Numerical Solution of Partial Differential Equations

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Lecture 11

The discrete maximum principle

Theorem (Discrete maximum principle for the θ-scheme)

The θ-scheme for the Dirichlet initial-boundary-value problem for the heat equation, with $0 \le \theta \le 1$ and $\mu(1-\theta) \le \frac{1}{2}$, yields a sequence of numerical approximations $\{U_j^m\}_{j=0,\dots,J;\ m=0,\dots,M}$ satisfying

$$U_{\min} \le U_j^m \le U_{\max}$$

where

$$U_{\min} = \min\left\{\min\{U_0^m\}_{m=0}^{M},\ \min\{U_j^0\}_{j=0}^{J},\ \min\{U_J^m\}_{m=0}^{M}\right\}$$

and

$$U_{\max} = \max\left\{\max\{U_0^m\}_{m=0}^{M},\ \max\{U_j^0\}_{j=0}^{J},\ \max\{U_J^m\}_{m=0}^{M}\right\}.$$

PROOF: We rewrite the θ-scheme as

$$(1+2\theta\mu)\,U_j^{m+1} = \theta\mu\left(U_{j+1}^{m+1} + U_{j-1}^{m+1}\right) + (1-\theta)\mu\left(U_{j+1}^{m} + U_{j-1}^{m}\right) + [1-2(1-\theta)\mu]\,U_j^m,$$

and recall that, by hypothesis,

$$\theta\mu \geq 0 \qquad (1-\theta)\mu \geq 0, \qquad 1-2(1-\theta)\mu \geq 0.$$

Suppose that U attains its maximum value U_j^{m+1} at an internal mesh point (x_j, t_{m+1}) where $j \in \{1,\ldots,J-1\}$, $m \in \{0,\ldots,M-1\}$. If this is not the case, the proof is complete.

We define

$$U^\star := \max\{U_{j+1}^{m+1},\, U_{j-1}^{m+1},\, U_{j+1}^{m},\, U_{j-1}^{m},\, U_j^m\}.$$

Then,

$$\begin{aligned}(1+2\theta\mu)\,U_j^{m+1} &\le 2\theta\mu U^\star + 2(1-\theta)\mu U^\star \\ &\quad + [1-2(1-\theta)\mu]U^\star = (1+2\theta\mu)\,U^\star,\end{aligned}$$

and therefore

$$U_j^{m+1} \le U^\star.$$

However, also,

$$U^\star \le U_j^{m+1},$$

as U_j^{m+1} is assumed to be the overall maximum value. Hence,

$$U_j^{m+1} = U^\star.$$

Thus the maximum value is also attained at all mesh points neighbouring (x_j, t_{m+1}) present in the stencil of the θ-scheme.

The same argument then applies to these neighbouring points, and we can then repeat this process until the boundary at $x = a$ or $x = b$ or at $t = 0$ is reached, in a finite number of steps.

The maximum is therefore attained at a boundary point.

By an identical argument the minimum is attained at a boundary point. $\diamond$

In summary then, for

$$\mu(1-\theta) \leq \frac{1}{2}$$

the θ-scheme satisfies the discrete maximum principle.

This condition is clearly more demanding than the ℓ_2-stability condition:

$$\mu(1-2\theta) \leq \frac{1}{2} \qquad \text{for} \qquad 0 \leq \theta \leq \tfrac{1}{2}.$$

E.g., the Crank-Nicolson scheme is unconditionally stable in the ℓ_2 norm, yet it only satisfies the discrete maximum principle when $\mu := \frac{\Delta t}{(\Delta x)^2} \leq 1$.

Convergence of the θ-scheme in the maximum norm

We close our discussion of finite difference schemes for the heat equation in one space-dimension with the convergence analysis of the θ-scheme for the Dirichlet initial-boundary-value problem.

We begin by rewriting the scheme as follows:

$$(1+2\theta\mu)\,U_j^{m+1} = \theta\mu\left(U_{j+1}^{m+1}+U_{j-1}^{m+1}\right) + (1-\theta)\mu\left(U_{j+1}^{m}+U_{j-1}^{m}\right) + [1-2(1-\theta)\mu]\,U_j^{m}.$$

The scheme is considered subject to the initial condition

$$U_j^0 = u_0(x_j), \qquad j = 0,\dots,J,$$

and the boundary conditions

$$U_0^{m+1} = A(t_{m+1}), \quad U_J^{m+1} = B(t_{m+1}), \quad m = 0,\dots,M-1.$$

The **consistency error** for the θ-scheme is defined by

$$T_j^m = \frac{u_j^{m+1} - u_j^m}{\Delta t} - (1-\theta)\,\frac{u_{j+1}^m - 2u_j^m + u_{j-1}^m}{(\Delta x)^2}$$
$$- \theta\,\frac{u_{j+1}^{m+1} - 2u_j^{m+1} + u_{j-1}^{m+1}}{(\Delta x)^2}, \quad \left\{ \begin{array}{l} j = 1, \ldots, J-1, \\ m = 0, \ldots, M-1, \end{array} \right.$$

where $u_j^m \equiv u(x_j, t_m)$, and therefore

$$(1+2\theta\mu)\,u_j^{m+1} = \theta\mu\left(u_{j+1}^{m+1} + u_{j-1}^{m+1}\right) + (1-\theta)\mu\left(u_{j+1}^m + u_{j-1}^m\right)$$
$$+ [1 - 2(1-\theta)\mu]\,u_j^m + \Delta t T_j^m, \quad \left\{ \begin{array}{l} j = 1, \ldots, J-1, \\ m = 0 \ldots, M-1. \end{array} \right.$$

Define the **global error**, that is the discrepancy at a mesh-point between the exact solution and its numerical approximation, by

$$e_j^m := u(x_j, t_m) - U_j^m, \quad \begin{cases} j = 0, \dots, J, \\ m = 0, \dots, M. \end{cases}$$

It then follows that

$$e_0^{m+1} = 0,\ e_J^{m+1} = 0,\ e_j^0 = 0, \quad j = 0, \dots, J,$$

and

$$\begin{aligned}(1 + 2\theta\mu)\, e_j^{m+1} = \theta\mu \left(e_{j+1}^{m+1} + e_{j-1}^{m+1}\right) + (1-\theta)\mu \left(e_{j+1}^m + e_{j-1}^m\right) \\ + [1 - 2(1-\theta)\mu]\, e_j^m + \Delta t T_j^m, \quad \begin{cases} j = 1, \dots, J-1, \\ m = 0, \dots, M-1. \end{cases}\end{aligned}$$

We define,

$$E^m = \max_{0 \le j \le J} |e_j^m| \quad \text{and} \quad T^m = \max_{1 \le j \le J-1} |T_j^m|.$$

As, by hypothesis,

$$\theta\mu \geq 0, \qquad (1-\theta)\mu \geq 0, \qquad 1 - 2(1-\theta)\mu \geq 0,$$

we have that

$$(1+2\theta\mu)E^{m+1} \leq 2\theta\mu E^{m+1} + E^m + \Delta t T^m.$$

Hence,

$$E^{m+1} \leq E^m + \Delta t\, T^m.$$

As $E^0 = 0$, upon summation,

$$\begin{aligned} E^m &\leq \Delta t \sum_{n=0}^{m-1} T^n \\ &\leq m\Delta t \max_{0\leq n\leq m-1} T^n \\ &\leq T \max_{0\leq m\leq M-1} \max_{1\leq j\leq J-1} |T_j^m|, \end{aligned}$$

which then implies that

$$\max_{0\leq j\leq J} \max_{0\leq m\leq M} |u(x_j, t_m) - U_j^m| \leq T \max_{1\leq j\leq J-1} \max_{0\leq m\leq M-1} |T_j^m|.$$

Recall that the consistency error of the θ-scheme is

$$T_j^m = \begin{cases} \mathcal{O}\left((\Delta x)^2 + (\Delta t)^2\right) & \text{for } \theta = 1/2, \\ \mathcal{O}\left((\Delta x)^2 + \Delta t\right) & \text{for } \theta \neq 1/2. \end{cases}$$

For the explicit/implicit Euler schemes, for which

$$T_j^m = \mathcal{O}\left((\Delta x)^2 + \Delta t\right),$$

one has the following bound on the global error:

$$\max_{0\le j\le J} \max_{0\le m\le M} |u(x_j, t_m) - U_j^m| \le \mathrm{Const.}\left((\Delta x)^2 + \Delta t\right),$$

while for the Crank–Nicolson scheme, which has consistency error

$$T_j^m = \mathcal{O}\left((\Delta x)^2 + (\Delta t)^2\right),$$

one has

$$\max_{0\le j\le J} \max_{0\le m\le M} |u(x_j, t_m) - U_j^m| \le \mathrm{Const.}\left((\Delta x)^2 + (\Delta t)^2\right).$$

Finite difference approximation in two space-dimensions

Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \qquad (x, y) \in \Omega := (a, b) \times (c, d), \;\; t \in (0, T],$$

subject to the initial condition

$$u(x, y, 0) = u_0(x, y), \qquad (x, y) \in [a, b] \times [c, d],$$

and the Dirichlet boundary condition

$$u|_{\partial\Omega} = B(x, y, t), \qquad (x, y) \in \partial\Omega, \;\; t \in (0, T],$$

where $\partial\Omega$ is the boundary of Ω.

We begin by considering the explicit Euler finite difference scheme for this problem.

The explicit Euler scheme

Let

$$\delta_x^2 U_{ij} := U_{i+1,j} - 2U_{ij} + U_{i-1,j},$$

and

$$\delta_y^2 U_{ij} := U_{i,j+1} - 2U_{ij} + U_{i,j-1}.$$

Let, further, $\Delta x := (b-a)/J_x$, $\Delta y := (d-c)/J_y$, $\Delta t := T/M$, and define

$$\begin{aligned} x_i &= a + i\Delta x, \qquad & i &= 0, \ldots, J_x, \\ y_j &= c + j\Delta y, \qquad & j &= 0, \ldots, J_y, \\ t_m &= m\Delta t, \qquad & m &= 0, \ldots, M. \end{aligned}$$

The explicit Euler finite difference scheme for the unsteady heat equation on the space-time domain $\overline{\Omega} \times [0, T]$ is then:

$$\frac{U_{ij}^{m+1} - U_{ij}^{m}}{\Delta t} = \frac{\delta_x^2 U_{ij}^m}{(\Delta x)^2} + \frac{\delta_y^2 U_{ij}^m}{(\Delta y)^2},$$

for $i = 1, \ldots, J_x - 1$, $j = 1, \ldots, J_y - 1$, $m = 0, 1, \ldots, M - 1$, subject to the initial condition

$$U_{ij}^0 = u_0(x_i, y_j), \quad i = 0, \ldots, J_x, \;\; j = 0, \ldots, J_y,$$

and the boundary condition

$$U_{ij}^m = B(x_i, y_j, t_m), \;\; \text{at the boundary mesh points, for } m = 1, \ldots, M.$$

The implicit Euler scheme

Let $\Delta x := (b-a)/J_x$, $\Delta y := (d-c)/J_y$, $\Delta t := T/M$, and define

$$\begin{aligned} x_i &= a + i\Delta x, \qquad & i &= 0,\ldots,J_x, \\ y_j &= c + j\Delta y, \qquad & j &= 0,\ldots,J_y, \\ t_m &= m\Delta t, \qquad & m &= 0,\ldots,M. \end{aligned}$$

The implicit Euler finite difference scheme for the problem is then

$$\frac{U_{ij}^{m+1} - U_{ij}^{m}}{\Delta t} = \frac{\delta_x^2 U_{ij}^{m+1}}{(\Delta x)^2} + \frac{\delta_y^2 U_{ij}^{m+1}}{(\Delta y)^2},$$

for $i = 1,\ldots,J_x - 1$, $j = 1,\ldots,J_y - 1$, $m = 0,1,\ldots,M-1$, subject to the initial condition

$$U_{ij}^{0} = u_0(x_i, y_j), \quad i = 0, \dots, J_x, \;\; j = 0, \dots, J_y,$$

and the boundary condition

$$U_{ij}^{m+1} = B(x_i, y_j, t_{m+1}), \qquad \text{at the boundary mesh points,}$$

$$\text{for } m = 0, \dots, M-1.$$

The θ-scheme

Let $\Delta x := (b-a)/J_x$, $\Delta y := (d-c)/J_y$, $\Delta t := T/M$, and, for $\theta \in [0,1]$, consider the finite difference scheme

$$\frac{U_{ij}^{m+1} - U_{ij}^{m}}{\Delta t} = (1-\theta)\left(\frac{\delta_x^2 U_{ij}^m}{(\Delta x)^2} + \frac{\delta_y^2 U_{ij}^m}{(\Delta y)^2}\right) + \theta\left(\frac{\delta_x^2 U_{ij}^{m+1}}{(\Delta x)^2} + \frac{\delta_y^2 U_{ij}^{m+1}}{(\Delta y)^2}\right)$$

for $i = 1, \ldots, J_x - 1$, $j = 1, \ldots, J_y - 1$, $m = 0, 1, \ldots, M-1$, subject to the initial condition

$$U_{ij}^0 = u_0(x_i, y_j), \quad i = 0, \ldots, J_x, \ \ j = 0, \ldots, J_y,$$

and the boundary condition

$$U_{ij}^{m+1} = B(x_i, y_j, t_{m+1}), \ \text{ at the boundary mesh points,}$$
$$\text{for } m = 0, \ldots, M-1.$$