Reminder on Quotient spaces

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1 Homeomorphisms

Definition 1. *Let X, Y be topological spaces. A map $f : X \to Y$ is a **homeomorphism** if f is 1-1, onto and both f and f^{-1} are continuous.*

Definition 2. *If there is a homeomorphism $f : X \to Y$, we say that X, Y are **homeomorphic** and denote this by $X \cong Y$.*

Remark 1. $\cong$ is an equivalence relation on any set of spaces

Spaces that are homeomorphic are the "**same**" from the point of view of topology. Intuitively, homeomorphic spaces have the *same "shape" if* we imagine our spaces to be *made of rubber*. In other wiords, two spaces are homeomorphic if we can stretch one so that it becomes the other; but we are not allowed to tear or to glue parts of the space.

For example a circle, a square loop, and an ellipse are all homeomorphic to each other. A football and a rugby ball are homeomorphic. A bagel and the surface of a mug are homeomorphic.

In the defintion, the condition on f^{-1} being continuous is actually necessary:

Example 1. Consider $X = [0, 1)$, $Y = S^1$ and $f : X \to Y$ given by $f(x) = (cos(2\pi x), sin(2\pi x))$. Then f is 1-1, onto continuous. But f^{-1} is not continuous, so f is not a homeomorphism.

However if X, Y are compact things are simpler.

Proposition 1. *Let X be a compact space, Y a Hausdorff space and $f : X \to Y$ a continuous map that is 1-1 and onto. Then f is a homeomorphism.*

Proof. To show that f^{-1} is continuous it is enough to show that if $K \subset X$ closed then $f(K)$ is also closed. Since K is closed and X is compact, K is compact. Therefore $f(K)$, the image of a compact space, is compact. Hence $f(K)$ is closed. $\square$

2 Quotient topology

Definition 3. *Let X be a topological space and $\sim$ an equivalence relation on X. For every $x \in X$, denote by $[x]$ its equivalence class.*
*The **quotient space** of X modulo $\sim$ is given by the set*

$$X/\sim \, = \{[x] : x \in X\}$$

*We have the **projection map** :*

$$p : X \to X/\sim \, , \, x \mapsto [x]$$

and we equip $X/\sim$ by the topology:
*$U \subseteq X/\sim$ is **open** iff $p^{-1}(U)$ is an open subset of X.*

Remark 2. This is the finest topology (i.e. the one with the greatest number of open sets) with respect to which p is continuous.

The quotient topology is a useful tool that allows us to construct easily interesting spaces, avoiding cumbersome constructions, using equations etc. For example it allows us to 'glue' spaces together.

A very important example of an equivalence relation comes from group actions:

Example 2. If a group G acts on a space X by homeomorphisms, then we have the *orbit equivalence relation*: $x \sim y$ if and only if $x = g \cdot y$ for some $g \in G$.

The nature of the quotient space in this case depends very much on the properties of the action: even if X is a very nice space, one needs some sort of "discreteness" for the action if the quotient space is to be a reasonable space.

Example 3.

1. Let $X = [0,1] \cup [2,3]$. We define an equivalence relation: $1 \sim 2$. Then $[1] = [2] = \{1,2\}$, while $[x] = \{x\}$, $\forall x \in X \setminus \{1,2\}$.

   Then $X/\sim$ is homeomorphic to $[0,1]$.

2. Let $X = [0,1]$ and $\sim$ an equivalence relation on X such that $0 \sim 1$ and $[x] = \{x\}$, $\forall x \in X \setminus \{0,1\}$. Then $X/\sim \, \cong S^1$.

   A homeomorphism is given by:

   $$f : X/\sim \to S^1, x \mapsto (\cos(2\pi x), \sin(2\pi x))$$

   This is well defined ($f(0) = f(1)$), 1-1, onto and its inverse is continuous.

3. Let $X = \mathbb{R}$ and $\sim$ equivalence relation on X, where for $x, y \in X$ we define

$$x \sim y \iff x - y \in \mathbb{Q}$$

Then $X/\sim$ is not *Hausdorff*. (prove this!)

4. Let $D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\}$, and define $\sim$ on D^2 by $a \sim b \iff a, b \in \partial D^2$ for all $a, b \in D^2$ (where $\partial D^2 = S^1$).

Then $D^2/\sim$ is homeomorphic to the sphere S^2.

We will come back to these examples later to give more detailed proofs of the homeomorphisms. Note that this is a very general definition and example 3 shows that things can go awry. In practice we will restrict to gentle equivalence relations.

Proposition 2. *If X is compact (connected), then the quotient space $X/\sim$ is also compact (connected).*

Proof. The projection map $p : X \to X/\sim$, is continuous and onto and the continuous image of a compact (connected) space is compact (connected). $\square$

Definition 4. *Let A be a subset of the topological space X and $\sim$ an equivalence relation on X.*

1. *The **saturation** of A with respect to $\sim$ is the set*

$$\hat{A} = \{x \in X / \exists a \in A : x \sim a\}$$

*If $\hat{A} = A$, then A is called **saturated**.*

2. *The relation $\sim$ is called **closed** if for every $A \subset X$ closed, $\hat{A}$ is also closed.*

3. *A Hausdorff space, X, is called **normal** if for any K_1, K_2 closed disjoint subsets of X, there are A_1, A_2 open disjoint subsets of X, such that $K_1 \subset A_1, K_2 \subset A_2$.*

Exercise 1. If A is open and saturated, show that $p(A)$ is an open subset of $X/\sim$.

We omit the proof of the following proposition. Informally what it says is that under a mild condition we can insure that the quotient space of a 'reasonable' topological space is also 'reasonable'.

Proposition 3. *Let X normal topological space and $\sim$ a closed equivalence relation on X. Then $X/\sim$ is normal.*

Example 4. Let X be a topological space and let $A \subset X$ be closed. We define the equivalence relation: $a \sim b \iff a, b \in A$.

We define $X/A := X/\sim$. From the previous proposition if X is normal, then $X/\sim$ is normal.

Proposition 4. *Let X, Y be topological spaces, $f : X \to Y$ continuous and $\sim$ an equivalence relation on X. If $f(x_1) = f(x_2), \forall x_1, x_2 \in X$ with $x_1 \sim x_2$, then the map $\bar{f} : X/\sim \to Y$, where $\bar{f}([x]) = f(x)$, is well defined and continuous.*

Proof. It is obvious that $\bar{f}$ is well defined. We show that $\bar{f}$ is continuous: let $U \subset Y$ open, then $\bar{f}^{-1}(U) \subset X/\sim$ and $p^{-1}(\bar{f}^{-1}(U)) = f^{-1}(U)$. $f^{-1}(U)$ is open since f is continuous and U is open. It follows that $\bar{f}^{-1}(U)$ is open, hence $\bar{f}$ is continuous. □

Example 5.

1. Let's show that $[0,1]/0 \sim 1$ is homeomorphic to S^1:

   We define $f : [0,1] \to S^1$ by $f(x) = e^{i2\pi x}$. This map continuous, onto and $f(0) = f(1)$. Hence $\bar{f}$ is continuous, 1-1 and since the spaces $[0,1]/0 \sim 1$ and S^1 are compact it follows that $\bar{f}$ is a homeomorphism.

2. Let's show that $D^2/\sim = D^2/\partial D^2$ is homeomorphic to S^2:

   It is easy to see (stereographic projection) that $S^2 \setminus \{N\} \cong \mathbb{R}^2 \cong \mathring{D}^2$.

   Let $\hat{f} : \mathring{D}^2 \to S^2 \setminus \{N\}$ a homeomorphism. We define $f : D^2 \to S^2$ as follows :

   $$f(x) = \begin{cases} \hat{f}(x) & \text{, if } x \in \mathring{D}^2 \\ N & \text{, if } x \in \partial D^2 \end{cases}$$

   For all $x, y \in \partial D$, $f(x) = f(y) = N$. Moreover f is continuous, onto, 1-1, and D, S^2 are compact, so $\bar{f} : D^2/\partial D^2 \to S^2$ is a homeomorphism.

   Similarly one shows that $D^n/\partial D^n \cong S^n$.

3. Let $X = S^1 \times I$ and $A = S^1 \times 1$. Then $X/A \cong D^2$. Indeed X is homeomorphic to the ring $C = \{x \in \mathbb{R}^2 \mid \frac{1}{2} \leqslant |x| \leqslant 1\}$ $A = \{x \in \mathbb{R}^2 \mid |x| = \frac{1}{2}\}$. We define

   $$f : C \to D^2 \text{ by } f(x) = 2(|x| - \frac{1}{2})x$$

   f is continuous and $f(x) = 0$, $\forall x \in A$, so it induces a continuous map $\bar{f} : C/A \to D^2$. Moreover $\bar{f}$ is 1-1 and onto so it gives a homeomorphism $C/A \cong D^2$.