j := y_j - y_{j-1}$, and let

$$\bar{k}_j := \frac{1}{2}(k_{j+1} + k_j).$$

Let

$$D_y^+ U_j := \frac{U_{j+1} - U_j}{k_j}, \quad D_y^- U_j := \frac{U_j - U_{j-1}}{k_j},$$

$$D_y^+ D_y^- U_j := \frac{1}{k_j} \left(\frac{U_{j+1} - U_j}{k_{j+1}} - \frac{U_j - U_{j-1}}{k_j} \right).$$

So on a general non-uniform mesh

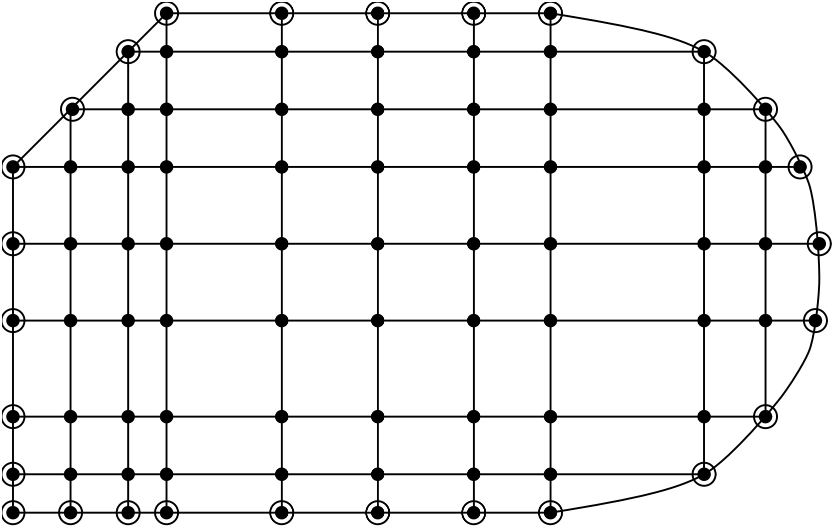
$$\overline{\Omega}_h := \{(x_i, y_j) : x_{i+1} - x_i = h_i,\ y_{j+1} - y_j = k_j\},$$

the Laplace operator, Δ, can be approximated by $D_x^+ D_x^- + D_y^+ D_y^-$, with the difference operators $D_x^+ D_x^-$, $D_y^+ D_y^-$ defined above.

Consider, for example, the Dirichlet problem

$$\begin{aligned} -\Delta u = f(x, y) \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \partial\Omega, \end{aligned}$$

where Ω and the non-uniform mesh $\overline{\Omega}_h$ are depicted in the next figure.



$\bullet\ \Omega_h;\quad \odot\ \Gamma_h,\qquad \overline{\Omega}_h = \Omega_h \cup \Gamma_h.$

Non-uniform mesh $\overline{\Omega}_h$.

The finite difference approximation of this problem is

$$\begin{aligned} -(D_x^+ D_x^- U_{i,j} + D_y^+ D_y^- U_{i,j}) = f(x_i, y_j) \quad &\text{in } \Omega_h, \\ U_{i,j} = 0 \qquad\qquad &\text{on } \Gamma_h. \end{aligned}$$

Equivalently,

$$\begin{aligned} &-\frac{1}{\hbar_i}\left(\frac{U_{i+1,j} - U_{i,j}}{h_{i+1}} - \frac{U_{i,j} - U_{i-1,j}}{h_i}\right) \\ &-\frac{1}{\bar{k}_j}\left(\frac{U_{i,j+1} - U_{i,j}}{k_{j+1}} - \frac{U_{i,j} - U_{i,j-1}}{k_j}\right) = f(x_i, y_j) \quad \text{in } \Omega_h, \\ &\qquad\qquad\qquad\qquad\qquad\qquad U_{i,j} = 0 \qquad\qquad \text{on } \Gamma_h. \end{aligned}$$

A typical difference stencil is shown below; clearly we still have a five-point difference scheme.

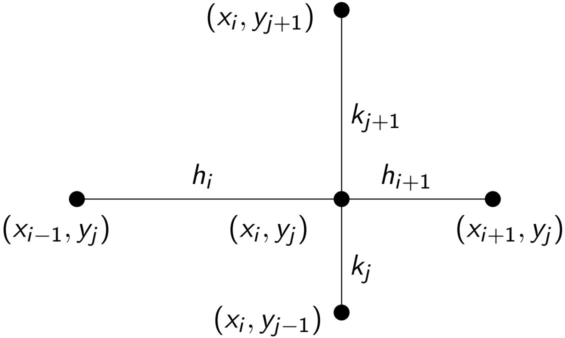


Figure: Five-point stencil on a non-uniform mesh.

The discrete maximum principle

The maximum principle is a key property of elliptic equations.

Under suitable sign-conditions imposed on the source term and the coefficients of the differential operator, it ensures that the maximum value of the solution is attained at the boundary of the domain rather than at an interior point, and if the maximum value of the solution is attained at an interior point, then the solution must be constant.

Our objective is to construct a finite difference approximation of the elliptic boundary-value problem $-\Delta u = f$, $u|_{\partial\Omega} = g$, and show that a discrete counterpart of the maximum principle satisfied by the function u holds for its finite difference approximation U.

For simplicity we shall confine ourselves to the case of two space dimensions and consider a general nonaxiparallel domain, such as the one in the figure from slide 4, and a general nonuniform mesh

$$\overline{\Omega}_h := \{(x_i, y_j) : x_{i+1} - x_i = h_i,\ y_{j+1} - y_j = k_j\}.$$

The Laplace operator, Δ, is approximated by $D_x^+ D_x^- + D_y^+ D_y^-$, with the difference operators $D_x^+ D_x^-$, $D_y^+ D_y^-$ on a nonuninform mesh. The approximation of the Dirichlet problem

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega \end{aligned}$$

is then given by

$$\begin{aligned} -(D_x^+ D_x^- U_{i,j} + D_y^+ D_y^- U_{i,j}) &= f(x_i, y_j) \qquad \text{in } \Omega_h, \\ U_{i,j} &= g(x_i, y_j) \qquad \text{on } \Gamma_h. \end{aligned} \tag{1}$$

Equivalently,

$$\begin{aligned} -\frac{1}{\hbar_i}\left(\frac{U_{i+1,j}-U_{i,j}}{h_{i+1}} - \frac{U_{i,j}-U_{i-1,j}}{h_i}\right) - \frac{1}{k\!\!\!-_j}\left(\frac{U_{i,j+1}-U_{i,j}}{k_{j+1}} - \frac{U_{i,j}-U_{i,j-1}}{k_j}\right) &= f(x_i, y_j) \;\text{ in } \Omega_h, \\ U_{i,j} &= g(x_i, y_j) \;\text{ on } \Gamma_h. \end{aligned}$$

Suppose (for contradiction) that $f(x_i, y_j) < 0$ for all $(x_i, y_j) \in \Omega_h$ and that the maximum value of U is attained at a point $(x_{i_0}, y_{j_0}) \in \Omega_h$. Clearly,

$$\left(\frac{1}{\hbar_i}\left(\frac{1}{h_{i+1}} + \frac{1}{h_i}\right) + \frac{1}{\hbar\!\!\!k_j}\left(\frac{1}{k_{j+1}} + \frac{1}{k_j}\right)\right) U_{i,j} = \frac{U_{i+1,j}}{\hbar_i\, h_{i+1}} + \frac{U_{i-1,j}}{\hbar_i\, h_i} + \frac{U_{i,j+1}}{\hbar\!\!\!k_j\, k_{j+1}} + \frac{U_{i,j-1}}{\hbar\!\!\!k_j\, k_j} + f(x_i, y_j)$$

for any $(x_i, y_j) \in \Omega_h$. Therefore, because $U_{i_0 \pm 1, j_0} \le U_{i_0, j_0}$ and $U_{i_0, j_0 \pm 1} \le U_{i_0, j_0}$, and $f(x_{i_0}, y_{j_0}) < 0$, it follows that

$$\left(\frac{1}{\hbar_{i_0}}\left(\frac{1}{h_{i_0+1}} + \frac{1}{h_{i_0}}\right) + \frac{1}{\hbar\!\!\!k_{j_0}}\left(\frac{1}{k_{j_0+1}} + \frac{1}{k_{j_0}}\right)\right) U_{i_0 j_0} < \frac{U_{i_0 j_0}}{\hbar_{i_0}\, h_{i_0+1}} + \frac{U_{i_0 j_0}}{\hbar_{i_0}\, h_{i_0}} + \frac{U_{i_0 j_0}}{\hbar\!\!\!k_{j_0}\, k_{j_0+1}} + \frac{U_{i_0 j_0}}{\hbar\!\!\!k_{j_0}\, k_{j_0}}.$$

Note, however, that the expressions on the two sides of this (strict!) inequality are equal, which means that we have run into a contradiction.

Thus we have shown that if $f(x_i, y_j) < 0$ for all $(x_i, y_j) \in \Omega_h$ then the maximum value of U must be attained on the boundary Γ_h of Ω_h, which completes the proof of the discrete maximum principle in this case:

$$\max_{(x_i,y_j)\in\Gamma_h} U_{i,j} = \max_{(x_i,y_j)\in\overline{\Omega}_h} U_{i,j}.$$

Now suppose that $f(x_i, y_j) \le 0$ for all $(x_i, y_j) \in \Omega_h$. Let $\varepsilon > 0$ and define

$$V_{i,j} := U_{i,j} + \frac{\varepsilon}{4}(x_i^2 + y_j^2) \qquad \text{for } (x_i, y_j) \in \overline{\Omega}_h.$$

Hence,

$$\begin{aligned} -(D_x^+ D_x^- V_{i,j} + D_y^+ D_y^- V_{i,j}) &= -(D_x^+ D_x^- U_{i,j} + D_y^+ D_y^- U_{i,j}) - \varepsilon \\ &= f(x_i, y_j) - \varepsilon < 0 \qquad \text{in } \Omega_h, \end{aligned}$$

which implies that the maximum of V is attained on Γ_h. Thus,

$$\begin{aligned} \max_{(x_i,y_j)\in\Gamma_h} U_{i,j} &= \max_{(x_i,y_j)\in\Gamma_h} \left[V_{i,j} - \frac{\varepsilon}{4}(x_i^2 + y_j^2) \right] \\ &\ge \max_{(x,y)\in\Gamma_h} V_{i,j} - \frac{\varepsilon}{4} \max_{(x_i,y_j)\in\Gamma_h} (x_i^2 + y_j^2) \\ &= \max_{(x_i,y_j)\in\overline{\Omega}_h} V_{i,j} - \frac{\varepsilon}{4} \max_{(x_i,y_j)\in\Gamma_h} (x_i^2 + y_j^2). \end{aligned}$$

As, by definition, $V_{i,j} \geq U_{i,j}$ for $(x_i, y_j) \in \overline{\Omega}_h$, it follows that

$$\max_{(x_i,y_j)\in\Gamma_h} U_{i,j} \geq \max_{(x_i,y_j)\in\overline{\Omega}_h} U_{i,j} - \frac{\varepsilon}{4} \max_{(x_i,y_j)\in\Gamma_h} (x_i^2 + y_j^2) \qquad \forall\, \varepsilon > 0.$$

By passing to the limit $\varepsilon \to 0_+$ it then follows that

$$\max_{(x_i,y_j)\in\Gamma_h} U_{i,j} \geq \max_{(x_i,y_j)\in\overline{\Omega}_h} U_{i,j}.$$

As $\Gamma_h \subset \overline{\Omega}_h$, trivially $\max_{(x_i,y_j)\in\overline{\Omega}_h} U_{i,j} \geq \max_{(x_i,y_j)\in\Gamma_h} U_{i,j}$, and therefore we have shown that if $f(x_i, y_j) \leq 0$ for all $(x_i, y_j) \in \Omega_h$, then the discrete maximum principle holds:

$$\boxed{\max_{(x_i,y_j)\in\Gamma_h} U_{i,j} = \max_{(x_i,y_j)\in\overline{\Omega}_h} U_{i,j}.}$$

Analogously, if $f(x_i, y_j) \geq 0$ for all $(x_i, y_j) \in \Omega_h$, then a discrete minimum principle holds:

$$\boxed{\min_{(x_i,y_j)\in\Gamma_h} U_{i,j} = \min_{(x_i,y_j)\in\overline{\Omega}_h} U_{i,j}.}$$

Our objective is now to use the discrete maximum/minimum principle to prove the stability of the finite difference scheme (1) with respect to perturbations in the boundary data.

Stability in the discrete maximum norm

We shall first prove the existence and uniqueness of a solution to (1).

Lemma

The finite difference scheme (1) *has a unique solution.*

PROOF. The finite difference scheme (1) has a unique solution if, and only if, its homogeneous counterpart (i.e. when we have a zero right-hand side and zero boundary datum) has the trivial solution as its unique solution. Let us therefore consider

$$\begin{aligned} -(D_x^+ D_x^- U_{i,j} + D_y^+ D_y^- U_{i,j}) &= 0 \qquad \text{in } \Omega_h, \\ U_{i,j} &= 0 \qquad \text{on } \Gamma_h. \end{aligned} \tag{2}$$

The existence of a solution to (2) is obvious: the mesh-function U, with $U_{i,j} = 0$ for all $(x_i, y_j) \in \overline{\Omega}_h$ is clearly a solution. We shall now show that this is the only solution to (2).

According to the discrete maximum principle, for any solution U of the finite difference scheme (2),

$$0 = \max_{(x_i,y_j)\in\Gamma_h} U_{i,j} = \max_{(x_i,y_j)\in\overline{\Omega}_h} U_{i,j},$$

while according to the discrete minimum principle

$$0 = \min_{(x_i,y_j)\in\Gamma_h} U_{i,j} = \min_{(x_i,y_j)\in\overline{\Omega}_h} U_{i,j}.$$

Therefore $U_{i,j} = 0$ for all $(x_i, y_j) \in \overline{\Omega}_h$, i.e., the only solution of (2) is the trivial solution. This then implies the existence of a unique solution to (1).
$\square$

Stability with respect to perturbations in the boundary data

Consider the mesh functions $U^{(1)}$ and $U^{(2)}$, which satisfy, respectively:

$$\begin{aligned} -(D_x^+ D_x^- U_{i,j}^{(1)} + D_y^+ D_y^- U_{i,j}^{(1)}) &= f(x_i, y_j) && \text{in } \Omega_h, \\ U_{i,j}^{(1)} &= g^{(1)}(x_i, y_j) && \text{on } \Gamma_h \end{aligned} \tag{3}$$

and

$$\begin{aligned} -(D_x^+ D_x^- U_{i,j}^{(2)} + D_y^+ D_y^- U_{i,j}^{(2)}) &= f(x_i, y_j) && \text{in } \Omega_h, \\ U_{i,j}^{(2)} &= g^{(2)}(x_i, y_j) && \text{on } \Gamma_h \end{aligned} \tag{4}$$

for given boundary data $g^{(1)}$ and $g^{(2)}$.

Let

$$U := U^{(1)} - U^{(2)} \quad \text{and} \quad g := g^{(1)} - g^{(2)}.$$

Then, by subtracting (4) from (3) we find that U solves

$$\begin{aligned} -(D_x^+ D_x^- U_{i,j} + D_y^+ D_y^- U_{i,j}) &= 0 && \text{in } \Omega_h, \\ U_{i,j} &= g(x_i, y_j) && \text{on } \Gamma_h. \end{aligned} \tag{5}$$

By the discrete maximum principle we have from (5) that

$$\max_{(x_i,y_j)\in\overline{\Omega}_h} U_{i,j} = \max_{(x_i,y_j)\in\Gamma_h} U_{i,j} = \max_{(x_i,y_j)\in\Gamma_h} g(x_i, y_j) \leq \max_{(x_i,y_j)\in\Gamma_h} |g(x_i, y_j)|.$$

In other words, for all $(x_i, y_j) \in \overline{\Omega}_h$,

$$U_{i,j} \leq \max_{(x_i,y_j)\in\Gamma_h} |g(x_i, y_j)|. \tag{6}$$

It follows from (5) that $-U$ solves

$$\begin{aligned} -(D_x^+ D_x^-(-U)_{i,j} + D_y^+ D_y^-(-U)_{i,j}) = 0 \qquad & \text{in } \Omega_h, \\ (-U)_{i,j} = -g(x_i, y_j) \qquad & \text{on } \Gamma_h, \end{aligned} \tag{7}$$

where $(-U)_{i,j} := -U_{i,j}$. Hence, also,

$$-U_{i,j} = (-U)_{i,j} \le \max_{(x_i,y_j)\in\Gamma_h} |-g(x_i, y_j)| = \max_{(x_i,y_j)\in\Gamma_h} |g(x_i, y_j)| \tag{8}$$

for all $(x_i, y_j) \in \overline{\Omega}_h$. By combining (6) and (8) we have the inequality

$$|U_{i,j}| \le \max_{(x_i,y_j)\in\Gamma_h} |g(x_i, y_j)|$$

for all $(x_i, y_j) \in \overline{\Omega}_h$. Hence,

$$\max_{(x_i,y_j)\in\overline{\Omega}_h} |U_{i,j}| \le \max_{(x_i,y_j)\in\Gamma_h} |g(x_i, y_j)|.$$

By recalling the definitions of U and g, we have thereby shown that

$$\max_{(x_i,y_j)\in\overline{\Omega}_h} |U_{i,j}^{(1)} - U_{i,j}^{(2)}| \leq \max_{(x_i,y_j)\in\Gamma_h} |g^{(1)}(x_i, y_j) - g^{(2)}(x_i, y_j)|. \qquad (9)$$

The inequality (9) expresses continuous dependence of the solution U to the finite difference scheme with respect to the boundary data g: it ensures that small perturbations in the boundary data result in small perturbations in the associated solution, a property that is referred to as stability of the solution with respect to perturbations in the boundary data (in the discrete maximum norm, in this case).