

Numerical Solution of Partial Differential Equations

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2024

Lecture 15

First-order hyperbolic equations: initial-boundary-value problem and energy estimate

Let Ω be a bounded open set in \mathbb{R}^n , $n \geq 1$, with boundary $\Gamma = \partial\Omega$, and let $T > 0$. In $Q = \Omega \times (0, T]$, we consider the initial boundary-value problem

$$\frac{\partial u}{\partial t} + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} + c(x, t)u = f(x, t), \quad x \in \Omega, \quad t \in (0, T], \quad (1)$$

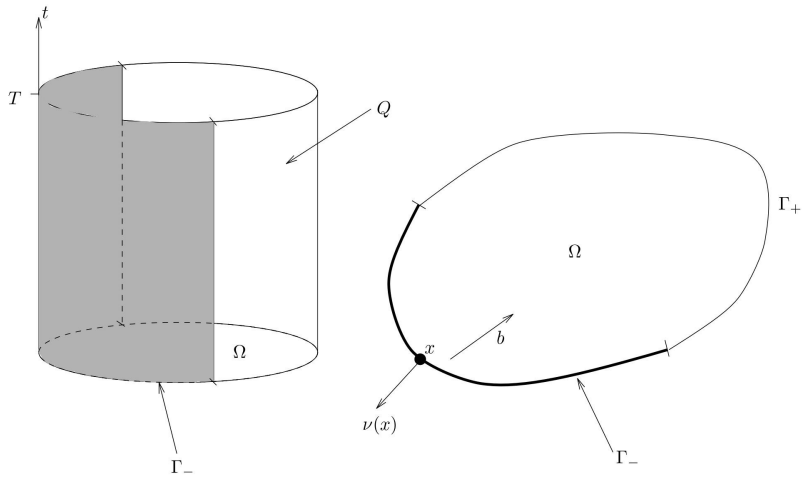
$$u(x, t) = 0, \quad x \in \Gamma_-, \quad t \in [0, T], \quad (2)$$

$$u(x, 0) = u_0(x) \quad x \in \overline{\Omega}, \quad (3)$$

where

$$\Gamma_- = \{x \in \Gamma : b(x) \cdot \nu(x) < 0\},$$

$b = (b_1, \dots, b_n)$ and $\nu(x)$ denotes the unit outward normal to Γ at $x \in \Gamma$. Γ_- will be called the *inflow boundary*. Its complement, $\Gamma_+ = \Gamma \setminus \Gamma_-$, will be referred to as the *outflow boundary*.



Continuous dependence of the solution on the data

We shall assume that

$$b_i \in C^1(\overline{\Omega}), \quad i = 1, \dots, n, \quad (4)$$

$$c \in C(\overline{Q}), \quad f \in L_2(Q), \quad (5)$$

$$u_0 \in L^2(\Omega). \quad (6)$$

In order to ensure consistency between the initial and the boundary condition, we shall suppose that $u_0(x) = 0$, $x \in \Gamma_-$.

We make the additional hypothesis:

$$c(x, t) - \frac{1}{2} \sum_{i=1}^n \frac{\partial b_i}{\partial x_i}(x) \geq 0, \quad x \in \overline{\Omega}, \quad t \in [0, T]. \quad (7)$$

By taking the inner product in $L_2(\Omega)$ of the equation (1) with $u(\cdot, t)$, performing partial integration and noting the boundary condition (2):

$$\begin{aligned} & \left(\frac{\partial u}{\partial t}(\cdot, t), u(\cdot, t) \right) + \left(c(\cdot, t) - \frac{1}{2} \sum_{i=1}^n \frac{\partial b_i}{\partial x_i}(\cdot), u^2(\cdot, t) \right) \\ & + \frac{1}{2} \int_{\Gamma_+} \left[\sum_{i=1}^n b_i(x) \nu_i(x) \right] u^2(x, t) \, ds(x) = (f(\cdot, t), u(\cdot, t)), \end{aligned} \quad (8)$$

where $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$ is the unit outward normal vector to Γ at $x \in \Gamma$.

By virtue of (7) and noting that

$$\begin{aligned}\left(\frac{\partial u}{\partial t}, u\right) &= \int_{\Omega} \frac{\partial u}{\partial t}(x, t) u(x, t) dx \\&= \int_{\Omega} \frac{1}{2} \frac{\partial}{\partial t} u^2(x, t) dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2(x, t) dx \\&= \frac{1}{2} \frac{d}{dt} \|u(\cdot, t)\|^2,\end{aligned}$$

it follows from (8) that

$$\frac{1}{2} \frac{d}{dt} \|u(\cdot, t)\|^2 + \frac{1}{2} \int_{\Gamma_+} \left[\sum_{i=1}^n b_i(x) \nu_i(x) \right] u^2(x, t) ds(x) \leq (f, u).$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned}(f, u) &\leq \|f(\cdot, t)\| \|u(\cdot, t)\| \\&\leq \frac{1}{2} \|f(\cdot, t)\|^2 + \frac{1}{2} \|u(\cdot, t)\|^2,\end{aligned}$$

and therefore, for any $t \in [0, T]$,

$$\frac{d}{dt} \|u(\cdot, t)\|^2 + \int_{\Gamma_+} \left[\sum_{i=1}^n b_i(x) \nu_i(x) \right] u^2(x, t) ds(x) - \|u(\cdot, t)\|^2 \leq \|f(\cdot, t)\|^2.$$

Multiplying both sides by e^{-t} , this inequality can be rewritten as follows:

$$\frac{d}{dt} (e^{-t} \|u(\cdot, t)\|^2) + e^{-t} \int_{\Gamma_+} \left[\sum_{i=1}^n b_i(x) \nu_i(x) \right] u^2(x, t) ds \leq e^{-t} \|f(\cdot, t)\|^2,$$

Integrating this inequality w.r.t. t and noting the initial condition (3),

$$\begin{aligned} e^{-t} \|u(\cdot, t)\|^2 + \int_0^t e^{-\tau} \int_{\Gamma_+} \left[\sum_{i=1}^n b_i(x) \nu_i(x) \right] u^2(x, \tau) ds(x) d\tau \\ \leq \|u_0\|^2 + \int_0^t e^{-\tau} \|f(\cdot, \tau)\|^2 d\tau, \quad t \in [0, T]. \end{aligned}$$

It therefore follows that

$$\begin{aligned} \|u(\cdot, t)\|^2 + \int_0^t e^{t-\tau} \int_{\Gamma_+} \left[\sum_{i=1}^n b_i(x) \nu_i(x) \right] u^2(x, \tau) ds(x) d\tau \\ \leq e^t \|u_0\|^2 + \int_0^t e^{t-\tau} \|f(\cdot, \tau)\|^2 d\tau, \quad t \in [0, T]. \end{aligned} \quad (9)$$

This ‘energy inequality’ expresses continuous dependence of the solution to (1)–(3) on the data. It also implies uniqueness of the solution.

Let us consider a particularly important case when

$$c \equiv 0, \quad f \equiv 0, \quad \text{and} \quad \operatorname{div} b = \sum_{i=1}^n \frac{\partial b_i}{\partial x_i} \equiv 0,$$

where $b(x) = (b_1(x), \dots, b_n(x))$. Then, thanks to the identity (8),

$$\frac{1}{2} \frac{d}{dt} \|u(\cdot, t)\|^2 + \frac{1}{2} \int_{\Gamma_+} [b(x) \cdot \nu(x)] u^2(x, t) \, ds(x) = 0,$$

and therefore,

$$\|u(\cdot, t)\|^2 + \int_0^t \int_{\Gamma_+} [b(x) \cdot \nu(x)] u^2(x, \tau) \, ds(x) \, d\tau = \|u_0\|^2,$$

which can be viewed as an identity expressing ‘conservation of energy’ for the initial-boundary-value problem (1)–(3).

Explicit finite difference approximation

We focus on a special case of the problem: the constant- coefficient hyperbolic equation in one space dimension

$$\frac{\partial u}{\partial t} + b \frac{\partial u}{\partial x} = f(x, t), \quad x \in (0, 1), \quad t \in (0, T], \quad (10)$$

subject to the boundary and initial conditions

$$u(x, t) = 0, \quad x \in \Gamma_-, \quad t \in [0, T], \quad (11)$$

$$u(x, 0) = u_0(x), \quad x \in [0, 1]. \quad (12)$$

If $b > 0$ then $\Gamma_- = \{0\}$, and if $b < 0$ then $\Gamma_- = \{1\}$. Let us assume, for example, that $b > 0$. Then the appropriate boundary condition is

$$u(0, t) = 0, \quad t \in [0, T]. \quad (13)$$

To construct a finite difference approximation of (10)–(13) let $\Delta x := 1/J$ be the mesh-size in the x -direction and $\Delta t := T/M$ the mesh-size in the time-direction, t . Let us also define

$$x_j := j\Delta x, \quad j = 0, \dots, J, \quad t_m := m\Delta t, \quad m = 0, \dots, M.$$

At the mesh-point (x_j, t_m) , (10) is approximated by the explicit finite difference scheme

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} + b D_x^- U_j^m = f(x_j, t_m), \quad j = 1, \dots, J, \quad (14)$$

$$m = 0, \dots, M - 1,$$

subject to the boundary and initial condition, respectively:

$$U_0^m = 0, \quad m = 0, \dots, M, \quad (15)$$

$$U_j^0 = u_0(x_j), \quad j = 0, \dots, J. \quad (16)$$

Equivalently, this can be written as follows:

$$U_j^{m+1} = (1 - \mu)U_j^m + \mu U_{j-1}^m + \Delta t f(x_j, t_m), \quad \begin{cases} j = 1, \dots, J, \\ m = 0, \dots, M - 1, \end{cases}$$

in conjunction with

$$\begin{aligned} U_0^m &= 0, & m &= 0, \dots, M, \\ U_j^0 &= u_0(x_j), & j &= 0, \dots, J, \end{aligned}$$

where

$$\mu = \frac{b\Delta t}{\Delta x};$$

μ is called the CFL (or Courant–Friedrichs–Lewy) number.

The explicit finite difference scheme (14) is frequently called the *first-order upwind scheme*.

We shall explore the stability of this scheme in the discrete maximum norm. Suppose that $0 \leq \mu \leq 1$; then

$$\begin{aligned} \left| U_j^{m+1} \right| &\leq (1 - \mu) \left| U_j^m \right| + \mu \left| U_{j-1}^m \right| + \Delta t \left| f(x_j, t_m) \right| \\ &\leq (1 - \mu) \max_{0 \leq j \leq J} \left| U_j^m \right| + \mu \max_{1 \leq j \leq J+1} \left| U_{j-1}^m \right| + \Delta t \max_{0 \leq j \leq J} \left| f(x_j, t_m) \right| \\ &= \max_{0 \leq j \leq J} \left| U_j^m \right| + \Delta t \max_{0 \leq j \leq J} \left| f(x_j, t_m) \right|. \end{aligned}$$

Thus we have that

$$\max_{0 \leq j \leq J} \left| U_j^{m+1} \right| \leq \max_{0 \leq j \leq J} \left| U_j^m \right| + \Delta t \max_{0 \leq j \leq J} \left| f(x_j, t_m) \right|.$$

Let us define the mesh-dependent norm

$$\|U\|_{\infty} := \max_{0 \leq j \leq J} |U_j|;$$

then

$$\|U^{m+1}\|_{\infty} \leq \|U^m\|_{\infty} + \Delta t \|f(\cdot, t_m)\|_{\infty}, \quad m = 0, \dots, M-1.$$

Summing through m , we get

$$\max_{1 \leq k \leq M} \|U^k\|_{\infty} \leq \|U^0\|_{\infty} + \sum_{m=0}^{M-1} \Delta t \|f(\cdot, t_m)\|_{\infty}, \quad (17)$$

which expresses the stability of the finite difference scheme (14)–(16) under the condition

$$0 \leq \mu = \frac{b\Delta t}{\Delta x} \leq 1. \quad (18)$$

Thus we have proved that the finite difference scheme (14)–(16) is conditionally stable, the condition being that the CFL number $\mu \in [0, 1]$.

It is possible to show that the scheme (14)–(16) is also stable in the mesh-dependent L_2 -norm, $\|\cdot\|$, defined by

$$\|V\|^2 = \sum_{i=1}^J \Delta x V_i^2.$$

The associated inner product is

$$(V, W) := \sum_{i=1}^J \Delta x V_i W_i.$$

Since

$$U_j^m = \frac{U_j^m + U_{j-1}^m}{2} + \frac{U_j^m - U_{j-1}^m}{2},$$

and $U_0^m = 0$, it follows that

$$\begin{aligned}(D_x^- U^m, U^m) &= \sum_{j=1}^J \Delta x \frac{U_j^m - U_{j-1}^m}{\Delta x} U_j^m \\&= \frac{1}{2} \sum_{j=1}^J \{(U_j^m)^2 - (U_{j-1}^m)^2\} + \frac{\Delta x}{2} \sum_{j=1}^J \Delta x \left(\frac{U_j^m - U_{j-1}^m}{\Delta x} \right)^2 \quad (19) \\&= \frac{1}{2} (U_J^m)^2 + \frac{\Delta x}{2} \|D_x^- U^m\|^2.\end{aligned}$$

In addition, since

$$U_j^m = \frac{U_j^{m+1} + U_j^m}{2} - \frac{U_j^{m+1} - U_j^m}{2}$$

for $m = 0, \dots, M-1$, we have for such m that

$$\left(\frac{U^{m+1} - U^m}{\Delta t}, U^m \right] = \frac{1}{2\Delta t} (\|U^{m+1}\|^2 - \|U^m\|^2) - \frac{\Delta t}{2} \left\| \frac{U^{m+1} - U^m}{\Delta t} \right\|^2. \quad (20)$$

By taking the $(\cdot, \cdot]$ -inner product of (14) with U^m and using (19) and (20):

$$\begin{aligned} & \|U^{m+1}\|^2 + \Delta t b(U_j^m)^2 + b \Delta x \Delta t \|D_x^- U^m\|^2 - \|U^m\|^2 \\ & - \Delta t^2 \left\| \frac{U^{m+1} - U^m}{\Delta t} \right\|^2 = 2\Delta t (f^m, U^m], \quad m = 0, \dots, M-1. \end{aligned} \quad (21)$$

First suppose that $f \equiv 0$; then,

$$\frac{U^{m+1} - U^m}{\Delta t} = -b D_x^- U^m,$$

and by substituting this into the last term on the left-hand side of the equality (21) we have that, for $m = 0, \dots, M-1$,

$$\|U^{m+1}\|^2 + \Delta t b |U_J^m|^2 + b \Delta x \Delta t (1 - \mu) \|D_x^- U^m\|^2 = \|U^m\|^2.$$

Summing through m ,

$$\|U^k\|^2 + \sum_{m=0}^{k-1} \Delta t b |U_J^m|^2 + b \Delta x (1 - \mu) \sum_{m=0}^{k-1} \Delta t \|D_x^- U^m\|^2 = \|U^0\|^2 \quad (22)$$

for $k = 1, \dots, M$, which proves the stability of the scheme in the case when $f \equiv 0$ whenever

$$0 \leq \mu = \frac{b \Delta t}{\Delta x} \leq 1.$$

In particular, if $\mu = 1$, we have from (22) that

$$\|U^k\|^2 + \sum_{m=0}^{k-1} \Delta t \, b \, |U_J^m|^2 = \|U^0\|^2, \quad k = 1, \dots, M,$$

which is the discrete version of the identity (9), and expresses ‘conservation of energy’ in the discrete sense.

More generally, for $0 \leq \mu \leq 1$, (22) implies

$$\|U^k\|^2 + \sum_{m=0}^{k-1} \Delta t \, b \, |U_J^m|^2 \leq \|U^0\|^2, \quad k = 1, \dots, M.$$

Now consider the question of stability in the $\|\cdot\|$ -norm for $f \neq 0$. Since

$$\begin{aligned} \left\| \frac{U^{m+1} - U^m}{\Delta t} \right\|^2 &= \|f^m - bD_x^- U^m\|^2 \leq \{\|f^m\| + b\|D_x^- U^m\|\}^2 \\ &\leq \left(1 + \frac{1}{\epsilon'}\right) \|f^m\|^2 + (1 + \epsilon')b^2 \|D_x^- U^m\|^2, \quad \epsilon' > 0, \end{aligned}$$

and

$$(f^m, U^m) \leq \|f^m\| \|U^m\| \leq \frac{1}{2} \|f^m\|^2 + \frac{1}{2} \|U^m\|^2,$$

it follows from the equality (21) that

$$\begin{aligned} \|U^{m+1}\|^2 + \Delta t b \|U_n^m\|^2 + b \Delta x \Delta t \left[1 - (1 + \epsilon') \frac{b \Delta t}{\Delta x} \right] \|D_x^- U^m\|^2 \\ \leq \Delta t \left[\left(1 + \frac{1}{\epsilon'}\right) \Delta t + 1 \right] \|f^m\|^2 + (1 + \Delta t) \|U^m\|^2. \end{aligned}$$

Letting $\epsilon = 1 - 1/(1 + \epsilon') \in (0, 1)$ and assuming that

$$0 \leq \mu = \frac{b \Delta t}{\Delta x} \leq 1 - \epsilon,$$

we have, for $m = 0, \dots, M - 1$, that

$$\|U^{m+1}\|^2 + \Delta t b \|U_J^m\|^2 \leq \|U^m\|^2 + \Delta t \left(1 + \frac{\Delta t}{\epsilon}\right) \|f^m\|^2 + \Delta t \|U^m\|^2.$$

Upon summation of this inequality over $m = 0, \dots, k - 1$, we deduce that

$$\begin{aligned} \|U^k\|^2 + \left(\sum_{m=0}^{k-1} \Delta t b \|U_J^m\|^2 \right) &\leq \|U^0\|^2 + \left(1 + \frac{\Delta t}{\epsilon}\right) \sum_{m=0}^{k-1} \Delta t \|f^m\|^2 \\ &\quad + \sum_{m=0}^{k-1} \Delta t \|U^m\|^2 \end{aligned} \tag{23}$$

for $k = 1, \dots, M$.

To complete the proof of stability of the finite difference scheme we require the next lemma, which is easily proved by induction.

Lemma (Discrete Gronwall lemma)

Let (a_k) , (b_k) , (c_k) and (d_k) be four sequences of non-negative numbers such that the sequence (c_k) is non-decreasing and

$$a_k + b_k \leq c_k + \sum_{m=0}^{k-1} d_m a_m, \quad k \geq 1; \quad a_0 + b_0 \leq c_0.$$

Then

$$a_k + b_k \leq c_k \exp \left(\sum_{m=0}^{k-1} d_m \right), \quad k \geq 1.$$

By applying this lemma to the inequality (23) with

$$a_k := \|U^k\|^2, \quad k \geq 0,$$

$$b_k := \sum_{m=0}^{k-1} \Delta t b \|U_J^m\|^2, \quad k \geq 1; \quad b_0 = 0,$$

$$c_k := \|U^0\|^2 + \left(1 + \frac{\Delta t}{\epsilon}\right) \sum_{m=0}^{k-1} \Delta t \|f^m\|^2, \quad k \geq 1; \quad c_0 = \|U^0\|^2,$$

$$d_k := \Delta t, \quad k = 1, 2, \dots, M,$$

we obtain for $k = 1, \dots, M$:

$$\|U^k\|^2 + \sum_{m=0}^{k-1} \Delta t b \|U_J^m\|^2 \leq e^{t_k} \left(\|U^0\|^2 + \left(1 + \frac{\Delta t}{\epsilon}\right) \sum_{m=0}^{k-1} \Delta t \|f^m\|^2 \right),$$

where $t_k = k\Delta t$. Hence we deduce stability of the scheme, in the sense that

$$\max_{1 \leq k \leq M} \left(\|U^k\|^2 + \sum_{m=0}^{k-1} \Delta t b \|U_J^m\|^2 \right) \leq e^T \left(\|U^0\|^2 + \left(1 + \frac{\Delta t}{\epsilon}\right) \sum_{m=0}^{M-1} \Delta t \|f^m\|^2 \right).$$