Numerical Solution of Partial Differential Equations

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Lecture 12

Finite difference approximation of hyperbolic equations

The simplest example of a second-order linear hyperbolic equation is the linear wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f(x, t),$$

where $c > 0$ is the wave speed and f is a given source term.

When $f \equiv 0$ and the equation is considered on the whole real line, $x \in \mathbb{R}$, by supplying two initial conditions

$$u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R},$$
$$\frac{\partial u}{\partial t}(x,0) = u_1(x) \quad \text{for } x \in \mathbb{R},$$

where $u_0 \in C^2(\mathbb{R})$ and $u_1 \in C^1(\mathbb{R})$, the solution is given by d'Alembert's formula:

$$u(x,t) = \frac{1}{2}\left[u_0(x-ct) + u_0(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(\xi)\,\mathrm{d}\xi.$$

More generally, if $f \in C(\mathbb{R} \times [0, \infty))$, then

$$u(x,t) = \frac{1}{2}\left[u_0(x-ct) + u_0(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(\xi)\,\mathrm{d}\xi \quad + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(s,\tau)\,\mathrm{d}s\,\mathrm{d}\tau.$$

We shall be interested in a problem of the above form in the physically more realistic setting where $x \in [a,b]$, with $a < b$, and $t \in [0,T]$, $T > 0$.

Then, in addition to the two initial conditions above, boundary conditions need to be prescribed at $x = a$ and $x = b$, and the problem thus becomes an initial-boundary-value problem.

Finite difference approximation of hyperbolic equations

We shall be concerned with the initial-boundary-value problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} &= f(x,t) && \text{for } (x,t) \in (a,b) \times (0,T], \\ u(x,0) &= u_0(x) && \text{for } x \in [a,b], \\ \frac{\partial u}{\partial t}(x,0) &= u_1(x) && \text{for } x \in [a,b], \\ u(a,t) = 0 \quad \text{and} \quad u(b,t) &= 0 && \text{for } t \in [0,T]. \end{aligned} \tag{1}$$

Here, $f \in C((a,b) \times (0,T])$, $u_0, u_1 \in C([a,b])$, and we shall assume compatibility of the initial data with the boundary conditions, in the sense that $u_0(a) = u_0(b) = 0$ and $u_1(a) = u_1(b) = 0$; $c > 0$ is the wave speed.

Energy (in)equality

The analysis of the finite difference scheme for (1) is based on a‘discrete energy inequality’, which will imply the stability of the scheme.

We begin by describing the derivation of the ‘energy inequality’ for the solution of the initial-boundary-value problem (1).

Note:
The proof of existence of a solution to the initial-boundary-value problem (1) is beyond the scope of this course; we shall suppose here that a solution u to (1) exists and that u is sufficiently smooth, so that our calculations are meaningful.

We begin by multiplying the PDE $(1)_1$ by the time derivative of u, and we then integrate over the interval $[a, b]$; thus,

$$\int_a^b \frac{\partial^2 u}{\partial t^2}(x,t)\,\frac{\partial u}{\partial t}(x,t)\,\mathrm{d}x - c^2 \int_a^b \frac{\partial^2 u}{\partial x^2}(x,t)\,\frac{\partial u}{\partial t}(x,t)\,\mathrm{d}x = \int_a^b f(x,t)\frac{\partial u}{\partial t}(x,t)\,\mathrm{d}x. \tag{2}$$

As $u(a, t) = 0$ and $u(b, t) = 0$ for all $t \in [0, T]$, it follows that

$$\frac{\partial u}{\partial t}(a,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(b,t) = 0 \quad \text{for all } t \in [0, T].$$

By partial integration with respect to x in the second term on the left-hand side of (2):

$$\begin{aligned} &\int_a^b \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}(x,t)\right)\frac{\partial u}{\partial t}(x,t)\,\mathrm{d}x \\ &\quad + c^2\int_a^b \frac{\partial u}{\partial x}(x,t)\,\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}(x,t)\right)\,\mathrm{d}x = \int_a^b f(x,t)\frac{\partial u}{\partial t}(x,t)\,\mathrm{d}x. \end{aligned} \tag{3}$$

Clearly,

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 \quad \text{and} \quad \frac{\partial u}{\partial x}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) = \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2,$$

and therefore

$$\begin{aligned} &\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \left(\frac{\partial u}{\partial t}\right)^2(x,t)\,\mathrm{d}x + \frac{c^2}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \left(\frac{\partial u}{\partial x}\right)^2(x,t)\,\mathrm{d}x \\ &\quad = \int_a^b f(x,t)\frac{\partial u}{\partial t}(x,t)\,\mathrm{d}x. \end{aligned} \tag{4}$$

When f is identically zero, the r.h.s. of (4) vanishes, and after integrating the expression from 0 to t, for any $t \in (0, T]$, we have

$$\begin{aligned} &\frac{1}{2}\int_a^b \left(\frac{\partial u}{\partial t}\right)^2 (x,t)\,\mathrm{d}x + \frac{c^2}{2}\int_a^b \left(\frac{\partial u}{\partial x}\right)^2 (x,t)\,\mathrm{d}x \\ &\quad = \frac{1}{2}\int_a^b \left(\frac{\partial u}{\partial t}\right)^2 (x,0)\,\mathrm{d}x + \frac{c^2}{2}\int_a^b \left(\frac{\partial u}{\partial x}\right)^2 (x,0)\,\mathrm{d}x. \end{aligned} \tag{5}$$

The l.h.s. side of the equality (5) can be viewed as the 'total energy' at time t and the r.h.s. as the 'initial total energy'. Thus, (5) expresses conservation of the total energy during the course of the evolution of the solution from time 0 to time $t \in (0, T]$, in the absence of a source term.

After multiplying (5) by 2 and defining

$$\mathcal{L}^2(u(\cdot,t)) := \int_a^b \left(\frac{\partial u}{\partial t}\right)^2 (x,t)\,\mathrm{d}x + c^2 \int_a^b \left(\frac{\partial u}{\partial x}\right)^2 (x,t)\,\mathrm{d}x$$

for $t \in [0, T]$, the equality (5) can be rewritten as

$$\mathcal{L}^2(u(\cdot,t)) = \mathcal{L}^2(u(\cdot,0)) \qquad \text{for all } t \in [0, T].$$

It is this argument that we shall try to mimic in our stability analysis of the finite difference approximation of the problem when $f \equiv 0$.

Note: The mapping

$$u \mapsto \max_{t\in[0,T]} [\mathcal{L}^2(u(\cdot,t))]^{1/2}$$

is a norm on the linear space of all functions $u \in C^1([a,b] \times [0,T])$ such that $u(a,t) = u(b,t) = 0$ for all $t \in [0,T]$, called the energy norm.

More generally, if f is not identically zero, then (4) implies that

$$\mathcal{L}^2(u(\cdot,t)) = \mathcal{L}^2(u(\cdot,0)) + 2\int_0^t \int_a^b f(x,t)\frac{\partial u}{\partial t}(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau.$$

As

$$2\alpha\beta \leq \alpha^2 + \beta^2, \qquad \text{for all } \alpha,\beta \in \mathbb{R},$$

it follows that

$$\begin{aligned}\mathcal{L}^2(u(\cdot,t)) &\leq \mathcal{L}^2(u(\cdot,0)) + \int_0^t \int_a^b f^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \int_a^b \left(\frac{\partial u}{\partial t}\right)^2 (x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leq \mathcal{L}^2(u(\cdot,0)) + \int_0^t \int_a^b f^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \mathcal{L}^2(u(\cdot,\tau))\,\mathrm{d}\tau.\end{aligned} \tag{6}$$

To proceed, we need the following result.

Lemma (Gronwall's Lemma)

Suppose that A and B are continuous real-valued nonnegative functions defined on $[0, T]$, and B is a nondecreasing function of its argument. Let

$$A(t) \leq B(t) + \int_0^t A(s)\,\mathrm{d}s$$

for all $t \in [0, T]$; then,

$$A(t) \leq \mathrm{e}^t\, B(t)$$

for all $t \in [0, T]$.

PROOF: Clearly,

$$\mathrm{e}^{-t} A(t) - \mathrm{e}^{-t} \int_0^t A(s)\,\mathrm{d}s \leq \mathrm{e}^{-t} B(t),$$

and thus, equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{-t} \int_0^t A(s)\,\mathrm{d}s\right] \leq \mathrm{e}^{-t} B(t).$$

By integrating this and noting that the expression in the square brackets vanishes at $t = 0$,

$$\mathrm{e}^{-t} \int_0^t A(s)\,\mathrm{d}s \leq \int_0^t \mathrm{e}^{-s} B(s)\,\mathrm{d}s.$$

Multiplying by e^t, and since B is a nondecreasing nonnegative function, whereby $B(s) \leq B(t)$ for all $s \in [0, t]$, we have that

$$\int_0^t A(s)\,\mathrm{d}s \leq \mathrm{e}^t B(t) \int_0^t \mathrm{e}^{-s}\,\mathrm{d}s = \mathrm{e}^t B(t)\,(1 - \mathrm{e}^{-t}) = \mathrm{e}^t B(t) - B(t).$$

Substituting this into the r.h.s. of the inequality in the statement of the lemma gives: $A(t) \leq B(t) + \mathrm{e}^t B(t) - B(t) = \mathrm{e}^t B(t)$. $\square$

We now return to (6) and set

$$A(t) := \mathcal{L}^2(u(\cdot,t)) \quad \text{and} \quad B(t) := \mathcal{L}^2(u(\cdot,0)) + \int_0^t \int_a^b f^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau.$$

It then follows from Gronwall's lemma that $A(t) \leq \mathrm{e}^t B(t)$, that is

$$\mathcal{L}^2(u(\cdot,t)) \leq \mathrm{e}^t \left(\mathcal{L}^2(u(\cdot,0)) + \int_0^t \int_a^b f^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \right),$$

with

$$\mathcal{L}^2(u(\cdot,t)) := \int_a^b \left(\frac{\partial u}{\partial t}\right)^2 (x,t)\,\mathrm{d}x + c^2 \int_a^b \left(\frac{\partial u}{\partial x}\right)^2 (x,t)\,\mathrm{d}x$$

and

$$\mathcal{L}^2(u(\cdot,0)) := \int_a^b \left(\frac{\partial u}{\partial t}\right)^2 (x,0)\,\mathrm{d}x + c^2 \int_a^b \left(\frac{\partial u}{\partial t}\right)^2 (x,0)\,\mathrm{d}x$$

$$= \|u_1\|^2_{L_2((a,b))} + c^2\,|u_0|^2_{H^1((a,b))}$$

This is the desired energy inequality satisfied by the solution.

It provides a bound on the (square of the) energy-norm of the solution in terms of the (square of the) norm of the initial data and the (square of the) L_2 norm of the source term f.

We shall mimic the derivation of this energy inequality in the stability analysis of the implicit and explicit finite difference schemes for the initial-boundary-value problem (1).