# Numerical Solution of Partial Differential Equations

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Lecture 9

# Stability of finite difference schemes

To replicate the stability property of the heat equation in the $L_2$ norm at the discrete level, we need a suitable notion of stability.

We shall say that a finite difference scheme for the unsteady heat equation is **(practically) stable in the $\ell_2$ norm**, if

$$\|U^m\|_{\ell_2} \leq \|U^0\|_{\ell_2}, \qquad m = 1, \dots, M,$$

where

$$\|U^m\|_{\ell_2} = \left( \Delta x \sum_{j=-\infty}^{\infty} |U_j^m|^2 \right)^{1/2}.$$

We shall use the semidiscrete Fourier transform to explore the stability of finite difference schemes.

### Definition

The semidiscrete Fourier transform of a function $U$ defined on the infinite mesh $x_j = j\Delta x$, $j = 0, \pm 1, \pm 2, \ldots$, is:

$$\hat{U}(k) = \Delta x \sum_{j=-\infty}^{\infty} U_j \, \mathrm{e}^{-\imath k x_j}, \qquad k \in [-\pi/\Delta x, \pi/\Delta x].$$

We shall also need the inverse semidiscrete Fourier transform, as well the discrete counterpart of Parseval's identity that connect these transforms, similarly as in the case of the Fourier transform and its inverse considered earlier.

### Definition

Let $\hat{U}$ be defined on the interval $[-\pi/\Delta x, \pi/\Delta x]$. The inverse semidiscrete Fourier transform of $\hat{U}$ is defined by

$$U_j := \frac{1}{2\pi} \int_{-\pi/\Delta x}^{\pi/\Delta x} \hat{U}(k)\, \mathrm{e}^{\imath k j \Delta x}\, \mathrm{d}k.$$

We then have the following result.

## Lemma (Discrete Parseval's identity)

*Let*

$$\|U\|_{\ell_2} = \left( \Delta x \sum_{j=-\infty}^{\infty} |U_j|^2 \right)^{1/2} \quad \textit{and} \quad \|\hat{U}\|_{L_2} = \left( \int_{-\pi/\Delta x}^{\pi/\Delta x} |\hat{U}(k)|^2 \, \mathrm{d}k \right)^{1/2}.$$

*If* $\|U\|_{\ell_2}$ *is finite, then also* $\|\hat{U}\|_{L_2}$ *is finite, and*

$$\|U\|_{\ell_2} = \frac{1}{\sqrt{2\pi}} \|\hat{U}\|_{L_2}.$$

The proof of this is similar to that of Parseval's identity discussed earlier, and we shall therefore leave its proof as an exercise.

# Stability analysis of the explicit Euler scheme

By inserting

$$U_j^m = \frac{1}{2\pi}\int_{-\pi/\Delta x}^{\pi/\Delta x} \mathrm{e}^{\imath kj\Delta x}\,\hat{U}^m(k)\,\mathrm{d}k$$

into the Euler scheme we deduce that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi/\Delta x}^{\pi/\Delta x} \mathrm{e}^{\imath kj\Delta x}\,\frac{\hat{U}^{m+1}(k)-\hat{U}^m(k)}{\Delta t}\,\mathrm{d}k\\
&\quad= \frac{1}{2\pi}\int_{-\pi/\Delta x}^{\pi/\Delta x} \frac{\mathrm{e}^{\imath k(j+1)\Delta x}-2\mathrm{e}^{\imath kj\Delta x}+\mathrm{e}^{\imath k(j-1)\Delta x}}{(\Delta x)^2}\,\hat{U}^m(k)\,\mathrm{d}k.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi/\Delta x}^{\pi/\Delta x} \mathrm{e}^{\imath kj\Delta x}\,\frac{\hat{U}^{m+1}(k)-\hat{U}^m(k)}{\Delta t}\,\mathrm{d}k\\
&\quad= \frac{1}{2\pi}\int_{-\pi/\Delta x}^{\pi/\Delta x} \mathrm{e}^{\imath kj\Delta x}\,\frac{\mathrm{e}^{\imath k\Delta x}-2+\mathrm{e}^{-\imath k\Delta x}}{(\Delta x)^2}\,\hat{U}^m(k)\,\mathrm{d}k.
\end{aligned}$$

By comparing the left-hand side with the right-hand side we get

$$\hat{U}^{m+1}(k) = \hat{U}^{m}(k) + \mu(\mathrm{e}^{\imath k \Delta x} - 2 + \mathrm{e}^{-\imath k \Delta x})\hat{U}^{m}(k)$$

for all **wave numbers** $k \in [-\pi/\Delta x, \pi/\Delta x]$. Thus we have

$$\hat{U}^{m+1}(k) = \lambda(k)\hat{U}^{m}(k),$$

where

$$\lambda(k) = 1 + \mu(\mathrm{e}^{\imath k \Delta x} - 2 + \mathrm{e}^{-\imath k \Delta x})$$

is the **amplification factor** and

$$\mu := \frac{\Delta t}{(\Delta x)^2}$$

is called the **CFL number**[1].

[1] After: Richard Courant, Kurt Friedrichs, and Hans Lewy (*Über die partiellen Differenzengleichungen der mathematischen Physik*. Mathematische Annalen, 100:32–74, 1928).

By the discrete Parseval identity stated in Lemma 3 we have that

$$\begin{aligned}
\|U^{m+1}\|_{\ell_2} &= \frac{1}{\sqrt{2\pi}}\|\hat{U}^{m+1}\|_{L_2} \\
&= \frac{1}{\sqrt{2\pi}}\|\lambda \hat{U}^{m}\|_{L_2} \\
&\leq \frac{1}{\sqrt{2\pi}}\max_k |\lambda(k)|\ \|\hat{U}^{m}\|_{L_2} \\
&= \max_k |\lambda(k)|\ \|U^{m}\|_{\ell_2}.
\end{aligned}$$

In order to mimic the $L_2$ norm bound, we would like to ensure that

$$\|U^{m+1}\|_{\ell_2} \leq \|U^{m}\|_{\ell_2}, \qquad m = 0, 1, \ldots, M-1.$$

Thus we demand that

$$\max_k |\lambda(k)| \leq 1,$$

i.e., that

$$\max_k |1 + \mu(\mathrm{e}^{\imath k \Delta x} - 2 + \mathrm{e}^{-\imath k \Delta x})| \leq 1.$$

Using Euler's formula

$$e^{\imath\varphi} = \cos\varphi + \imath\sin\varphi$$

and the trigonometric identity

$$1 - \cos\varphi = 2\sin^2\frac{\varphi}{2}$$

we can restate this as follows:

$$\max_k \left|1 - 4\mu\sin^2\left(\frac{k\Delta x}{2}\right)\right| \leq 1.$$

Equivalently, we need to ensure that

$$-1 \leq 1 - 4\mu\sin^2\left(\frac{k\Delta x}{2}\right) \leq 1 \quad \forall k \in [-\pi/\Delta x, \pi/\Delta x].$$

This holds if, and only if, $\mu = \frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2}$.

Thus we have shown the following result.

**Theorem**

*Suppose that $U_j^m$ is the solution of the explicit Euler scheme*

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} = \frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{(\Delta x)^2}, \quad j = 0, \pm 1, \pm 2, \ldots,$$

$$U_j^0 = u_0(x_j), \qquad j = 0, \pm 1, \pm 2, \ldots,$$

*and* $\mu = \frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2}$. *Then,*

$$\|U^m\|_{\ell_2} \leq \|U^0\|_{\ell_2}, \qquad m = 1, 2, \ldots, M. \tag{1}$$

Hence, the explicit Euler scheme is **conditionally practically stable**, the condition for stability being that $\mu = \Delta t/\Delta x^2 \leq 1/2$. One can also show that if $\mu > 1/2$, then (1) will fail.

# Stability analysis of the implicit Euler scheme

We shall now perform a similar analysis for the **implicit Euler scheme** for the heat equation:

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} = \frac{U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}}{(\Delta x)^2}, \quad j = 0, \pm 1, \pm 2, \ldots$$

$$U_j^0 = u_0(x_j), \qquad j = 0, \pm 1, \pm 2, \ldots.$$

Equivalently,

$$U_j^{m+1} - \mu(U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}) = U_j^m$$

$$U_j^0 = u_0(x_j), \qquad j = 0, \pm 1, \pm 2, \ldots,$$

where, again,

$$\mu = \frac{\Delta t}{(\Delta x)^2}.$$

Using an identical argument as for the explicit Euler scheme, we find that the amplification factor is now

$$\lambda(k) = \frac{1}{1 + 4\mu \sin^2\left(\frac{k\Delta x}{2}\right)}.$$

Clearly,

$$\max_k |\lambda(k)| \le 1$$

for all values of

$$\mu = \frac{\Delta t}{(\Delta x)^2}.$$

Thus we have the following result.

### Theorem

*Suppose that $U_j^m$ is the solution of the implicit Euler scheme*

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} = \frac{U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}}{(\Delta x)^2}, \quad j = 0, \pm 1, \pm 2, \ldots,$$

$$U_j^0 = u_0(x_j), \qquad j = 0, \pm 1, \pm 2, \ldots.$$

*Then, for all $\Delta t > 0$ and $\Delta x > 0$,*

$$\|U^m\|_{\ell_2} \leq \|U^0\|_{\ell_2}, \qquad m = 1, 2, \ldots, M. \tag{2}$$

Thus, the implicit Euler scheme is **unconditionally practically stable**, meaning that the bound (2) holds without any restrictions on $\Delta x$ and $\Delta t$.

# Stability analysis of the $\theta$-scheme

Consider the $\theta$-scheme:

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} = (1-\theta)\,\frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{(\Delta x)^2} + \theta\,\frac{U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}}{(\Delta x)^2},$$

$$U_j^0 = u_0(x_j), \qquad j = 0, \pm 1, \pm 2, \ldots,$$

where $\theta \in [0,1]$ is a parameter.

For $\theta = 0$ it is the explicit Euler scheme, for $\theta = 1$ it is the implicit Euler scheme, and for $\theta = 1/2$ it is the arithmetic average of the two Euler schemes, and is called the **Crank–Nicolson scheme**.

Using an identical argument as in the case of the two Euler methods, we find that

$$\lambda(k) - 1 \quad = \quad -4(1-\theta)\,\mu\,\sin^2\left(\frac{k\Delta x}{2}\right) - 4\theta\,\mu\,\lambda(k)\sin^2\left(\frac{k\Delta x}{2}\right).$$

Therefore,

$$\lambda(k) = \frac{1 - 4(1-\theta)\mu\sin^2\left(\frac{k\Delta x}{2}\right)}{1 + 4\theta\mu\sin^2\left(\frac{k\Delta x}{2}\right)}.$$

For practical stability, we demand that

$$|\lambda(k)| \leq 1 \qquad \forall k \in [-\pi/\Delta x, \pi/\Delta x],$$

which holds if, and only if,

$$2(1-2\theta)\mu \leq 1.$$

Thus we have shown that:

- For $\theta \in [1/2, 1]$ the $\theta$-scheme is **unconditionally practically stable**;
- For $\theta \in [0, 1/2)$ the $\theta$-scheme is **conditionally practically stable**, the stability condition being that

$$\mu \leq \frac{1}{2(1-2\theta)}.$$