B8.3: Mathematical Modelling of Financial Derivatives

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This version: March 4, 2024

Option pricing: binomial model

Overview

- Arbitrage pricing
- Binomial trees
- Risk-neutral valuation

Financial Options

Definition

A **European call (put) option** gives the right to the holder of the option to purchase (sell) the underlying, for example a stock S, at a pre-specified time, called the expiration date T, for a pre-specified amount known as the strike price K.

Definition

An **American call/put option** is like a European option with the difference that it can be exercised (i.e., buy or sell the underlying) at any time up until, and including, expiration T.

Pricing

Simple example of binomial tree setup to price a call option

- Assume that there are two possible states.
  - A stock is trading at 100 and tomorrow it will either
    - go up to 101 or
    - go down to 99.
- What is the value of a European call option with strike price $K = 100$?

Pricing

Simple example of binomial tree setup to price a call option

- Assume that there are two possible states.
  - A stock is trading at 100 and tomorrow it will either
    - go up to 101 or
    - go down to 99.
- What is the value of a European call option with strike price $K = 100$?
- What happens if the probability of landing in the down state is $q = \{.25\,, 0.5\,, 0.95\}$?

Simple world

- Two states of nature that occur with probability p and $q = 1 - p$, and two traded assets
- Asset 1 (A_1) pays 1 in state 1 and 1 in state 2, i.e., pays $(1, 1)$
- Asset 2 (A_2) pays 0 in state 1 and 3 in state 2, i.e., pays $(0, 3)$
- Price of A_1 is p_1 and of A_2 is p_2

Now, assume that there is a third asset in this simple economy paying (2,3). What is its initial price p_3?

Pricing asset A_3

- Set up a portfolio $\Pi(t=0)$ consisting of a units of A_1 and b units of A_2. Find a and b such that $\Pi(t=1) = A_3(1)$.

$$\Pi_u(1) = a \times 1 + b \times 0$$

and

$$\Pi_d(1) = a \times 1 + b \times 3\,.$$

- We require that $\Pi_u(1) = 2$ and $\Pi_d(1) = 3$, i.e., we replicate A_3's payoff.
- Therefore, $a = 2$ and $b = 1/3$ and at time $t = 0$,

$$p_3 = 2\,p_1 + \frac{1}{3}\,p_2\,.$$

Pricing a Call option in a Binomial model

- Two states of the world, up and down, with probabilities p and $q = 1 - p$, respectively.
- Starting value of stock is S.
- In the 'up' state, with probability p, asset becomes $u\,S$ where u is a constant.
- In the 'down' state asset becomes $d\,S$ where d is a constant.
- There is a risk-free bond that pays a constant interest rate r.
- In the up state the payoff of the call is

$$C_u^E = \max(u\,S - K, 0)\,.$$

- In the down state the payoff of the call is

$$C_d^E = \max(d\,S - K, 0)\,.$$

Pricing the call at $t = 0$

- As above, set up a portfolio with B cash in a bond and Δ amount of the stock to replicate the payoff of option:

$$\Pi(0) = B + \Delta\, S\,.$$

- Choose Δ such that

$$\Delta\, u\, S + R\, B = C_u^E\,,$$

  and

$$\Delta\, d\, S + R\, B = C_d^E\,,$$

  where the gross risk free rate is $R = 1 + r$.

In matrix form, we solve the system of equations

$$\begin{bmatrix} u\,S & R \\ d\,S & R \end{bmatrix} \begin{bmatrix} \Delta \\ B \end{bmatrix} = \begin{bmatrix} C_u^E \\ C_d^E \end{bmatrix},$$

therefore

$$\begin{bmatrix} \Delta \\ B \end{bmatrix} = \frac{1}{R\,(u\,S - d\,S)} \begin{bmatrix} R & -R \\ -d\,S & u\,S \end{bmatrix} \begin{bmatrix} C_u^E \\ C_d^E \end{bmatrix},$$

so

$$\Delta = \frac{C_u^E - C_d^E}{u\,S - d\,S} \qquad \text{and} \qquad B = \frac{-d\,C_u^E + u\,C_d^E}{R\,(u - d)}\,.$$

Hence, the value of the portfolio at time $t = 0$ is, by **no arbitrage**, the same value as that of the call, i.e., $\Pi(0) = C^E(S, t = 0; K, 1)$.

$$\begin{aligned} C^E(S, t=0; K, 1) &= \Delta\, S + B \\ &= \frac{C^E_u - C^E_d}{u\,S - d\,S}\, S + \frac{-d\, C^E_u + u\, C^E_d}{R\,(u-d)} \\ &= \frac{1}{R}\left[\frac{R-d}{u-d}\, C^E_u + \frac{u-R}{u-d}\, C^E_d\right]. \end{aligned}$$

Risk-neutral valuation

The value of the call can be seen as the discounted weighted average of the payoff at expiry, with weights

$$p^\star = \frac{R-d}{u-d} \qquad \text{and} \qquad q^\star = \frac{u-R}{u-d}\,,$$

and write the price of the call as the expectation (under the new measure) as

$$C^E(S, t=0; K, 1) = \frac{1}{R}\,\left[p^\star C_u^E + q^\star C_d^E\right].$$

Can we, in this risk-neutral world, calculate the discounted expected value of the stock price $R^{-1}\,\mathbb{E}^\star[S_1]$?

- First, note that

$$C^E(S, t = 0; K = 0, 1) = S\,.$$

- Then

$$\begin{aligned}
C^E(S, t = 0; K = 0, 1) &= \frac{1}{R}\left[p^\star\, C^E_u + q^\star\, C^E_d\right] \\
S &= \frac{1}{R}\,[p^\star\, u\, S + q^\star\, d\, S] \\
&= \frac{1}{R}\,[p^\star\, u\, S + (1 - p^\star)\, d\, S] \\
&= \frac{1}{R}\,[p^\star\, u\, S + (1 - p^\star)\, d\, S] \\
&= \frac{1}{R}\,\mathbb{E}^\star[S_1]\,.
\end{aligned}$$

Model independent properties

Call prices satisfy the following inequalities

1. $$C^A(S,t;K,T) \geq C^E(S,t;K,T)\,,$$

2. $$C^A(S,t;K_1,T) \leq C^A(S,t;K_2,T)\,, \qquad \text{if } K_1 \geq K_2\,,$$

3. $$C^A(S,t;K,T_1) \geq C^A(S,t;K,T_2)\,, \qquad \text{if } T_1 \geq T_2\,,$$

4. $$C^A(S,t;K,T) \leq S\,,$$

5. $$C^A(0,t;K,T) = C^E(0,t;K,T) = 0\,.$$

Early Exercise

Proposition

Let S be an underlying security that pays no dividends. Then an American call written on S is **never** *exercised early.*

First we establish the inequality

$$C^A(S,t;K,T) \geq S - K\,e^{-r\,(T-t)}\,.$$

Consider the portfolio $C^A(S,t;K,T) - S + K\,e^{-r\,(T-t)}$. If the American call is exercised early we obtain

$$S - K - S + K\,e^{-r\,(T-t)} = K\,(e^{-r\,(T-t)} - 1) < 0\,.$$

If we wait until T we exercise if $S \geq K$ and obtain 0 profit; if $S < K$ we do not exercise the option and obtain $K - S > 0$.

Therefore we are better off waiting until T, hence we have shown

$$C^A(S,t;K,T) \geq S - K\,e^{-r\,(T-t)}\,.$$

To show that an American call written on a stock that pays no dividend is never exercised we observe that a call yields $S - K$ if exercised but

$$S - K \leq S - K\,e^{-r\,(T-t)} \leq C^A(S,t;K,T)\,.$$

QED

Proposition

Put-call-parity *for European options:*

$$C^E(S,t;K,T) - P^E(S,t;K,T) = S - K\,e^{-r\,(T-t)}\,.$$

Brownian Motion, Stochastic Integrals, Ito's Lemma

Overview

- Brownian Motion, Wiener Process
- Stochastic Integrals
- Itô's Lemma
- Modelling returns

Wiener process, Brownian motion

Definition

A stochastic process W is called a Wiener process or Brownian motion if the following conditions hold.

1. $W_0 = 0$.
2. The process W has independent increments, i.e., if $r < s \leq t < u$ then $W_u - W_t$ and $W_s - W_r$ are independent stochastic variables.
3. For $s < t$ the distribution of the stochastic variable $W_t - W_s$ is $N(0, t-s)$.
4. W has continuous trajectories (almost surely, i.e., with probability one).

Note: It is not immediately obvious that we can rigorously construct a process W which satisfies these four properties, but it can be done.

Elementary properties of Brownian motion

Proposition

Let W_t be a Brownian motion and let $u > 0$, then

$$W_u \sim N(0, u) \tag{1}$$

and therefore

$$\mathbb{E}[W_u] = 0 \qquad \text{and} \qquad \mathrm{Var}(W_u) = u\,.$$

Proof.

The result in (1) is a consequence of the third property for $t = u$ and $s = 0$ together with the property that $W_0 = 0$. □

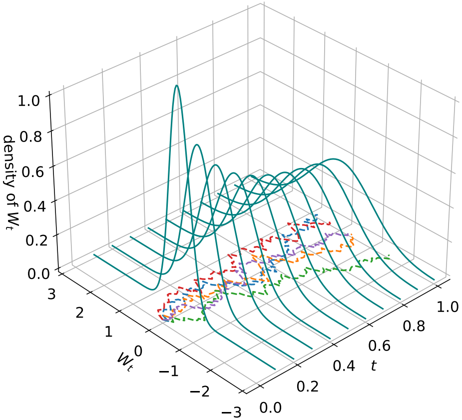


Figure: Five paths of Brownian motion and its density at various points in time.

Proposition

Let W_t be a Brownian motion. Given that $W_t \sim N(0, t)$ we have

$$\mathbb{P}(W_t > x) = \Phi^c\left(\frac{x}{\sqrt{t}}\right) \qquad \textit{and} \qquad \mathbb{P}(W_t \leq x) = \Phi\left(\frac{x}{\sqrt{t}}\right). \tag{2}$$

Proof.

We have that

$$\mathbb{P}(W_t > x) = \mathbb{P}\left(\frac{W_t - 0}{\sqrt{t}} > \frac{x - 0}{\sqrt{t}}\right) = \mathbb{P}\left(Z > \frac{x}{\sqrt{t}}\right) = \Phi^c\left(\frac{x}{\sqrt{t}}\right) \tag{3}$$

where $\Phi^c(x) = \int_x^\infty \frac{1}{\sqrt{2\,\pi}}\, e^{-\,z^2}\, dz$, and we are using that $Z = (W_t - 0)/\sqrt{t}$ is a standard Normal random variable. The second equality follows from the identity $\mathbb{P}(A) = 1 - \mathbb{P}(A^c)$. □

Proposition

Let W_t be a Brownian motion, then

$$\mathbb{E}[W_s\, W_t] = \min(s, t)\,. \tag{4}$$

Proof.

Let $0 \le s \le t$. Then

$$\mathbb{E}[W_s W_t] \;=\; \mathbb{E}[W_s(W_s + W_t - W_s)] \;=\; \mathbb{E}[W_s^2] \;=\; \mathrm{Var}(W_s) \;=\; s$$

because
$\mathbb{E}[W_s\,(W_t - W_s)] = \mathbb{E}[(W_s - W_0)\,(W_t - W_s)] = \mathbb{E}[W_s - W_0]\,\mathbb{E}[W_t - W_s] = 0$, (last step is because of independent increments). This means that in general, for $s, t \ge 0$

$$R(s,t) \;:=\; \mathbb{E}[W_s\, W_t] \;=\; \min(s,t)\,. \tag{5}$$

This is known as the **covariance function** of Brownian motion. □

Proposition

Let W_t be a Brownian motion, and let $0 < s < t$, then

$$\mathbb{P}\left(W_t \leq x | W_s = y\right) = \Phi\left(\frac{x-y}{\sqrt{t-s}}\right) .$$

Proof.

$$\begin{aligned}
\mathbb{P}\left(W_t \leq x | W_s = y\right) &= \mathbb{P}\left(W_t - W_s + W_s \leq x | W_s = y\right) \\
&= \mathbb{P}\left(W_t - W_s + y \leq x | W_s = y\right) \\
&= \mathbb{P}\left(W_t - W_s \leq x - y | W_s = y\right) \\
&= \mathbb{P}\left(W_t - W_s \leq x - y\right) ,
\end{aligned}$$

because $W_t - W_s$ is independent from W_s. Lastly,

$$\begin{aligned}
\mathbb{P}\left(W_t \leq x | W_s = y\right) &= \mathbb{P}\left(\frac{W_t - W_s}{\sqrt{t-s}} \leq \frac{x-y}{\sqrt{t-s}}\right) \\
&= \Phi\left(\frac{x-y}{\sqrt{t-s}}\right) .
\end{aligned}$$

Corollary

Let W_t be a Brownian motion, and let $0 < s < t$, then

$$\mathbb{E}\,[W_t \mid W_s = y] = y\,. \tag{6}$$

Corollary

Let W_t be a Brownian motion, and let $0 < s < t$, then

$$f_{W_t \mid W_s = y}(x) = \frac{1}{\sqrt{2\,\pi(t-s)}}\,e^{-\frac{(x-y)^2}{2\,(t-s)}}\,. \tag{7}$$

Corollary

Let $(W_t)_{t\geq 0}$ be a standard Brownian motion then $(-W_t)_{t\geq 0}$ is also a standard Brownian motion.

Quadratic Variation

Partitions and QV

A partition of the time interval $[0, t]$ is a set of the form $\Pi = t_0 = 0 < t_1 < ... < t_n = t$. The size of the partition is

$$\|\Pi\| = \max_{0 \le i \le n-1} (t_{i+1} - t_i)\,,$$

i.e., equal to the largest interval of the partition. The **quadratic variation** (QV) of a random process X over a fixed time interval $[0, t]$ is

$$[X, X]_t = \lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2$$

if this limit exists and does not depend on the choice of the sequence of partitions Π.[1]

[1] I follow closely the material in "Stochastic Calculus for Finance. II Continuous-time models", by S. Shreve

QV of deterministic function

Let $f(t)$ be a continuous function defined on $0 \leq t \leq T$. The QV of f up to T is

$$[f,f]^n_T = \lim_{\|\Pi\|\to 0} \sum_{i=0}^{n-1} [f(t_{i+1}) - f(t_i)]^2 \,, \tag{8}$$

where the partition Π is $\{t_0,\, t_1,\, \cdots\,,\, t_n\}$, $0 = t_0 < t_1 < \,\cdots < t_n = T$, and $n = n(\Pi)$ denotes the number of partition points in Π.

Next, we show that the QV of the function f is zero.

QV of deterministic function

$$
\begin{aligned}
\sum_{i=0}^{n-1}[f(t_{i+1}) - f(t_i)]^2 &= \sum_{i=0}^{n-1} f'(t_i^*)^2\,(t_{i+1} - t_i)^2 \le \|\Pi\| \sum_{i=0}^{n-1} |f'(t_i^*)|^2\,(t_{i+1} - t_i) \\
&= \lim_{\|\Pi\|\to 0} \|\Pi\| \sum_{i=0}^{n-1} |f'(t_i^*)|^2\,(t_{i+1} - t_i) \\
&= \lim_{\|\Pi\|\to 0} \|\Pi\| \lim_{\|\Pi\|\to 0} \sum_{i=0}^{n-1} |f'(t_i^*)|^2\,(t_{i+1} - t_i) \\
&= \lim_{\|\Pi\|\to 0} \|\Pi\| \int_0^T |f'(t)|^2\,dt = 0\,.
\end{aligned}
$$

In last step $\int_0^T |f'(t)|^2\,dt$ is finite because f is continuous.

Sampled QV of Brownian motion

Let W denote a standard Brownian motion. We define

$$[W,W]_t^n = \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2 \tag{9}$$

to be the sampled quadratic variation for a single partition Π.

Proposition

The following holds true

$$\begin{aligned} \mathbb{E}[[W,W]_t^n] &= t\,, \\ \mathrm{Var}([W,W]_t^n) = \mathbb{E}[([W,W]_t^n - t)^2] &\to 0\,, \end{aligned}$$

as $n \to \infty$.

proof We first note that

$$
\begin{aligned}
\mathbb{E}\left[[W,W]_t^n\right] &= \mathbb{E}\left[\sum_{i=0}^{n-1}(W_{t_{i+1}}-W_{t_i})^2\right] \\
&= \sum_{i=0}^{n-1}\mathbb{E}[(W_{t_{i+1}}-W_{t_i})^2] \\
&= \sum_{i=0}^{n-1}(t_{i+1}-t_i) \\
&= t\,,
\end{aligned}
$$

because $W_{t_{i+1}} - W_{t_i} \sim N(0, t_{i+1} - t_i)$. Thus the expected sampled quadratic variation is independent of the partition, and trivially $\lim_{n\to\infty} \mathbb{E}[[W,W]_t^n] = t$, because the expectation here does not depend on n. Next,

$$
\begin{aligned}
\mathrm{Var}([W,W]_t^n) &= \mathbb{E}\left[([W,W]_t^n - t)^2\right] = \mathbb{E}\left[\left(\sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2 - t\right)^2\right] \\
&= \mathbb{E}\left[\left(\sum_{i=0}^{n-1}\left[(W_{t_{i+1}} - W_{t_i})^2 - (t_{i+1} - t_i)\right]\right)^2\right]
\end{aligned}
$$

(all cross products when squaring the above have expectation zero)

$$
\begin{aligned}
&= \sum_{i=0}^{n-1}\mathbb{E}\left[\left((W_{t_{i+1}} - W_{t_i})^2 - (t_{i+1} - t_i)\right)^2\right] \\
&= \sum_{i=0}^{n-1}\mathbb{E}\left[(W_{t_{i+1}} - W_{t_i})^4 - 2\,(t_{i+1} - t_i)\,(W_{t_{i+1}} - W_{t_i})^2 + (t_{i+1} - t_i)^2\right] \\
&= \sum_{i=0}^{n-1} 3\,(t_{i+1} - t_i)^2 - 2\,(t_{i+1} - t_i)^2 + (t_{i+1} - t_i)^2
\end{aligned}
$$

(use $W_{t_{i+1}} - W_{t_i} \sim \sqrt{t_{i+1} - t_i}\, Z$ and $\mathbb{E}[Z^4] = 3$ where $Z \sim N(0,1)$)

$$
\begin{aligned}
&= 2\sum_{i=0}^{n-1}(t_{i+1} - t_i)^2 \le 2\,\|\Pi\|\sum_{i=0}^{n-1}(t_{i+1} - t_i) \\
&= 2\,\|\Pi\|\,t \qquad \text{which tends to zero if } \|\Pi\| \to 0.
\end{aligned}
$$

QV of Brownian motion

We proved the following theorem.

Theorem

Let W denote a Brownian motion. Then $[W,W]_T = T$ for all $T \geq 0$ almost surely.

- We proved convergence in mean square, also called L^2 convergence.
- In general the quadratic variation $[X,X]_t$ of a process X is a random process, but for Brownian motion W, $[W,W]_t = t$ **almost surely** (a.s.).
  - Almost surely means that there are some paths of the Brownian motion for which $[W,W]_t = t$ is not true.
  - The probability of the set of paths for which $[W,W]_t = t$ is not true is zero.

To bear in mind

Above we used

$$\mathbb{E}[(W_{t_i+1} - W_{t_i})^2] = t_{i+1} - t_i \qquad \text{and} \qquad \mathrm{Var}[(W_{t_i+1} - W_{t_i})^2] = 2\,(t_{i+1} - t_i)^2\,.$$

Intuitively, one would like to claim that

$$(W_{t_i+1} - W_{t_i})^2 \sim t_{i+1} - t_i\,,$$

which makes sense because for a small time increment both sides are very small.

However, best to think about this as the square of a Normal r.v.

$$Y_{i+1} = \frac{W_{t_i+1} - W_{t_i}}{\sqrt{t_{i+1} - t_i}}\,; \tag{10}$$

the distribution of both sides is the same regardless of the time interval.

Now, take time interval $t_{i+1} - t_i = T/n$ and write

$$T\,\frac{Y_{i+1}^2}{n} = (W_{t_i+1} - W_{t_i})^2\,. \tag{11}$$

- By the LLN

$$\frac{1}{n}\sum_{i=0}^{n-1} Y_{i+1}^2 \to \mathbb{E}[Y_{i+1}^2] = 1 \qquad \text{as} \qquad n \to \infty\,.$$

- Thus,

$$\sum_{i=0}^{n-1}(W_{t_i+1} - W_{t_i})^2 \to T\,. \tag{12}$$

- Each term of the sum above can be different from its mean

$$t_{i+1} - t_i = T/n\,,$$

but when we sum many of them the differences average out to zero.

However, informally, one uses as rule-of-thumb

- $dW_t\, dW_t = dt$,
- $dW_t\, dt = 0$,
- $dt\, dt = 0$.

Stochastic Differential Equations

Stochastic Integrals

Objective: define a stochastic integral for a large class of integrands with respect to Brownian motion. The main challenge is to overcome the fact that the

Brownian motion is not of bounded variation which means that

$$\lim_{\|\Pi_n\|\to 0} \sum_{i=0}^{n-1} |W_{t_{i+1}^n} - W_{t_i^n}| = \infty\,, \tag{13}$$

where Π_n denotes a sequence of partitions with $\|\Pi_n\| \to 0$ as before. We want to

study the object

$$\int_0^t \alpha_s dW_s\,, \tag{14}$$

which is known as a stochastic integral; here, α_t can depend on W_t.

Riemann Integrals

The standard approach to define an integral for a deterministic function $f : [a, b] \to \mathbb{R}$ is the Riemann integral

$$\int_a^b f(s)\, ds := \lim_{\|\Pi_n\| \to 0} \sum_{i=0}^{n-1} f(\xi_i)\, (t_{i+1} - t_i)\ , \tag{15}$$

and $\Pi_n = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of $[a, b]$ and $\xi_i \in [t_i, t_{i+1}]$.

This definition makes sense if the right-hand side converges for every sequence of partitions and choices of ξ.

In this case, the function f is called Riemann integrable and the left-hand side is the Riemann integral.

Riemann–Stieltjes

Consider the Riemann–Stieltjes integral

$$\int_a^b f(s)g(ds) := \lim_{\|\Pi_n\|\to 0} \sum_{i=0}^{n-1} f(\xi_i)\,\left(g(t_{i+1}) - g(t_i)\right), \tag{16}$$

where as before, the definition makes sense if the right-hand side converges for every sequence of partitions and choices of ξ.

Here, the function f is called Riemann–Stieltjes integrable with respect to g and the unique limit is called the Riemann–Stieltjes integral of f with respect to g.

One can show that at least every continuous function $f : [a, b] \to \mathbb{R}$ is Riemann–Stieltjes integrable with respect to every function g of finite variation.

A simple stochastic process

Definition

A stochastic process $(\alpha_t)_{t\in[0,T]}$ is called a simple stochastic process if it is of the form

$$\alpha_t = \sum_{i=1}^{N} \alpha_i \, 1_{t\in(t_{i-1},t_i]} \,, \tag{17}$$

where α_i is random but only depends on the history of the Brownian motion W up to time t_i, and $0 = t_0 < t_1 < \cdots < t_N = T$.

A simple stochastic process

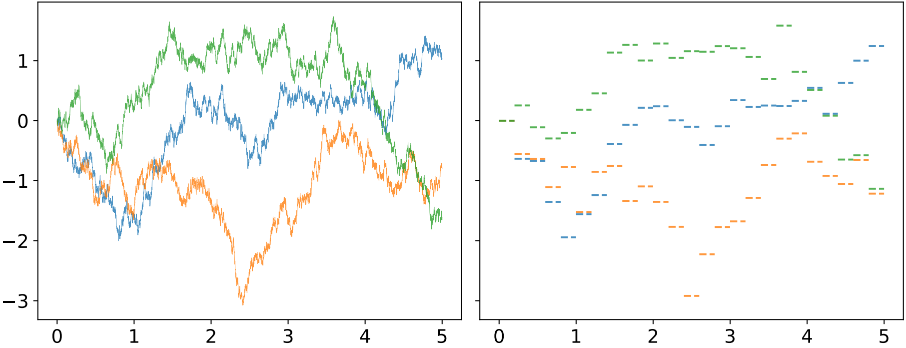


Figure: Example of a simple stochastic process α_t. Here, $t_0 = 0$, $t_N = 5$, equal spacing $\Delta t = 0.2$, and $\alpha_t = W_{t_{i-1}}$ for $t \in (t_{i-1}, t_i]$. The left panel has three sample paths of a standard Brownian motion, and the right panel shows the simple stochastic process.

A simple stochastic process

Definition

Let $(\alpha_t)_{t\in[0,T]}$ be a simple stochastic process, we define the stochastic integral of α_t with respect to W_t by

$$I(\alpha) := \int_0^T \alpha_s \, dW_s \quad = \quad \sum_{i=1}^{N} \alpha_{t_{i-1}} \left(W_{t_i} - W_{t_{i-1}} \right).$$

To generalize the stochastic integral to more general (non-simple) process α_t, we approximate the process to arbitrary accuracy with a simple process and use arguments involving L^2-convergence. One can show that the integral $I(\alpha)$ satisfies the so-called Ito's isometry for simple stochastic processes, that is

$$\mathbb{E}\left[(I(\alpha))^2 \right] = \int_0^T \mathbb{E}\left[(\alpha_s)^2 \right] ds \,. \tag{18}$$

SDEs

Consider a stochastic differential equation (SDE) which we write informally as

$$dX_t \quad = \quad b(X_t)\, dt + \sigma(X_t)\, dW_t\,, \qquad X_0 = x_0\,. \tag{19}$$

Can we assign a rigorous meaning to this equation? Note that if $b(x) = 0$ and $\sigma(x) = 1$, then $dX_t = dW_t$, so $X_t = x_0 + W_t$.

Definition

A solution to the SDE (19) with $X_0 = x_0$ is a process X_t with a continuous sample path which satisfies the integral equation

$$X_t \quad = \quad x_0 + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s\,, \tag{20}$$

here $b(\cdot)$ is known as the drift, and $\sigma(\cdot)$ is known as the volatility.

SDEs

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous if $\exists\, K \in \mathbb{R}^+$ such that for $x, y \in \mathbb{R}$

$$|f(x) - f(y)| \leq K\,|x - y|\,. \tag{21}$$

Proposition

If b and σ are Lipschitz continuous, then a solution to (19) exists.

Ito's Lemma

Change of variable

If $f(S,t)$ is a deterministic function of S and time t we approximate the change in f due to a change in both S and t as

$$df(S,t) = \frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial S}\,dS\,. \tag{22}$$

Another way is to start here (assume f only depends on S):

$$\begin{aligned}\Delta f &\equiv f(S+\Delta S) - f(S)\\ &= f'(S)\,\Delta S + \frac{1}{2!}\,f''(S)\,(\Delta S)^2 + \frac{1}{3!}\,f'''(S)\,(\Delta S)^3 + \cdots\,.\end{aligned}$$

However, what happens if S is not deterministic? For example, assume that

$$dS = dW\,.$$

Informally,

$$
\begin{aligned}
\Delta f(S) &= f'(S)\,\Delta S + \frac{1}{2!}\,f''(S)\,(\Delta S)^2 \\
&\quad + \frac{1}{3!}\,f'''(S)\,(\Delta S)^3 + \cdots \\
&= f'(S)\,\Delta W + \frac{1}{2!}\,f''(S)\,(\Delta W)^2 \\
&\quad + \frac{1}{3!}\,f'''(S)(\Delta W)^3 + \cdots \\
&= f'(S)\,\phi\,\Delta t^{1/2} + \frac{1}{2!}\,f''(S)\,\phi^2\,\Delta t^{2/2} \\
&\quad + \frac{1}{3!}\,f'''(S)\,\phi^3\,\Delta t^{3/2} + \cdots\,,
\end{aligned}
$$

where $\Delta W = \phi\,\Delta t^{1/2}$, $\phi \sim N(0,1)$.

Itô's Lemma

Assume that the process S satisfies the following stochastic differential equation

$$dS_t = \mu(S,t)\,dt + \sigma(S,t)\,dW_t\,,$$

where $\mu(S,t)$ and $\sigma(S,t)$ are adapted processes. Let f be once differentiable in t and twice differentiable in S.[2] be a twice continuous differentiable function then

$$\begin{aligned} df(S,t) &= \left(\frac{\partial f}{\partial t} + \mu(S,t)\,\frac{\partial f}{\partial S} + \frac{1}{2}\,\sigma^2(S,t)\,\frac{\partial^2 f}{\partial S^2}\right)dt \\ &\quad + \sigma(S,t)\,\frac{\partial f}{\partial S}\,dW_t\,. \end{aligned}$$

[2]Mathematically we write this as $f \in C^{1,2}([0,T]\times\mathbb{R})$.

Modelling Returns

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t\,. \tag{23}$$

Here, the drift μ and volatility $\sigma \geq 0$ are constants.

Note that (23) is as in (19) with $X_t = S_t$, $b(S_t) = \mu\, S_t$ and $\sigma(S_t) = \sigma\, S_t$

Naive solution of the SDE would be:

$$\begin{aligned} \int_0^t \frac{dS_t}{S_t} &= \int_0^t \mu\, ds + \sigma \int_0^t dW_s \\ \ln S(t)/S(0) &= \mu\, t + \sigma\, W_t\,. \end{aligned}$$

Hence (it seems that)

$$S_t = S_0\, e^{\mu\, t + \sigma\, W_t}\,. \tag{24}$$

Let $f = \ln S$, what is df? First rewrite (23) as

$$dS_t = S_t\,\mu\,dt + S_t\,\sigma\,dW_t\,. \tag{25}$$

We can use Itô's lemma with $\mu(S,t) = \mu\,S$ and $\sigma(S,t) = \sigma\,S$ hence, use (23) with

$$f = \ln S, \qquad \frac{\partial f}{\partial S} = 1/S, \qquad \frac{\partial^2 f}{\partial S^2} = -1/S^2, \qquad \frac{\partial f}{\partial t} = 0$$

to obtain

$$\begin{aligned} d(\ln S) &= \left(S\,\mu\,\frac{1}{S} - \frac{1}{2}\,S_t^2\,\sigma^2\,\frac{1}{S^2}\right)dt + S\,\sigma\,\frac{1}{S}\,dW_t \\ &= \left(\mu - \frac{1}{2}\,\sigma^2\right)dt + \sigma\,dW_t\,. \end{aligned}$$

Hence, by integrating both sides we obtain

$$S_t = S_0 \, e^{(\mu - \frac{1}{2}\,\sigma^2)\,t + \sigma\,W_t}\,. \tag{26}$$

Note that the naive solution was 'nearly' correct but it was missing the term

$$\frac{1}{2}\,\sigma^2\,t\,.$$

Where does this 'extra term' come from?

Black–Scholes

Black–Scholes PDE

Problem:

- Assume $dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dW_t$.
- Objective: Price a European-style option, $V(S, t; K, T)$, written on S with payoff $V(S, T; K, T) = V(S, T)$.

Steps:

- Form a hedge portfolio long a call and 'delta' amount of the underlying.
- See what is the change in the value of the portfolio over a 'small' time step. That is, write the stochastic dynamics of the value of the portfolio.
- Try to choose the amount of the underlying in the portfolio so that risk or randomness is reduced.

Proceed as in the binomial case and form a hedge portfolio

$$\Pi(S,t) = V(S,t;K,T) - \Delta\, S_t\,,$$

where Δ is the number of shares we choose to hold in the portfolio.

- The change in the value of the portfolio $d\Pi = dV_t - \Delta\, dS_t$.
- Use Ito's lemma

$$\begin{aligned} d\Pi \;&=\; \left(\frac{\partial V}{\partial t} + \mu\, S_t\,\frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2\,S_t^2\,\frac{\partial^2 V}{\partial S^2} - \Delta\,\mu\,S_t\right)\,dt \\ &\quad + \left(\frac{\partial V}{\partial S} - \Delta\right)\,S_t\,\sigma\,dW_t\,. \end{aligned} \tag{27}$$

Choose, at every instant in time, the following amount of stock:

$$\Delta = \frac{\partial V}{\partial S} \tag{28}$$

so that the change in the portfolio is deterministic. Therefore, substituting (28) into (27) yields

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\,\sigma^2\, S_t^2\, \frac{\partial^2 V}{\partial S^2} \right) dt\,.$$

What can we do next?

By no-arbitrage, the portfolio grows like a risk-free bond

$$d\Pi = r\,\Pi\,dt\,.$$

Putting these results together we have

$$\begin{aligned} r\,\Pi\,dt &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2 V}{\partial S^2}\right)\,dt \\ r\,\left(V_t - \frac{\partial V}{\partial S}\,S_t\right)\,dt &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2 V}{\partial S^2}\right)\,dt \\ r\,V_t &= \frac{\partial V}{\partial t} + r\,S\,\frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2 V}{\partial S^2}\,, \end{aligned}$$

which is the Black–Scholes PDE — you need boundary conditions to solve for a particular European-style option.

Solving the Black–Scholes PDE

Ee explore in more detail the Black–Scholes PDE

$$r\,V = V_t + r\,S\,V_S + \frac{1}{2}\,\sigma^2\,S^2\,V_{SS}\,. \qquad (29)$$

- How general is equation (29)?
- Can we price any type of European option? How?
- Can we price any American option?
- How come the drift μ of the returns process is nowhere to be seen?
- What happens if the stock pays a dividend?
- Is it really possible to assume continuous hedging?

Pricing a European call

The Black–Scholes PDE for a European call is

$$C^E_t + \frac{1}{2}\,\sigma^2\,S^2\,C^E_{SS} + r\,S\,C^E_S - r\,C^E = 0\,, \qquad (30)$$

subject to $C^E(S_T, T) = \max(S_T - K, 0)$ and

$$C(0,t) = 0, \qquad C(S,t) \sim S \quad \text{as } S \to \infty\,.$$

Here, with a slight abuse of notation, $C^E_t = \partial C^E/\partial t$, $C^E_S = \partial C^E/\partial S$ and so on.

How can we solve this PDE?

Heat equation and Black–Scholes PDE

- We want to relate this problem to the heat equation, which we know how to solve.
- For the heat equation we have an initial condition and here we have a 'final' condition.
- We have terms like C_S^E and C^E.
- The domain of the heat equation is $(-\infty, \infty)$ and here is $[0, \infty)$.
- Via suitable variable changes we might get to a PDE that we know how to solve.

Change time direction

$$t = T - \tau / \tfrac{1}{2}\sigma^2.$$

Get rid of the S coefficients by letting

$$S = K\, e^x, \qquad \text{i.e.,} \qquad x = \ln S/K\,.$$

Finally, let $C = K\, v(x, \tau)$ to obtain

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - k\, v\,, \tag{31}$$

where $k = r/\frac{1}{2}\,\sigma^2$ and I.C. $v(x, 0) = \max(e^x - 1, 0)$.

But we are not there yet...

We let

$$v(x,\tau) = e^{\alpha\, x + \beta\, \tau}\, u(x,\tau)\,,$$

for some constants α and β, then

$$\beta\, u + \frac{\partial u}{\partial \tau} = \alpha^2\, u + 2\,\alpha\,\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - k\, u\,.$$

Next, choose

$$\beta = \alpha^2 + (k-1)\,\alpha - k\,,$$

to eliminate the u term. Moreover, the choice

$$2\,\alpha + (k-1) = 0$$

eliminates the $\partial u/\partial x$ term.

Therefore, choose

$$\alpha = -\frac{1}{2}(k-1) \qquad \text{and} \qquad \beta = -\frac{1}{4}(k+1)^2$$

and write

$$v(x,\tau) = e^{\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau} u(x,\tau),$$

where

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \qquad \text{for } -\infty < x < \infty, \ \tau > 0,$$

with

$$u(x,0) = \max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0). \tag{32}$$

Black–Scholes Formula

The solution is

$$\begin{aligned} u(x,\tau) &= \int_{-\infty}^{\infty} u(s,0)\, e^{-\frac{(x-s)^2}{4\,\tau}}\, ds \\ &= e^{\frac{1}{2}\,(k+1)\,x+\frac{1}{4}\,(k+1)^2\,\tau}\,\Phi(d_1) - e^{\frac{1}{2}\,(k-1)\,x+\frac{1}{4}\,(k-1)^2\,\tau}\,\Phi(d_2)\,, \end{aligned}$$

where

$$d_2 = \frac{\ln(S_t/K) + (r - \frac{1}{2}\,\sigma^2)(T-t)}{\sigma\,\sqrt{T-t}}\,,$$

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\,\sigma^2)\,(T-t)}{\sigma\,\sqrt{T-t}}\,.$$

The last step is to go back to the original variables, i.e.,

$$x = \ln(S/K), \qquad \tau = \frac{1}{2}\,\sigma^2\,(T-t), \qquad C = K\,v(x,\tau)\,,$$

to write the Black–Scholes formula to price the European call

$$C^E(S,t;K,T) = S_t\,\Phi(d_1) - e^{-r\,(T-t)}\,K\,\Phi(d_2)\,. \tag{33}$$

Continuous dividends

$$\frac{dS_t}{S_t} = (\mu - D_0)\,dt + \sigma\,dW_t\,. \qquad (34)$$

To obtain he Black–Scholes PDE, we proceed as before but note that the change in the value of the portfolio is given by

$$d\Pi = dV_t - \Delta\,dS_t - D_0\,\Delta S_t\,dt\,.$$

Hence the Black–Scholes PDE becomes

$$r\,V \quad = \quad \frac{\partial V}{\partial t} + (r - D_0)\,S\,\frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2 V}{\partial S^2}\,. \qquad (35)$$

Black–Scholes Formula with dividends

$$C^E = e^{-D_0\,(T-t)}\,S_t\,\Phi(d_{10}) - e^{-r\,(T-t)}\,K\,\Phi(d_{20})\,,$$

where

$$d_{10} = \frac{\ln(S_t/K) + (r - D_0 + \frac{1}{2}\,\sigma^2)\,(T-t)}{\sigma\,\sqrt{T-t}}$$

and

$$d_{20} = d_{10} - \sigma\,\sqrt{T-t}\,.$$

Risk-neutral valuation

More generally, we could derive the following expression to price **any** European-style option with payoff $\Pi(S, T)$

$$
\begin{aligned}
V(S,t) \quad = \quad & \frac{e^{-r\,(T-t)}}{\sqrt{\sigma^2\,2\,\pi\,(T-t)}} \\
& \times \int_0^\infty V(u,T) \\
& \times e^{-\left(\ln(u/S)-(r-\frac{1}{2}\,\sigma^2)\,(T-t)\right)^2/(2\,\sigma^2\,(T-t))}\ \frac{du}{u}\,.
\end{aligned}
$$

Log-normal distribution

Proposition

Let $(S_t)_{t\geq 0}$ be a geometric Brownian motion. Then, the density function of S_t denoted by $p_{S_t}(S)$ is given by

$$p_{S_t}(S) = \frac{1}{S\,\sigma\,\sqrt{2\,\pi\,t}}\ \exp\left(-\frac{\left(\ln\frac{S}{S_0} - \left(\mu - \sigma^2\right)\ t\right)^2}{2\,\sigma^2\,t}\right)$$

for $S > 0$.

Proof

$$\mathbb{P}(S_t \leq S) = \mathbb{P}(\ln S_t \leq \ln S) = F(\ln S)\,,$$

where F is the distribution function of $\ln S_t$. Differentiate both sides wrt S to write the density $p_{S_t}(S)$ of S_t:

$$p_{S_t}(S) = \frac{d}{dS}\mathbb{P}(S_t \leq S) = \frac{1}{S}\,F'(\ln S) = \frac{1}{S}\,p_{X_t}(x)\,,$$

where $x = \ln S$ and $p_{X_t}(x)$ is the density of $X_t = \ln S_t$, which is given by

$$p_{X_t}(x) = \frac{1}{\sqrt{2\,\pi\,\sigma^2\,t}}\ \exp\left(-\frac{(x - x_0 - (\mu - \frac{1}{2}\sigma^2)\,t)^2}{2\,\sigma^2\,t}\right)\,, \tag{36}$$

because $X_t \sim N(X_0 + (\mu - \frac{1}{2}\sigma^2)\,t, \sigma^2\,t)$, so

$$p_{S_t}(S) = \frac{1}{S\,\sigma\,\sqrt{2\,\pi t}}\ \exp\left(-\frac{\left(\ln\frac{S}{S_0} - \left(\mu - \frac{1}{2}\sigma^2\right)\ t\right)^2}{2\,\sigma^2\,t}\right) \qquad \text{for} \quad S > 0\,.$$

Feynman–Kac

Theorem

Let $V(S,t)$ be a solution to the PDE

$$r\,V(S,t) - \partial_t V(S,t) - r\,S\,\partial_S V(S,t) - \frac{1}{2}\,\sigma^2\,S^2\,\partial_{SS} V(S,t) = 0\,, \qquad (37)$$

with terminal condition $V(S,t) = f(S)$. Then, $V(S,t)$ admits the following probabilistic representation

$$V(S,t) = e^{-r\,(T-t)}\,\mathbb{E}^{\mathbb{Q}}\left[f(S_T)\middle|S_t = S\right]\,, \qquad (38)$$

for all $(S,t) \in [0,T] \times \mathbb{R}$ and where

$$dS_t = r\,S_t\,dt + \sigma\,S_t\,dW_t^{\mathbb{Q}}\,, \qquad S_0 > 0\,. \qquad (39)$$

Feynman–Kac call option

Thus, the price of a European call option $C^E(S,t)$ which pays $\max(S_T - K, 0)$ at time T in the Black–Scholes model has the *probabilistic* representation

$$C^E(S,t) = e^{-r\,(T-t)}\,\mathbb{E}^{\mathbb{Q}}\left[\max(S_T - K, 0)\,|\,S_t = S\right]\,,$$

where $\mathbb{Q}$ is a new probability measure under which S satisfies

$$dS_t = S_t\,\left(r\,dt + \sigma\,dW_t^{\mathbb{Q}}\right)\,, \tag{40}$$

and $W^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$.

In general, the Feynman–Kac Theorem implies the following:

- **the option price (i.e., the cost of replicating the option) is the discounted expected value of $f(S_T)$ in the risk-neutral world $\mathbb{Q}$ where the drift is r not μ.**

- We refer to this world as the **risk neutral measure**.

Self-financing derivation of Black–Scholes PDE

- As in the one-step binomial example, let us derive the Black–Scholes PDE with a self-financing strategy
- Form a portfolio with an amount of cash and an amount of underlying
  - As time evolves, rebalance how much is held in each asset (stock and bond)
  - Rebalancing without investing nor withdrawing wealth from the portfolio, i.e., a self-financing portfolio.

Self-financing strategy

Definition

A trading strategy $(\phi_t, \psi_t)_{t\in[0,T]}$ is a pair of adapted stochastic processes such that ϕ_t is the number of shares the investor holds at time t and ψ_t is the number of risk-free bonds held at time t. Thus, the value of the position of the agent at time t is

$$V_t = \phi_t\, S_t + \psi_t\, B_t\,. \tag{41}$$

The strategy is called self-financed if

$$dV_t = \phi_t\, dS_t + \psi_t\, dB_t\,. \tag{42}$$

Note: here, V_t is the value (stochastic) of a portfolio, or the value of an asset that we wish to replicate with positions in the stock S and a risk-free bond B, where $dB_t = r\,B_t\,dt$.

Let $(\phi_t, \psi_t)_{t\in[0,T]}$ be a self-financed trading strategy. Then, the value of the portfolio V is that in (41). We want to find a self-financed strategy (ϕ_t, ψ_t) such that $V_T = f(S_T)$.

$$\begin{aligned} dV_t &= \phi_t\, dS_t + \psi_t\, dB_t \\ &= \phi_t\, dS_t + \psi_t\, r\, B_t\, dt \\ &= \phi_t\, dS_t + r\, (V_t - \phi_t\, S_t)\, dt \\ &= \phi_t\, (\mu\, S_t\, dt + \sigma\, S_t\, dW_t) + r\, (V_t - \phi_t\, S_t)\, dt\,, \end{aligned}$$

thus,

$$dV_t = (\phi_t\, \mu\, S_t + (V_t - \phi_t\, S_t)\, r)\, dt + \phi_t\, \sigma\, S_t\, dW_t\,. \tag{43}$$

Write $V_t = v(t, S_t)$ for $v \in^{1,2} ([0, T] \times \mathbb{R})$. Then, by Itô's lemma we have that

$$dV_t = \partial_t v(t, S_t)dt + \partial_s v(t, S_t)\ (\mu\, S_t\, dt + \sigma\, S_t\, dW_t) + \frac{1}{2}\,\partial_{ss} v(t, S_t)\,\sigma^2\, S_t^2\, dt \tag{44}$$

$$= \left(\partial_t v(t, S_t) + \partial_s v(t, S_t)\,\mu\, S_t + \frac{1}{2}\partial_{ss} v(t, S_t)\sigma^2\, S_t^2\right) dt + \partial_s v(t, S_t)\,\sigma\, S_t\, dW_t\,. \tag{45}$$

Choose the amount in the stock $\phi_t = \partial_s v(t, S_t)$ so that (43) becomes

$$dV_t = (\partial_s v(t, S_t)\,\mu\, S_t + (v(t, S_t) - \partial_s v(t, S_t)\, S_t)\, r)\, dt + \partial_s v(t, S_t)\,\sigma\, S_t\, dW_t\,. \tag{46}$$

Compare (45) and (46) to see that v satisfies

$$\partial_s v(t,S)\,\mu\,S + (v(t,S) - \partial_s v(t,S)\,S)\,r = \partial_t v(t,S) + \partial_s v(t,S)\,\mu\,S + \frac{1}{2}\partial_{ss} v(t,S)\sigma^2$$

and we see that the terms $\partial_s v(t,S)\,\mu\,S$ cancels out and the above PDE reduces to

$$v(t,S)\,r - \partial_s v(t,S)\,S\,r - \partial_t v(t,S) - \frac{1}{2}\partial_{ss} v(t,S)\sigma^2\,S^2 = 0\,.$$

Proposition

There is a unique solution $v(t,S)$ to the PDE

$$r\,v(t,S) - \partial_t v(t,S) - r\,S\,\partial_s v(t,S) - \frac{1}{2}\,\sigma^2\,S^2\,\partial_{ss} v(t,S) = 0\,, \tag{47}$$

with terminal boundary condition $v(T,S) = f(S)$.

Next,

Theorem

Let $v(t,S)$ be the solution to (47) with terminal condition $f(S)$. The strategy

$$\phi_t = \partial_S v(t,S_t)\,, \qquad \psi_t = \left(v(t,S_t) - S_t\,\partial_S v(t,S_t)\right) e^{-r\,t}\,,$$

is self-financing and

$$V_T = \phi_T\,S_T + \psi_T\,e^{r\,T} = f(S_T)\,.$$

Proof

Let $v(t,S)$ be the solution to (47) with terminal condition $f(S)$ and define

$$\phi_t = \partial_S v(t,S_t)\,, \qquad \psi_t = (v(t,S_t) - S_t\,\partial_S v(t,S_t))\,e^{-r\,t}\,,$$

with $V_t = \phi_t\,S_t + \psi_t\,B_t$. Then,

$$\begin{aligned} V_t &= \phi_t\,S_t + \psi_t\,B_t \\ &= \partial_S v(t,S_t)\,S_t + (v(t,S_t) - S_t\,\partial_S v(t,S_t))\,e^{-r\,t}\,e^{r\,t} \\ &= v(t,S_t)\,, \end{aligned}$$

and by Itô's lemma

$$dV_t = \partial_s v(t,S_t)\,dS_t + \left(\partial_t v(t,S_t) + \frac{1}{2}\,\partial_{ss} v(t,S_t)\,\sigma^2\,S_t^2\right)dt\,, \tag{48}$$

and given that v satisfies the PDE in (47) then,

$$\partial_t v(t,S_t) + \frac{1}{2}\,\partial_{ss} v(t,S_t)\,\sigma^2\,S_t^2 = v(t,S_t)\,r - \partial_s v(t,S_t)\,S_t\,r\,,$$

and thus, (48) becomes

$$\begin{aligned} dV_t &= \partial_s v(t, S_t)\, dS_t + (v(t, S_t)\, r - \partial_s v(t, S_t)\, S_t\, r)\, dt \\ &= \partial_s v(t, S_t)\, dS_t + (v(t, S_t) - S_t\, \partial_s v(t, S_t))\, e^{-r\,t}\, e^{r\,t}\, r\, dt \\ &= \phi_t\, dS_t + \psi_t\, B_t\, r\, dt \\ &= \phi_t\, dS_t + \psi_t\, dB_t\,, \end{aligned}$$

which proves that (ϕ_t, ψ_t) is a self-financed trading strategy. Then, we have that

$$V_T = v(T, S_T) = f(S_T)\,,$$

because $v(T, S) = f(S)$.

Forward Contracts

Forward contract

- A forward contract is an agreement between two parties. One party commits to delivering an asset, say S, at time T. The other party commits to delivering K cash, at time T. At the inception of the contract there is no exchange of money.
- The payoff at time T of such derivative is $S_T - K$.
- Compute arbitrage-free price with Feynman–Kac:

$$
\begin{aligned}
P(S,t) &= e^{-r(T-t)}\,\mathbb{E}^{\mathbb{Q}}\left[S_T - K \mid S_t = S\right] \\
&= e^{-r(T-t)}\left(S\,e^{r\,(T-t)} - K\right) \\
&= S - K\,e^{-r(T-t)}\,.
\end{aligned}
$$

The above formula at time zero is

$$
S - K\,e^{-r\,T}\,,
$$

so the forward price (which is defined as the strike K that makes the value of the contract equal to zero) is

$$
K = S\,e^{r\,T}\,.
$$

Discrete Dividends

- Assume that the asset pays a lump-sum dividend at time t_d.
- The dividend is known in advance, D_d $(0 \leq D_d < 1)$.
- At the time of payment the holder of the stock receives $D_d\, S(t_d^-)$, where $S(t_d^-)$ is the asset's price immediately before the dividend is paid.
- By no-arbitrage

$$S(t_d^+) = S(t_d^-) - D_d\, S(t_d^-) = S(t_d^-)\,(1 - D_d)\,. \tag{49}$$

What happens to the value of an option when the underlying jumps due to a discrete dividend payment. By no-arbitrage

$$V(S(t_d^-), t_d^-) = V(S(t_d^+), t_d^+)\,. \tag{50}$$

We break the problem in three steps:

- First we value the European call option in the interval $[t_d^+, T]$;
- Second, we implement the jump condition (50) at $t = t_d^-$;
- Finally, we solve the Black–Scholes equation backwards from $t = t_d^-$ including the two previous values.

The value of the option right after the dividend is given by

$$C_D(S, t; K, T) = C(S, t; K, T) \qquad \text{for } t_d^+ \leq t \leq T.$$

Next, use the jump condition (50) to write

$$\begin{aligned} C(S(t_d^-), t_d^-) &= C(S(t_d^+), t_d^+) \\ &= C(S(t_d^-)\,(1 - D_d), t_d^+)\,. \end{aligned} \tag{51}$$

Now let us see what is the value of (51) at maturity T

$$\begin{aligned} C(S\,(1-D_d),T;K,T) &= \max(S(\,1-D_d)-K,0) \\ &= (1-D_d) \\ &\quad \times \max\left(S-\frac{K}{1-D_d},0\right). \end{aligned} \tag{52}$$

This is the price of $1-D_d$ calls struck at $K/(1-D_d)$.
Therefore for $t < t_d$

$$C_d(S,t;K,T) = (1-D_d)\,C(S,t;K/(1-D_d),T)\,,$$

and for $t \geq t_d$

$$C_d(S,t;K,T) = C(S,t;K,T)\,.$$

Binary and other Options

Binary option

The payoff of a binary call is

$$B_c(S,T;K,T) = \begin{cases} 1\,, & S \geq K\,, \\ 0\,, & S \leq K\,. \end{cases}$$

Recall

$$V(S,t) = \frac{e^{-r\,(T-t)}}{\sqrt{\sigma^2 2\pi(T-t)}} \int_0^\infty \Pi(u,T) e^{-\left(\ln(u/S)-(r-\frac{1}{2}\sigma^2)(T-t)\right)^2/(2\sigma^2(T-t))}\,\frac{du}{u}\,. \tag{53}$$

with payoff $\Pi(u,T) = B_c(S,T;K,T)$. Thus,

$$B_c(S,t;K,T) = e^{-r\,(T-t)}\,\Phi(d_2)\,, \tag{54}$$

where

$$d_2 = \frac{\ln(S_t/K) + (r - \frac{1}{2}\,\sigma^2)\,(T-t)}{\sigma\,\sqrt{T-t}}\,.$$

What is the delta of a binary call?

$$\Delta = \frac{e^{-r\,(T-t)}\,\Phi(d_2)}{\sigma\, S\,\sqrt{T-t}}\,.$$

To price a Binary put we proceed as above with payoff

$$B_p(S,T;K,T) = \begin{cases} 0\,, & S > K\,, \\ 1\,, & S \le K\,. \end{cases}$$

Using (53) we obtain

$$B_p(S,t;K,T) = e^{-r\,(T-t)}\,\Phi(-d_2)\,. \tag{55}$$

Note that we could also price a binary put by observing that if we hold a binary call and put the payoff at maturity is 1, in other words

$$B_c(S,T;K,T) + B_p(S,T;K,T) = \mathcal{H}(S-K) + \mathcal{H}(K-S) = 1\,,$$

where $\mathcal{H}(S-K)$ is the Heaviside function. Hence, at any time $t \leq T$

$$B_c(S,T;K,T) + B_p(S,T;K,T) = e^{-r\,(T-t)}. \tag{56}$$

Thus, the fair price of a binary put is

$$\begin{aligned} B_p(S,t;K,T) &= e^{-r\,(T-t)} - B_c(S,t;K,T) \\ &= e^{-r\,(T-t)}\,(1-\Phi(d_2)) \\ &= e^{-r\,(T-t)}\,\Phi(-d_2)\,. \end{aligned}$$

General payoffs

Synthesizing a general payoff $\Lambda(S)$ from vanilla call options means that we can approximate the payoff by a sum of n delta functions times the payoff of different call options.

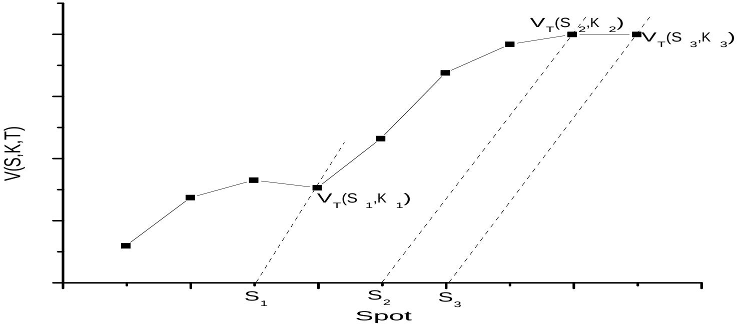


Figure: Synthesising a European payoff.

Thus, if the price of the option is given by

$$V(S,t) = \int_0^\infty f(K)\, C(S,t;K)\, dK \tag{57}$$

then the value of the option at expiry is

$$\begin{aligned} \Lambda(S) &= \int_0^\infty f(K)\, \max(S-K)\, dK \\ &= \int_0^S f(K)\,(S-K)\, dK\,. \end{aligned} \tag{58}$$

How do we obtain the function $f(K)$?

$$\begin{aligned} \Lambda'(S) &= f(S)(S-S) + \int_0^S f(K)\,\frac{d(S-K)}{dS}\,dK \\ &= \int_0^S f(K)\,dK\,, \end{aligned} \tag{59}$$

and differentiating again we arrive to

$$\Lambda''(S) = f(S)\,. \tag{60}$$

Example I

Let us find the synthesizing portfolio when $\Lambda(S) = \max(S - K, 0)$. First, note that we can write

$$\begin{aligned} \Lambda(S) &= \max(S - K, 0) \\ &= (S - K)\,\mathcal{H}(S - K)\,, \end{aligned} \tag{61}$$

where $\mathcal{H}$ is the Heaviside function. Recall that the Heaviside function is

$$\int_{-\infty}^{x} \delta(s)\, ds = \mathcal{H}(x)\,. \tag{62}$$

Hence

$$\mathcal{H}'(x) = \delta(x). \tag{63}$$

From (61) and (63) we arrive to

$$\begin{aligned} \Lambda'(S) &= \mathcal{H}(S-K); \\ \Lambda''(S) &= \delta(S-K). \end{aligned} \tag{64}$$

Thus, we may write the option's price as

$$\begin{aligned} V(S,t) &= \int_0^\infty \delta(K'-K)\, C(S,t;K')\, dK' \\ &= C(S,t;K)\,. \end{aligned} \tag{65}$$

Example II

Assume that $\Lambda(S) = S$.

- What is the synthesising portfolio here?
- We proceed as above and we can readily verify that $\Lambda''(S) = \delta(S)$, hence the the synthesising portfolio is just S, or a call with exercise price zero.

Example III

Assume we want to replicate a payoff with binary calls, i.e., pay 1 or nothing. Find $f(K)$

$$V(S,t) = \int_0^\infty f(K)\, B(S,t;K,T)\, dK\,.$$

At expiry

$$\begin{aligned}\Lambda(S) &= \int_0^\infty f(K)\, \mathcal{H}(S-K)\, dK \\ &= \int_0^S f(K)\, dK\,.\end{aligned}$$

Then

$$\Lambda'(S) \;=\; f(S)\,.$$

So to replicate a call we must use $f(K) = \mathcal{H}(S - K)$, hence

$$\begin{aligned} C(S,t;K,T) &= \int_0^\infty \mathcal{H}(K'-K)\,B(K')\,dK' \\ &= \int_0^\infty \mathcal{H}(K'-K)\,B(K')dK'\,. \end{aligned}$$

Power options

Assume stock price dynamics are given by

$$dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dW_t\,.$$

The payoff of the power options is $\max(S^n - K^n, 0)$ for a call and $\max(K^n - S^n, 0)$ for a put where $n \geq 1$.

One approach to price the option is:

As before, set up a hedge portfolio and derive the Black–Scholes PDE

$$r\,V \quad = \quad \frac{\partial V}{\partial t} + r\,S\,\frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2 V}{\partial S^2}\,,$$

and solve the PDE subject to the relevant terminal condition.

Power options

Another approach is to use Feyman–Kac so that the price of the call (similarly for put) is:

$$C^E(S,t) = e^{-r\,(T-t)}\,\mathbb{E}^{\mathbb{Q}}[\max(S^n - K^n, 0)]$$

with stock dynamics

$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dW_t^{\mathbb{Q}}\,.$$

Power options

Another approach

Under the pricing measure, the stock price follows

$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dW_t^{\mathbb{Q}}\,.$$

Let $\xi = S^n$ and use Ito's lemma to write

$$\frac{d\xi}{\xi} = r_n\,dt + \sigma_n\,dW_t^{\mathbb{Q}}\,,$$

where

$$r_n = n\,r + \frac{1}{2}\,\sigma^2\,n\,(n-1) \qquad \text{and} \qquad \sigma_n = n\,\sigma\,.$$

We rewrite the drift so that it has the form of the risk-free rate r plus a 'dividend payment':

$$\begin{aligned} r_n &= n\,r + \tfrac{1}{2}\,\sigma^2\,n\,(n-1) \\ &= r - r + n\,r + \tfrac{1}{2}\,\sigma^2\,n\,(n-1) \\ &= r - \left(-(n-1)\,(r + \tfrac{1}{2}\,n\,\sigma^2)\right)\,. \end{aligned}$$

Hence, the Black–Scholes PDE becomes

$$V_t + \frac{1}{2}\,n^2\,\sigma^2\,\xi^2\,V_{\xi\xi} + \left(r - \left(-(n-1)\,(r + \frac{1}{2}\,n\,\sigma^2\right)\right)\,\xi\,V_\xi = -r\,V\,.$$

Yet another approach to is: To change variables: $\xi = S^n$, so

$$\frac{d\xi}{dS} = n\, S^{n-1}\,, \qquad \frac{d^2\xi}{dS^2} = n\,(n-1)\,S^{n-2}$$

and

$$V_\xi = \frac{1}{n\,S^{n-1}}\,V_{SS}\,, \qquad V_{\xi\xi} = \frac{1-n}{n^2\,S^{2\,n-1}}\,V_S + \frac{1}{n^2\,S^{2\,(n-1)}}\,V_{SS}\,.$$

Next, substitute in the Black–Scholes PDE to obtain

$$V_t + \frac{1}{2}\,n^2\,\sigma^2\,\xi^2\,V_{\xi\xi} + \left(r - \left(-(n-1)\,(r + \frac{1}{2}\,n\,\sigma^2\right)\right)\,\xi\,V_\xi = -r\,V\,.$$

In both cases above, instead of solving the Black–Scholes PDE we can take a shortcut. Let the 'dividend' be

$$D_n = -(n-1)\left(r + \tfrac{1}{2} n\sigma^2\right).$$

Thus, we use the Black–Scholes formula with r and D_n to write the value of the power option:

$$C(S^n, t; K^n, T) = S^n\, e^{-D_n\,(T-t)}\,\Phi(d_{1n}) - e^{-r\,(T-t)}\,K^n\,\Phi(d_{2n})\,, \qquad (66)$$

with

$$d_{1n} = \frac{\ln(S_t^n/K^n) + (r - D_n + \frac{1}{2}\,n^2\,\sigma^2)\,(T-t)}{n\,\sigma\,\sqrt{T-t}}$$

and

$$d_{2n} = \frac{\ln(S_t^n/K^n) + (r - D_n - \frac{1}{2}n^2\sigma^2)\,(T-t)}{n\,\sigma\,\sqrt{T-t}}.$$

Barrier Options

Down-and-out

Let the stock price satisfy

$$dS_t = (\mu - D)\, S_t\, dt + \sigma\, S_t\, dW_t\,.$$

- Assume there is a barrier B that if the stock price reaches it the option becomes worthless.
- Consider a down-and-out call where the strike is above the barrier; $K > B$.

Apply the usual Black–Scholes hedging argument to show that $C_{d/o}$ satisfies

$$\frac{\partial C_{d/o}}{\partial t} + \frac{1}{2}\,\sigma^2\, S^2\, \frac{\partial^2 C_{d/o}}{\partial S^2} + (r - D)\, S\, \frac{\partial C_{d/o}}{\partial S} - r\, C_{d/o} = 0\,, \qquad (67)$$

while the option is alive, i.e., $B < S < \infty$.

The boundary conditions are similar to those used for vanilla options.

$$C_{d/o} = \max(S - K, 0).$$

As $S \to \infty$ the probability of hitting the barrier becomes negligible hence

$$C_{d/o}(S, t) \sim S\, e^{-D\, t}\,.$$

In the knock-out case we have that

$$C_{d/o}(S = B, t) = 0\,, \qquad \text{for } t \leq T\,.$$

Solution; change of variables

$$S = B\,e^{x}, \qquad t = T - \tau/\frac{1}{2}\,\sigma^2, \qquad C_{d/o} = B\,e^{\alpha\,x + \beta\,\tau}\,u(x,\tau),$$

with $\alpha = \frac{1}{2}\,(1-k')$, $\beta = -\frac{1}{4}\,(k'-1)^2 - k$, $k = r/\frac{1}{2}\,\sigma^2$, and $k' = (r-D)/\frac{1}{2}\,\sigma^2$. In these new variables the barrier transforms to the point $x = 0$, and the barrier option problem becomes

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \tag{68}$$

for $0 < x < \infty$, $\tau > 0$, with initial condition

$$u(x,0) = U(x) = \max\left(e^{\frac{1}{2}\,(k'+1)\,x} - (K/B)e^{\frac{1}{2}\,(k'-1)\,x}\,,\,0\right) \qquad x \geq 0\,, \tag{69}$$

and spatial condition (i.e., the barrier)

$$u(0,\tau) = 0\,. \tag{70}$$

By reflection we know that if $u(x,\tau)$ is a solution to the problem so is $u(-x,\tau)$. We must reflect the initial condition and solve the problem

$$u(x,0) = \begin{cases} U(x) & \text{for } x > 0, \\ -U(-x) & \text{for } x < 0, \end{cases} \tag{71}$$

that is $u(x,0) =$

$$\begin{cases} \max\left(e^{\frac{1}{2}(k'+1)x} - (K/B)\, e^{\frac{1}{2}(k'-1)x}, 0\right) & \text{for } x > 0\,, \\ -\max\left(e^{-\frac{1}{2}(k'+1)x} - (K/B)\, e^{-\frac{1}{2}(k'-1)x}, 0\right) & \text{for } x < 0\,. \end{cases} \tag{72}$$

Note that in this way we satisfy the boundary condition $u(0,\tau) = 0$. Now instead of solving problem (68) subject to (69) and (70) we solve (68) subject to (72).

The Trick

We can also solve problem (68) subject to (72) by relating this to vanilla options.

- Consider a vanilla call, with the same expiry and exercise price but no barrier.
- Write its value as $C_v(S, t; K, T)$ and $U_v(x, \tau)$ for the corresponding solution to the heat equation (i.e., the one where have already transformed variables).

Inspect initial condition (72). The first part says that for $x > \ln(K/B)$

$$e^{\frac{1}{2}(k'+1)x} \quad > \quad (K/B)\, e^{\frac{1}{2}(k'-1)\, x}\,, \tag{73}$$

i.e., the transformed payoff is positive. Hence, according to this condition we have that for $0 < x \leq \ln(K/B)$ the payoff is zero and positive for $x > \ln(K/B)$.

Similarly, the second part of the initial condition (72) says that the payoff is zero for $-\ln(K/B) \le x < 0$ and negative for $x < -\ln(K/B)$.

- Extend the payoff (recall that the initial condition (72) is the transformed payoff) so (72) becomes $u(x,0) =$

$$\begin{cases} \max\left(e^{\frac{1}{2}(k'+1)x} - (K/B)\, e^{\frac{1}{2}(k'-1)x}, 0\right), & -\infty < x < \infty\,, \\ -\max\left(e^{-\frac{1}{2}(k'+1)x} - (K/B)\, e^{-\frac{1}{2}(k'-1)x}, 0\right), & -\infty < x < \infty\,. \end{cases} \tag{74}$$

- This extended payoff is basically the initial condition for a call $U(x)$ minus the initial condition of a call but evaluated at $-x$.

$$u(x,0) = U(x) - U(-x) \qquad \text{for } -\infty < x < \infty\,. \tag{75}$$

The solution to the problem of the transformed option value is

$$u(x, \tau) = U_v(x, \tau) - U_v(-x, \tau)\,.$$

- Note that the right-hand side satisfies the heat equation and that it has the correct initial value.
- Moreover, the most important (since it is new for us) condition $u(0, t) = 0$ for all t.

Recall that $U(x, \tau)$ is the solution to the vanilla call option in the transformed variables, in other words

$$C_v(S, t; K, T) = C_v\left(B\,e^x, T - \tau/\frac{1}{2}\,\sigma^2; K, T\right) = B\,e^{\alpha\,x + \beta\,\tau}\,U_v(x, \tau)\,.$$

Hence

$$U_v(x,\tau) = \frac{e^{-\alpha\, x - \beta\, \tau}}{B}\, C_v\left(B\, e^{x}, T - \tau/\frac{1}{2}\,\sigma^2; K, T\right) ,$$

and so

$$U_v(-x,\tau) = \frac{e^{\alpha\, x - \beta\, \tau}}{B}\, C_v\left(B\, e^{-x}, T - \tau/\frac{1}{2}\,\sigma^2; K, T\right) .$$

Finally, $C_{d/o}(S,t;K,T)$

$$\begin{aligned}
&= B\, e^{\alpha\, x + \beta\, \tau}\, u(x,\tau) \\
&= B\, e^{\alpha\, x + \beta\, \tau}(U_v(x,\tau) - U_v(-x,\tau)) \\
&= C_v\left(B\, e^{x}, T - \tau/\frac{1}{2}\,\sigma^2; K, T\right) - e^{2\,\alpha x}\, C_v\left(B\, e^{-x}, T - \tau/\frac{1}{2}\,\sigma^2; K, T\right) \\
&= C_v(S,t;K,T) - \left(\frac{S}{B}\right)^{2\alpha} C_v(B^2/S, t; K, T)\,.
\end{aligned}$$

American Options

Overview

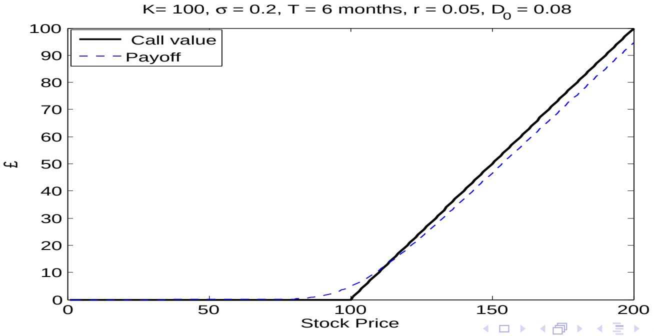
In this lecture we explore in more detail the Black–Scholes PDE

$$r\,V = V_t + r\,S\,V_S + \frac{1}{2}\,\sigma^2\,S^2\,V_{SS}\,. \tag{76}$$

- What PDE do American options satisfy?
- Can the value of a European put fall below its intrinsic value?
- Can the value of an American put fall below its intrinsic value?
- Perpetual options.

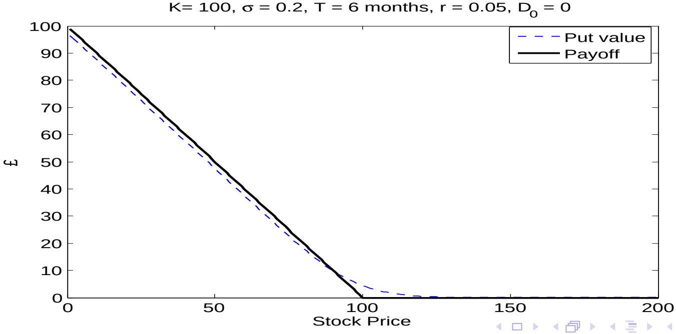
European call

The value of the European call can fall below its intrinsic value



European put

The value of the European put can fall below its intrinsic value



Can $P^A(S,t;K,T) < \max(K-S,0)$?

- Suppose that S is such that $P^A(S,t;K,T) < \max(K-S,0)$, and consider exercising the option
- Arbitrage opportunity: buy the asset in the market for S
- at the same time buy the option for P^A
- then exercise the option
- sell the asset for K
- will yield a risk-less profit of $K - P^A - S$
- Therefore we must have that

$$P^A(S,t;K,T) \geq \max(K-S,0)\,.$$

American-style PDE

- We want to hedge an American option and do it the usual way

$$\Pi(S,t) = V(S,t) - \Delta\, S\,.$$

- Delta-hedge the portfolio $\Delta = \partial V / \partial S$ to obtain

$$d\Pi(S,t) = \left(V_t + \frac{1}{2}\,\sigma^2\, S^2\, V_{SS} \right)\, dt\,.$$

- All we can expect then is that the holder of the portfolio cannot make more than the risk-free rate

$$\begin{aligned} d\Pi &\leq r\,\Pi\, dt \\ \frac{\partial V}{\partial t} + \frac{1}{2}\,\sigma^2\, S^2\, \frac{\partial^2 V}{\partial S^2} + r\, S\, \frac{\partial V}{\partial S} - r\, V &\leq 0\,, \end{aligned}$$

because only if the option is held till maturity does the delta-hedge work and the equality $d\Pi = r\,\Pi\, dt$ holds – like a European option.

Example: American put

For example, the pricing PDE for an American put satisfies

$$\frac{\partial P^A}{\partial t} + \frac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2 P^A}{\partial S^2} + r\,S\,\frac{\partial P^A}{\partial S} - r\,P^A \leq 0\,.$$

- We can check that if we exercise early, ie $P^A = K - S_t$, then

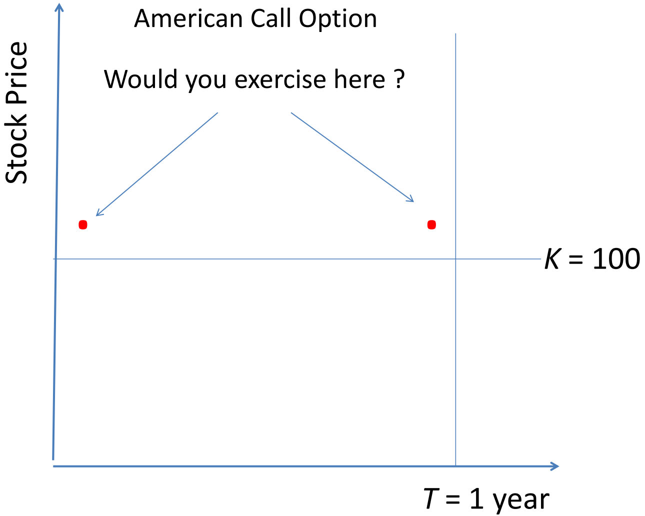
$$\frac{\partial P^A}{\partial t} + \frac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2 P^A}{\partial S^2} + r\,S\,\frac{\partial P^A}{\partial S} - r\,P^A = -r\,K < 0\,.$$

- And if it is optimal not to exercise, then

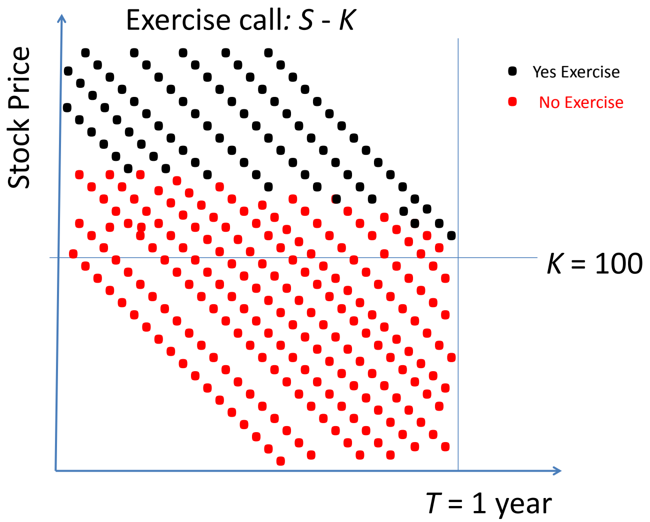
$$\frac{\partial P^A}{\partial t} + \frac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2 P^A}{\partial S^2} + r\,S\,\frac{\partial P^A}{\partial S} - r\,P^A = 0\,.$$

To Hold or Not to Hold

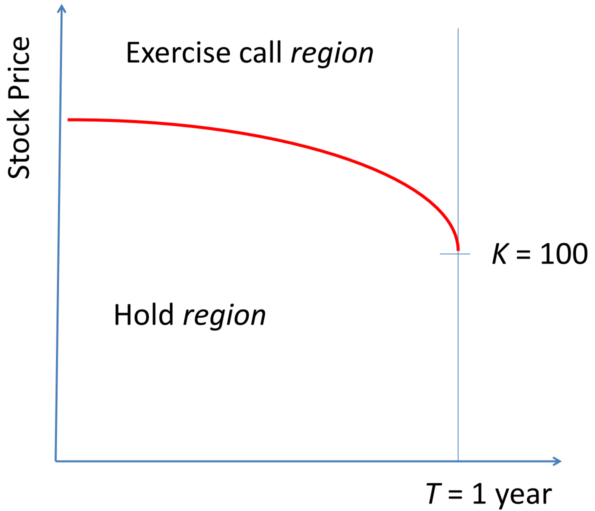
Exercise Region American Call



Exercise Region American Call



Free Boundary American Call



Perpetual Options

Perpetual option

- Perpetual options are American options with no expiry; $T = \infty$
- The value of the perpetual call and put can be determined in closed-form
- The Black–Scholes PDE becomes an ODE because the value of the option does not change with time t
  - In other words, if $T = \infty$, $V_t = 0$
  - This makes the valuation problem simpler to solve because regardless of t the problem is 'always the same'.
  - The exercise boundary can be determined explicitly and is fixed

Perpetual option

Since the value of the option V, which could be a call or a put, does not decay with time, $V_t = 0$, the pricing equation satisfies

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)\, S \frac{\partial V}{\partial S} - r\, V \leq 0\,, \tag{77}$$

with equality when it is optimal to hold.

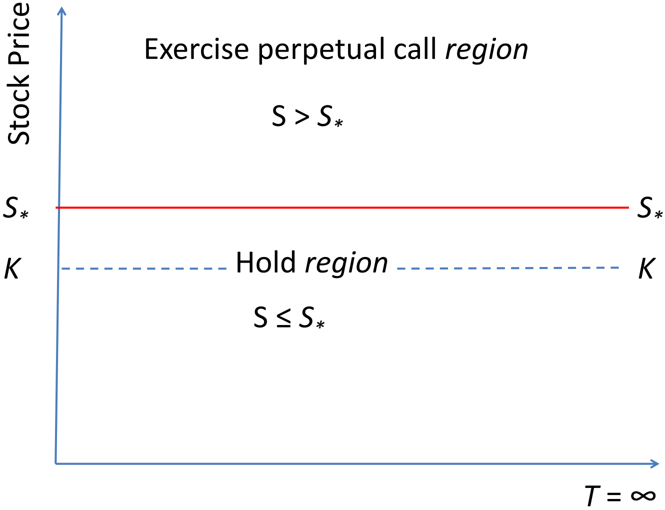
Thus, when it is not optimal to exercise the ODE we must solve

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)\, S \frac{\partial V}{\partial S} - r\, V = 0\,, \tag{78}$$

where the hold region for the call is $S < S_\star$ and for the put $S_\star < S$.

What is the price of a perpetual call if $D_0 = 0$?

Exercise Perpetual Call



Trial solution $V(S,t) = S^{\beta}$ and obtain the characteristic equation

$$\begin{aligned} \beta\pm &= \frac{1}{2}\left(1 - \frac{2\,(r - D_0)}{\sigma^2}\right) \\ &\pm \frac{1}{2}\sqrt{\left(\left(1 - \frac{2\,(r - D_0)}{\sigma^2}\right)^2 + \frac{8\,r}{\sigma^2}\right)}. \end{aligned}$$

Hence the solution to (78) is given by the linear combination

$$V(S) = A\,S^{\beta+} + B\,S^{\beta-}\,,$$

where A and B are constants (that we need to determine).

Perpetual call

- Payoff of call is
$$C^A(S_\star) = \max(S_\star - K, 0)\,.$$
- We also know that as $S \to 0$ the price $C^A(S) \to 0$.
- We must rule out the solution that contains the negative root β_-.
- At exercise we have that
$$\begin{aligned} C(S_\star) &= S_\star - K\,, \\ A\,S_\star^{\beta_+} &= S_\star - K\,, \end{aligned}$$
which in the literature is known as the 'value matching' condition.

The holder must choose the largest

$$A = \frac{S_\star - K}{S_\star^{\beta_+}}\,,$$

such that the option value is maximal. Hence

$$\max_{S_\star} \frac{S_\star - K}{S_\star^{\beta_+}}\,. \tag{79}$$

The first order condition is

$$\begin{aligned} \frac{S_\star^{\beta_+} - \beta_+\,(S_\star - K)\,S_\star^{\beta_+-1}}{S_\star^{2\beta_+}} &= 0 \\ -1 - \beta_+\,(S_\star - K)\,S_\star^{-1} &= 0\,, \end{aligned} \tag{80}$$

which is known as the ‘smooth pasting’ condition.

Therefore

$$S_\star = \frac{\beta_+}{\beta_+ - 1}\, K \,,$$

and

$$A = K^{1-\beta_+}\left(\frac{\beta_+ - 1}{\beta_+}\right)^{\beta_+} \frac{1}{\beta_+ - 1} \,.$$

Finally, the value of the perpetual call is

$$C(S) = \begin{cases} \left(\frac{\beta_+ - 1}{\beta_+\, K}\right)^{\beta_+} \frac{K}{\beta_+ - 1}\, S^{\beta_+} & \text{for} \qquad S < S_\star \,, \\ S - K & \text{for} \qquad S \geq S_\star \,. \end{cases}$$

Implied Volatility

Overview

- Examine how the prices of European calls and puts depend on the volatility of the underlying stock
- Back out the implied volatility (IV) using the prices of traded options
- Study different shapes of IV
- Derive analytical formulae for the bounds of the slope of the IV curve
- Revisit Black–Scholes with time-varying deterministic volatility
- Non-traded assets
- Stochastic volatility models
- The leverage effect

Implied Volatility

- Let $\overline{C}(S, t; K, T)$ denote the observed (market) European call option price
- IV is the value of the volatility parameter that must go into the Black–Scholes formula

$$C(S, t; K, T, \sigma) \quad = \quad S_t\, \Phi(d_1) - e^{-r\,(T-t)}\, K\, \Phi(d_2)\,,$$

where

$$d_1 \quad = \quad \frac{\ln(S_t/K) + (r + \frac{1}{2}\,\sigma^2)\,(T-t)}{\sigma\,\sqrt{T-t}}$$

and

$$d_2 = \frac{\ln(S_t/K) + (r - \frac{1}{2}\,\sigma^2)\,(T-t)}{\sigma\,\sqrt{T-t}}\,,$$

to match the market price

$$C(S, t; K, T, I) = \overline{C}\,. \tag{81}$$

1. The Vega of a European call is positive, i.e.,

$$\frac{\partial C}{\partial \sigma} = \frac{S\, e^{-\frac{d_1^2}{2}} \sqrt{T-t}}{\sqrt{2\,\pi}} > 0\,,$$

   so there is a unique $I > 0$ because of the monotonicity of the Black–Scholes formula in the volatility parameter σ.

2. The implied volatility from the put and call options of the same strike and time to maturity are the same because put-call-parity. (True in practice?)

Questions

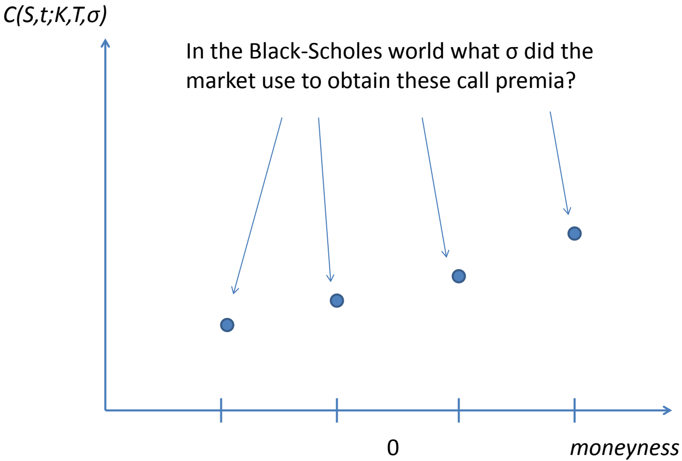
- What should happen to IV if the world was as assumed by the Black–Scholes model?
- What happens in practice?
- What is the interpretation?
- What do we expect to happen to IV across time?

Plotting IV

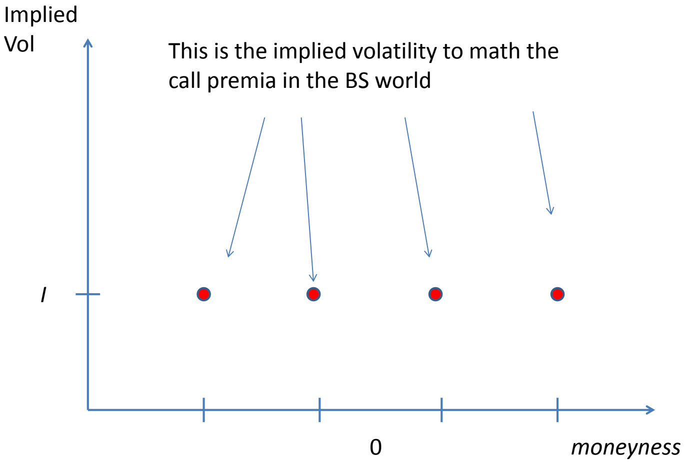
- Plot IV as a function of moneyness
- For moneyness we use
  - $$m = \frac{\ln F/K}{ATMV\sqrt{T}}$$

    where $ATMV$ is at-the-money implied volatility, T expiry of option, K strike, F forward price of stock (with same expiry)
- This measure of moneyness allows for comparisons across maturities and assets

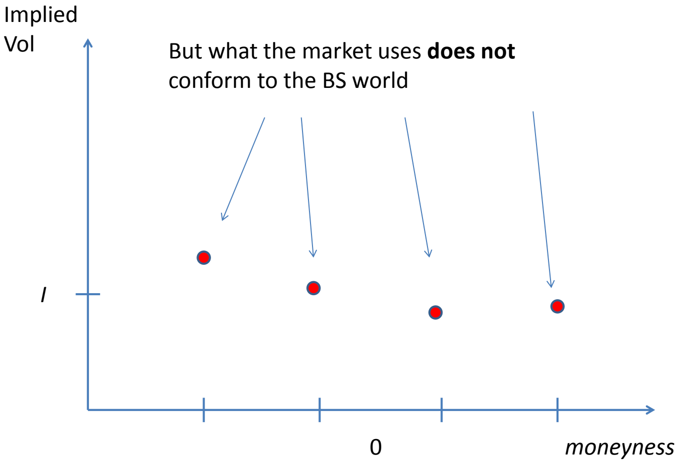
Call premia


C(S,t;K,T,σ)
In the Black-Scholes world what σ did the market use to obtain these call premia?
0
moneyness

Black–Scholes IV

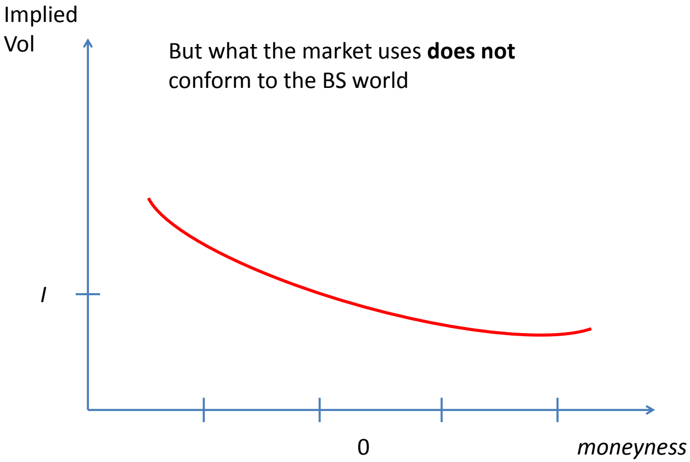


IV


Implied
Vol
But what the market uses **does not** conform to the BS world
I
0
moneyness

IV Smile

IV Smirk


Implied
Vol
But what the market uses **does not** conform to the BS world
I
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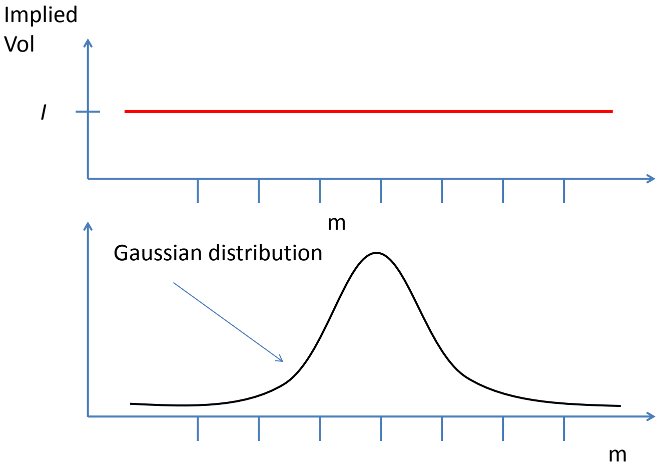
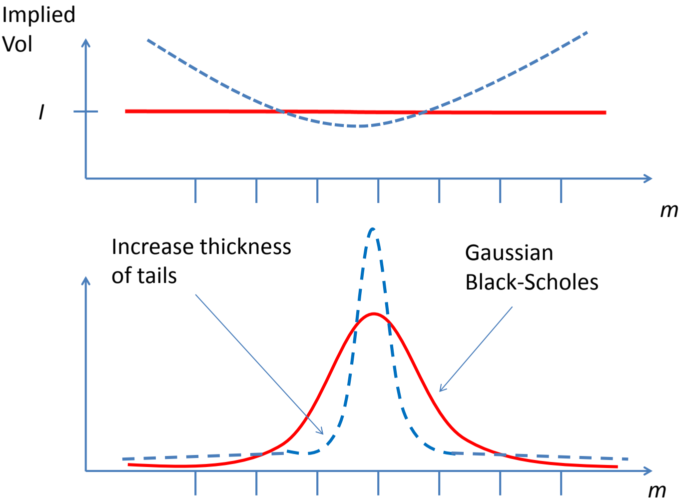
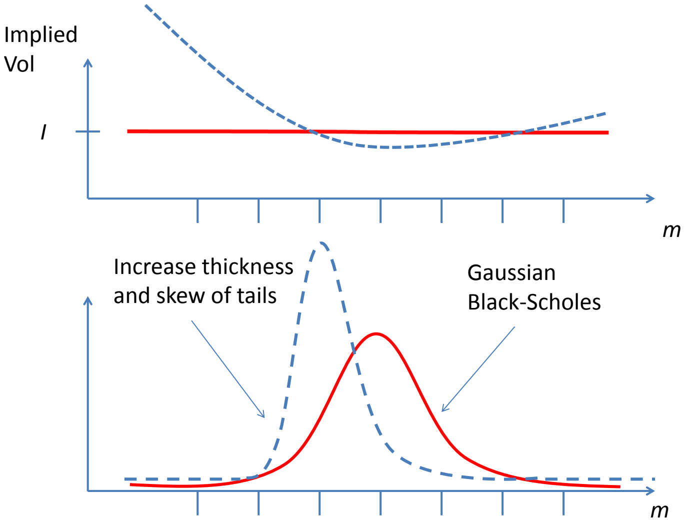
IV & Dist



IV & Dist


Implied
Vol
I
m
Increase thickness
of tails
Gaussian
Black-Scholes
m

IV & Dist



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- The smile shows that there is a premium charged for
  - out-of-the-money put options, and
  - for in-the-money calls

  above their Black–Scholes price computed with at-the-money volatility.
- In other words, the market prices options as though the lognormal model **underestimates** the probability of large movements in the underlying.
- The downward slope of the implied volatility is a consequence of the asymmetry in the risk-neutral distribution of the underlying stock return.
- The convexity shown by the implied volatility is a consequence of the thickness of the tails of the distribution.

Bounds

We can derive bounds on the slope of the volatility smile by differentiating (81) with respect to K.

$$\frac{\partial \overline{C}}{\partial K} = \frac{\partial C}{\partial K} + \frac{\partial C}{\partial \sigma}\,\frac{\partial I}{\partial K} \leq 0\,, \tag{82}$$

hence we have that

$$\frac{\partial I}{\partial K} \leq -\frac{\partial C/\partial K}{\partial C/\partial \sigma}\,. \tag{83}$$

Similarly, put prices must be increasing in K. Moreover, since puts and calls must have the same implied volatility I we must have

$$\frac{\partial I}{\partial K} \geq -\frac{\partial P/\partial K}{\partial P/\partial \sigma}\,. \tag{84}$$

We can substitute and rearrange to obtain

$$-\frac{\sqrt{2\,\pi}}{S\,\sqrt{T-t}}(1-\Phi(d_2))\,e^{-r\,(T-t)+d_1^2/2} \leq \frac{\partial I}{\partial K} \leq \frac{\sqrt{2\,\pi}\,\Phi(d_2)}{S\,\sqrt{T-t}}\,e^{-r\,(T-t)+d_1^2/2}\,,$$

where d_1 and d_2 are as usual but with σ replaced by I.

Implied Deterministic Volatility

Assume that the stock price follows the SDE

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma(S,t)\, dW_t\,,$$

were $\sigma(S,t)$ is a deterministic function of (S,t). Obtain pricing PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\,\sigma^2(S,t)\,S^2\,\frac{\partial^2 V}{\partial S^2} + r\,S\,\frac{\partial V}{\partial S} - r\,V = 0\,. \tag{85}$$

Simplify this approach and let $\sigma(S,t)$ depend on time only. Thus

$$\frac{dS_t}{S_t} = r\, dt + \sigma(t)\, dW_t\,.$$

Use Ito's lemma to write

$$S_T = S_t\, e^{r\,(T-t) - \frac{1}{2}\,\int_t^T \sigma^2(s)\,ds + \int_t^T \sigma(s)\,dW_s}\,.$$

As an example let us price a European call with payoff $\max(S_T - K, 0)$. It suffices to observe that the distribution of log-returns is

$$\ln(S_T/S_t) \sim N\left((r - \tfrac{1}{2}\,\overline{\sigma^2})\,(T-t), \overline{\sigma^2}\,(T-t)\right)\,,$$

where

$$\overline{\sigma^2} = \frac{1}{T-t}\int_t^T \sigma^2(s)\,ds\,.$$

Therefore the price of the call is given by??

$$C(S, t; K, T, \sqrt{\overline{\sigma^2}}) \quad = \quad S_t \, \Phi(d_1) - e^{-r\,(T-t)} \, K \, \Phi(d_2) \,,$$

where

$$d_1 \quad = \quad \frac{\ln(S_t/K) + (r + \frac{1}{2}\overline{\sigma^2})\,(T-t)}{\sqrt{\overline{\sigma^2}}\sqrt{T-t}} \,,$$

and

$$d_2 = \frac{\ln(S_t/K) + (r - \frac{1}{2}\,\overline{\sigma^2})\,(T-t)}{\sqrt{\overline{\sigma^2}}\,\sqrt{T-t}} \,.$$

In other words, we use the average variance over the interval $[t, T]$.

Stochastic Volatility Models

Stein and Stein proposed

$$\begin{aligned} dS &= r\,S\,dt + \sigma\,S\,dW_1\,, &(86)\\ d\sigma &= \alpha\,dt + \beta\,dW_2\,, &(87)\end{aligned}$$

for constants α and β.
Another very popular model is Heston's

$$\begin{aligned} dS_t &= \mu\,S_t\,dt + \sqrt{v_t}\,S_t\,dW_1 &(88)\\ d\sqrt{v_t} &= -\beta\,\sqrt{v_t}\,dt + \delta\,dW_2\,. &(89)\end{aligned}$$

We can use Ito's lemma to show that the variance v_t follows the process

$$dv_t = \kappa\,(\theta - v_t)\,dt + \sigma\,\sqrt{v_t}\,dW_2\,, \tag{90}$$

where ρ is the correlation between W_1 and W_2.

The remainder of the material in non-examinable

Non-traded asset

$$\frac{dX_t}{X_t} = \mu\,dt + \sigma\,dW_t\,,$$

where W_t is the usual Brownian Motion and X is not traded.

- How can we price a European-style option written on X?
- Can we proceed as in the Black–Scholes case above?
- How are we going to hedge this option?

Let $\Pi(X,t)$ be a portfolio containing a number of V, with expiry T, and b number of F, with expiry $T_1 > T$, and $a + b = 1$,

$$\Pi(X,t) = a\,V(X,t) + b\,F(X,t)$$

and its returns process given by

$$\frac{d\Pi}{\Pi} = a\,\frac{dV}{V} + b\,\frac{dF}{F}\,.$$

Now using Ito's lemma write

$$\begin{aligned}\frac{d\Pi}{\Pi} \quad = \quad & \frac{(1-b)}{V}\left(\left(V_t + \mu\,XV_X + \frac{1}{2}\,\sigma^2\,X^2\,V_{XX}\right)\,dt + \sigma\,X\,V_X\,dW\right) \\ & + \frac{b}{F}\left(\left(F_t + \mu\,XF_X + \frac{1}{2}\,\sigma^2\,X^2\,F_{XX}\right)\,dt + \sigma\,X\,F_X\,dW\right)\,. \qquad (91)\end{aligned}$$

Make portfolio risk-neutral:

$$b^* = \frac{F\, V_X}{F\, V_X - V\, F_X}\,,$$

hence

$$\begin{aligned}\frac{d\Pi}{\Pi} &= \frac{1-b^*}{V}\left(V_t + \mu\, X V_X + \frac{1}{2}\,\sigma^2\, X^2\, V_{XX}\right)\, dt \\ &\quad + \frac{b^*}{F}\left(F_t + \mu\, X\, F_X + \frac{1}{2}\,\sigma^2\, X^2\, F_{XX}\right)\, dt\,.\end{aligned}$$

then $d\Pi/\Pi = r\, dt$, so

$$\begin{aligned}r &= \frac{1-b^*}{V}\left(V_t + \mu\, X\, V_X + \frac{1}{2}\,\sigma^2\, X^2\, V_{XX}\right) \\ &\quad + \frac{b^*}{F}\left(F_t + \mu\, X\, F_X + \frac{1}{2}\,\sigma^2\, X^2\, F_{XX}\right)\,.\end{aligned}$$

After some algebra we have

$$\frac{F}{F_X}\left(\frac{F_t + \mu\, X\, F_X + \frac{1}{2}\,\sigma^2\, X^2\, F_{XX}}{F} - r\right)$$
$$= \frac{V}{V_X}\left(\frac{V_t + \mu\, X\, V_X + \frac{1}{2}\,\sigma^2\, X^2\, V_{XX}}{V} - r\right)\,. \tag{92}$$

By looking at equation (91), note that if we held a portfolio with only one option $V(X,t)$, i.e., $a = 1$ and $b = 0$, the returns would be given by

$$\frac{dV}{V} \quad = \quad \frac{1}{V}\left(\left(V_t + \mu\, X V_X + \frac{1}{2}\,\sigma^2\, X^2\, V_{XX}\right)\, dt + \sigma\, X\, V_X\, dW\,;\right)$$

hence the 'volatility' coefficient of this portfolio would be given by

$$\sigma_V = \frac{\sigma X\, V_X}{V}\,, \tag{93}$$

and the drift coefficient is given by

$$\mu_V = \left(V_t + \mu\, X V_X + \frac{1}{2}\,\sigma^2\, X^2\, V_{XX}\right) / V\,. \tag{94}$$

We can write equation (92) as

$$\frac{1}{\frac{\sigma\, X\, F_X}{F}}\left(\frac{F_t + \mu\, X\, F_X + \frac{1}{2}\,\sigma^2\, X^2\, F_{XX}}{F} - r\right) =$$

$$\frac{1}{\frac{\sigma\, X\, V_X}{V}}\left(\frac{V_t + \mu\, X V_X + \frac{1}{2}\,\sigma^2\, X^2\, V_{XX}}{V} - r\right).$$

Note that the left-hand side of this equation only depends on F that expires at time T_1 and the right-hand side only depends on V that expires at time $T < T_1$...
Any suggestions?

... hence we can write

$$\frac{1}{\sigma\, X\, V_X / V}\left(\frac{V_t + \mu\, X V_X + \frac{1}{2}\,\sigma^2\, X^2\, V_{XX}}{V} - r\right) = \lambda(t).$$

Next, use (93) and (94) to write

$$\frac{1}{\sigma_V}\,(\mu_V - r\,) = \lambda(t)\,.$$

How may we interpret this relationship?

$\lambda(t)$: Market Price of Risk

- $\lambda(t)$ measures the excess return $\mu_V - r$ per unit of volatility σ_V an investor expects to obtain from holding a risky asset.
- We can rewrite this expression as

$$\begin{aligned} \mu_V - r &= \sigma_V\,\lambda(t) \\ V_t + \mu\,X\,V_X + \frac{1}{2}\,\sigma^2 X^2 V_{XX} - rV &= \sigma\,X\,V_X\,\lambda(t)\,. \end{aligned}$$

Hence we obtain a pricing PDE of the form

$$V_t + (\mu - \sigma\,\lambda(t))\,X\,V_X + \frac{1}{2}\,\sigma^2\,X^2\,V_{XX} - r\,V = 0\,. \tag{95}$$

What happens if X is traded?

We must stress that the market price of risk "appears" because X is not a traded asset, so we cannot hedge instruments written on X. For suppose that $X = S$ where S is traded, hence $V = S$ must satisfy (95). Straightforward substitution implies

$$(\mu - \sigma\lambda(t))\, S = r\, S\,,$$

hence

$$\mu - \sigma\lambda(t) = r\,,$$

and equation (95) becomes the usual Black–Scholes PDE.

Pricing PDE with stochastic volatility

Let us derive a pricing PDE when we have a two factor model.

$$dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dW_1\,,$$

$$d\sigma = p(S,\sigma,t)\, dt + q(S,\sigma,t)\, dW_2\,, \qquad (96)$$

where p and q are deterministic functions and the correlation between W_1 and W_2 is ρ.

Form hedge portfolio

$$\Pi = V - \Delta S - \Delta^1 V^1. \qquad (97)$$

$$
\begin{aligned}
d\Pi \quad = \quad & \left(V_t + \mu\, S\, V_S + \frac{1}{2}\,\sigma^2\, S^2\, V_{SS} + p\, V_\sigma + \frac{1}{2}\, q^2\, V_{\sigma\sigma} - \Delta\, \mu\, S + \rho\,\sigma\, q\, S\, V_{S\sigma} \right) dt \\
& - \Delta^1 \left(V_t^1 + \mu\, S\, V_S^1 + \frac{1}{2}\,\sigma^2\, S^2\, V_{SS}^1 + p\, V_\sigma^1 + \frac{1}{2}\, q^2\, V_{\sigma\sigma}^1 + \rho\,\sigma\, q\, S\, V_{S\sigma}^1 \right) dt \\
& + \sigma\, S \left(V_S - \Delta - \Delta^1\, V_S^1 \right) dW_1 \\
& + q \left(V_\sigma - \Delta^1\, V_\sigma^1 \right) dW_2\,.
\end{aligned}
$$

To eliminate randomness we choose

$$
V_S - \Delta - \Delta^1 V_S^1 = 0 \qquad \text{and} \qquad V_\sigma - \Delta^1 V_\sigma^1 = 0.
$$

Thus

$$
\begin{aligned}
d\Pi &= \left(V_t + \frac{1}{2}\,\sigma^2\, S^2\, V_{SS} + \frac{1}{2}\, q^2\, V_{\sigma\sigma} + \rho\,\sigma\, q\, S\, V_{S\sigma} \right) dt \\
&\quad - \frac{V_\sigma}{V^1_\sigma} \left(V^1_t + \frac{1}{2}\,\sigma^2\, S^2\, V^1_{SS} + \frac{1}{2}\, q^2\, V^1_{\sigma\sigma} + \rho\,\sigma\, q\, S\, V^1_{S\sigma} \right) dt\,.
\end{aligned} \tag{98}
$$

The portfolio must evolve like a bond hence $d\Pi = r\,\Pi\, dt$. Therefore

$$
\frac{V_t + \frac{1}{2}\,\sigma^2\, S^2\, V_{SS} + \frac{1}{2}\, q^2\, V_{\sigma\sigma} + \rho\,\sigma\, q\, S\, V_{S\sigma} + rSV_S - rV}{V_\sigma} = \tag{99}
$$

$$
\frac{V^1_t + \frac{1}{2}\,\sigma^2\, S^2\, V^1_{SS} + \frac{1}{2}\, q^2\, V^1_{\sigma\sigma} + \rho\,\sigma\, q\, SV^1_{S\sigma} + r\, S\, V^1_S - r\, V}{V^1_\sigma}\,. \tag{100}
$$

Proceeding as above (market price of risk)

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + \frac{1}{2} q^2 V_{\sigma\sigma} + \rho\,\sigma\,q\,S\,V_{S\sigma} + r\,S\,V_S - r\,V = -(p - \lambda q)\,V_\sigma\,, \qquad (101)$$

or alternatively

$$\begin{aligned}
&V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + r\,S\,V_S - r\,V \qquad \text{Black–Scholes}\\
&+\rho\,\sigma\,q\,S\,V_{S\sigma} \qquad \text{correlation}\\
&+\frac{1}{2} q^2 V_{\sigma\sigma} + p\,V_\sigma \qquad \text{operator of volatility process on } V\\
&-\lambda\,q\,V_\sigma \qquad \text{market price of volatility risk}\\
&= 0\,.
\end{aligned}$$

It is straightforward to see that if $p = 0$ and $q = 0$ we are back to the usual Black–Scholes PDE.

Hull–White Formula

Assume we want to price a call option written on a stock where the volatility also follows a stochastic process, say $\sigma_t = f(Y_t)$ where Y is stochastic, that is independent of the Brownian motion of the stock.

$$C(S,t;K,T) = \mathbb{E}\left[C_{BS}\left(S,t;K,T,\sqrt{\overline{\sigma^2}}\right) \mid \sigma_s, t \le s \le T)\right], \tag{102}$$

where

$$\overline{\sigma^2} = \frac{1}{T-t}\int_t^T f(Y_s)^2\, ds\,.$$

The leverage effect

- Financial data suggest that returns are skewed rather than symmetric (more so under the pricing measure).
- Assume that the Brownian motion driving the returns process, say W_t, is correlated with the Brownian motion of the volatility process, say Z_t.
- We can write $\tilde{Z}_t = \rho\, W_t + \sqrt{1-\rho^2}\, Z_t$, where $\tilde{Z}_t$ is independent of W_t,
- Pricing with the Black–Scholes framework becomes relatively more simple.
- ρ is also known in the literature as the leverage effect and empirical studies suggest that $\rho < 0$.
- The financial interpretation is that in periods of high volatility prices go down and vice-versa.

Jumps in Prices

Overview

- Poisson Processes
- Example: modelling stock prices with diffusion and Poisson jumps

Poisson process

- A Poisson process is a process subject to jumps of fixed size or random size.
- λ denotes the mean arrival rate of an event, during a time interval dt.
- The probability that an event will occur is $\lambda\, dt$, and that it will not occur is $1 - \lambda\, dt$.
- The event is a jump of size u, which can itself be a random variable. The simplest is $u = 1$, so the Poisson process is a counting process.

Formally, we let N_t be the number of events that occur by time t then $N_t, t > 0$ is called a Poisson process, and it can be shown that

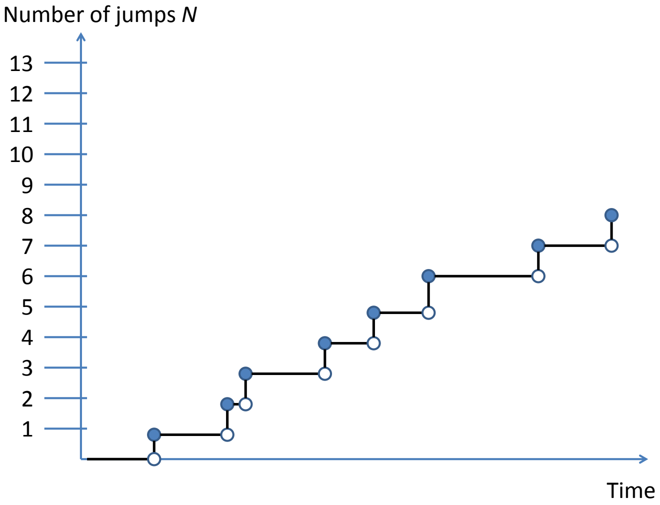
$$\mathbb{P}\{N_t = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \qquad n = 0, 1, \ldots . \tag{103}$$

1. The probability of at least one event $(ALOE)$ happening in a time period of duration Δt is

$$\mathbb{P}[ALOE] = \lambda \Delta t + o(\Delta t) \qquad \text{as } \Delta t \to 0, \lambda > 0 .$$

2. The probability of two or more events happening in a period of time Δt is $o(\Delta t)$. In other words, we do not see two events happening at the same time.

Poisson Process



Interarrival times

Another way to see a Poisson counting process is by looking at the interarrival times; i.e., how long it takes between each Poisson event. Let T_j be the time of the jth arrival, then

$$\mathbb{P}[T_{n+1} - T_n > s \mid T_1, \cdots, T_n] = 1 - e^{-\lambda s}\,.$$

In other words, the interarrival times $T_1,\ T_2 - T_1,\ \cdots$ of a Poisson process are iid with cdf $1 - e^{-\lambda s}$. Moreover, the pdf of the interarrival times is

$$\mathbb{P}[\tau > t] = \lambda\, e^{-\lambda\, t}\,.$$

Exercise: Show that

$$\mathbb{P}[\tau > t + s \mid \tau > t] = \mathbb{P}[\tau > s]\,.$$

That is, that the probability of observing an event does not depend on the past events.

A stock model: diffusion and jumps

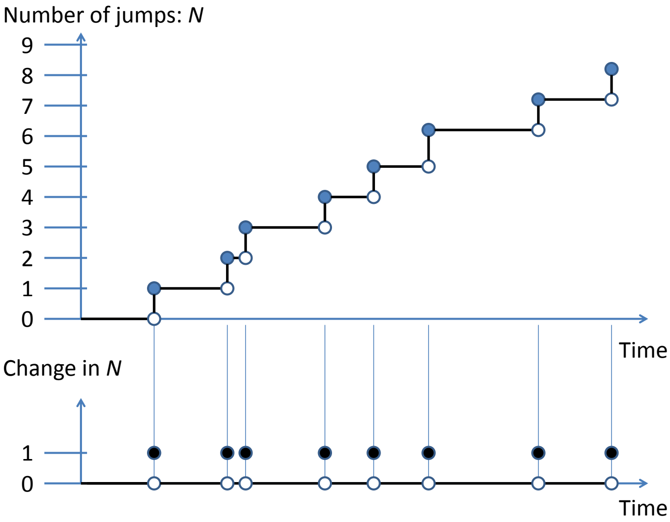
$$\frac{dS_t}{S_{t^-}} = \mu\, dt + \sigma\, dW_t - \delta\, dN_t\,, \tag{104}$$

where W_t and N_t are independent and $0 \leq \delta \leq 1$ is a constant. (Note: We have S_{t^-} because is the price 'just before the jump'. Below, for simplicity of notation we write S_t.)

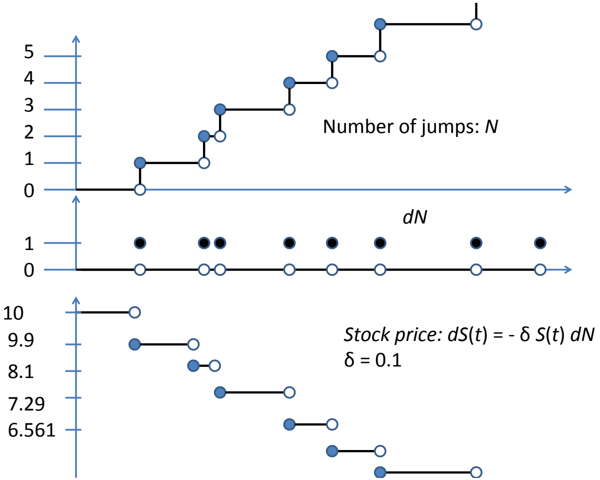
- What is the solution of this SDE?
- Assume that $\mu = 0$ and $\sigma = 0$ and solve

$$\frac{dS_t}{S_t} = -\delta\, dN_t\,. \tag{105}$$

Poisson Process



Stock Price with Fixed Jump Size $\delta = 0.1$



A stock model: diffusion and jumps

- Thus, the solution to

$$\frac{dS_t}{S_t} = -\delta\, dN_t\,,$$

is

$$S_t = S_0\,(1-\delta)^{N_t}.$$

- All that matters is the number of jumps N_t and not when they occurred!
- And the solution to

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t - \delta\, dN_t\,, \tag{106}$$

is therefore

$$S_t = S_0\, e^{\left(\mu-\frac{1}{2}\,\sigma^2\right)t+\sigma\, W_t}(1-\delta)^{N_t}\,.$$

Classical Model: MJD

Merton's Jump Diffusion. Assume that the returns process follows

$$\frac{dS_t}{S_t} = (\alpha - \lambda\, k)\, dt + \sigma\, dW_t + (Y - 1)\, dN_t\,. \tag{107}$$

Another way to express (107) is

$$\frac{dS_t}{S_t} = (\alpha - \lambda\, k)\, dt + \sigma\, dW\,, \qquad \text{if no jumps occur,}$$

$$\frac{dS_t}{S_t} = (\alpha - \lambda\, k)\, dt + \sigma\, dW_t + Y - 1\,, \qquad \text{if a jump occurs.}$$

We can apply Ito's lemma to $f = \ln S$ to solve (107).

$$S_t = S_0\, e^{(\alpha - \frac{1}{2}\,\sigma^2 - \lambda\, k)\, t + \sigma\, W_t}\, Y(n)\,, \tag{108}$$

where $Y(n) = 1$ if $n = 0$, $Y(n) = \Pi_{j=1}^{n} Y_j$ for $n \geq 1$, where Y_j are iid and n is Poisson distributed with parameter $\lambda\, t$.
Note that

$$d(\ln S) = \left(\alpha - \frac{1}{2}\,\sigma^2 - \lambda\, k\right)\, dt + \sigma\, dW_t + \ln Y\, dN_t\,.$$

Do we need any restriction on the support of Y?

Exercise: $\mathbb{E}[e^{-rt} S_t] =$

$$
\begin{aligned}
&= \mathbb{E}\left[S_0\, e^{-rt} e^{(\alpha-\frac{1}{2}\sigma^2-\lambda k)t+\sigma W_t}\, Y(n)\right] \\
&= \mathbb{E}_{N,Y}\left[\mathbb{E}_W\left[S_0\, e^{-rt} e^{(\alpha-\frac{1}{2}\sigma^2-\lambda\, k)t+\sigma W_t} \prod_{j=1}^{n} Y_j \middle| N_t = n\,,\, Y\right]\right] \\
&= S_0\, e^{-rt} e^{(\alpha-\lambda k)t} \mathbb{E}_{N,Y}[\textstyle\prod_{j=1}^{n} Y_j] \\
&= S_0\, e^{-rt} e^{(\alpha-\lambda k)t} \mathbb{E}_N\left[\mathbb{E}_Y[\textstyle\prod_{j=1}^{n} Y_j | N_t = n]\right] \\
&= S_0\, e^{-rt} e^{(\alpha-\lambda k)t} \mathbb{E}_N\left[\mathbb{E}_Y[Y]^n | Y\right] \\
&= S_0\, e^{-rt} e^{(\alpha-\lambda\, k)t} \sum_{n=0}^{\infty} \frac{e^{-\lambda\, t}\left(\lambda\, t\mathbb{E}_Y[Y]\right)^n}{n!} \\
&= S_0\, e^{-rt} e^{(\alpha-\lambda\, k)t} e^{-\lambda\, t} e^{\lambda\, t\mathbb{E}_Y[Y]}\,.
\end{aligned}
$$

Exercise

Let

$$d(\ln S) = \left(\alpha - \frac{1}{2}\sigma^2 - \lambda\, k\right) dt + \sigma dW + \ln Y dN.$$

where α, σ, k are constants, W is a standard Brownian motion, N is a counting process with intensity λ, and Y are iid. W, N, Y are independent.
Assume that $\alpha = r$ and $k = \mathbb{E}_Y[Y] - 1$. Show that

$$\mathbb{E}\left[e^{-rT}\max(S_T - K, 0)\right] = \sum_{n=0}^{\infty}\left[\frac{e^{-\lambda\, t}(\lambda\, t)^n}{n!}\mathbb{E}_Y\left[C^E(S_0 Y^n e^{-\lambda(\mathbb{E}_Y[Y]-1)T}, 0; T, K)\right]\right]$$

where C^E denotes the value of a European call option in the Black–Scholes model.
Moreover, show that

$$\mathbb{E}\left[e^{-rT}\max(S_T - K, 0)\right] \geq C^E(S_0, 0; T, K)\,.$$

When do we get strict equality in the above equation?

$C^E(S,0;T,K) =$

$$
\begin{aligned}
&= \mathbb{E}^\star[e^{-rT}(S_T - K)^+] \\
&= \mathbb{E}^\star[e^{-rT}\max(S_0\, e^{(r-\frac{1}{2}\sigma^2-\lambda(\mathbb{E}_Y[Y]-1))T+\sigma W_T} Y(n) - K, 0)] \\
&= \mathbb{E}^\star_{N,Y}\left[\mathbb{E}^\star_W[e^{-rT}(S_0 e^{(r-\frac{1}{2}\sigma^2-\lambda(\mathbb{E}_Y[Y]-1))T+\sigma W_T} Y(n) - K)^+ | N_T, Y]\right] \\
&= \mathbb{E}^\star_{N,Y}\left[\mathbb{E}^\star_W[e^{-rT}(S_0 e^{(r-\frac{1}{2}\sigma^2-\lambda(\mathbb{E}_Y[Y]-1))T+\sigma W_T} Y(n) - K)^+ | N_T, Y]\right] \\
&= \mathbb{E}^\star_{N,Y}\left[\mathbb{E}^\star_W[e^{-rT}(S_0 Y(n) e^{-\lambda(\mathbb{E}_Y[Y]-1)T} e^{(r-\frac{1}{2}\sigma^2)T+\sigma W_T} - K)^+ | N_T, Y]\right] \\
&= \mathbb{E}^\star_{N,Y)}\left[\mathbb{E}^\star_W[e^{-rT}\max(\hat{S} e^{(r-\frac{1}{2}\sigma^2)T+\sigma W_T} - K, 0) | N_T, Y]\right] \\
&= \mathbb{E}^\star_{N,Y}\left[C^E(\hat{S}, 0; T, K)\right],
\end{aligned}
$$

where $\hat{S} = S_0 Y(n) e^{-\lambda(\mathbb{E}_Y[Y]-1)T}$.

In other words

$$C^E = \sum_{n=0}^{\infty} \left[\frac{e^{-\lambda\, t}(\lambda\, t)^n}{n!} \mathbb{E}_Y \left[C^E(S_0\, Y^n e^{-\lambda(\mathbb{E}_Y[Y]-1)T}, 0; T, K) \right] \right].$$

We can verify

$$\begin{aligned} \mathbb{E}^{\star}_{N,Y}\left[C^E(\hat{S}, 0; T, K) \right] &\geq C^E(\mathbb{E}^{\star}_{N,Y}[\hat{S}], 0; T, K) \\ &= C^E(S_0, 0; T, K). \end{aligned}$$

Note that we would get strict inequality if $\mathbb{P}(Y_k = 1) \neq 1$.

Ito's lemma with jumps

Consider a process X_t for $t > t_0$ of the form

$$X_t = X_0 + \int_{t_0}^{t} b(u, X_{u-})du + \int_{t_0}^{t} \sigma(u, X_{u-})\, dW_u + \sum_{n}^{N(t)} \Delta X_n\,, \qquad (109)$$

where $\Delta X_n = X_{\tau_n} - X_{\tau_n -}$ and τ_n denotes the jump times of the Poisson process. Here (I am being very loose with notation and conditions) the minus sign is there to denote that the variable is right before a jump occurs (if it occurs of course).
Now we want a more formal statement of Ito's lemma with Poisson jumps. Thus, assume that $f(t, X)$ is $C^{1,2}$. Then df is given by

$$\begin{aligned} df &= \left(f_t(t, X_{t-}) + f_X(t, X_{t-}) b(t, X_{t-}) + \frac{1}{2}\, f_{XX}(t, X_{t-})\, \sigma^2(t, X_{t-}) \right)\, dt \\ &\quad + f_X(t, X_{t-})\, \sigma(t, X_{t-})\, dW_t \\ &\quad + \left(f(t, X_{t-} + \Delta X_t) - f(t, X_{t-}) \right)\, dN_t \end{aligned}$$

and recall that $\Delta X_t = X_t - X_{t-}$.

Jump-Diffusion: option prices

Assume that the stock price S satisfies the dynamics

$$dS_t = \tilde{\mu}\, S_t\, dt + \sigma\, S_t\, dW_t + S\,(Y_{N_t} - 1)\, dN_t\,,$$

where W is Brownian motion, N is a Poisson process with intensity λ, Y is a random variable (W, N, J are independent) and $\tilde{\mu} = \mu - \lambda\, k$ where μ, and k are constants.

1. Easy to see that

$$\mathbb{E}[dS_t/S_t] = \mu\, dt \qquad \text{if and only if} \qquad k = \mathbb{E}[Y - 1]\,.$$

2. Integrate the SDE between 0 and t to write

$$S_t = S_0\, e^{\hat{\mu}\, t + \sigma\, W_t}\, \tilde{Y}(N_t)\,,$$

where $\hat{\mu} = \mu - \lambda\, k - \frac{\sigma^2}{2}$ and

$$\tilde{Y}(N_t) = \begin{cases} 1 & \text{if } N_t = 0\,, \\ \prod_{i=1}^{N_t} Y_i & \text{if } n \geq 1\,, \end{cases}$$

where the J_i are independently and identically distributed.

Can we hedge an option?

1. Choose the portfolio in the usual way,

$$\Pi(S,t) = V(S,t) - \Delta S_t \,,$$

2. Hedge the diffusion risk with

$$\Delta = V_S \,,$$

   but the jump risk still remains:

$$d\Pi = V_t\, dt + \frac{1}{2}\,\sigma^2\, S^2\, V_{SS}\, dt + \{V(Y\,S) - V(S) - \Delta(Y\,S - S)\}\, dN_t \,.$$

   Note that the S inside the braces $\{\ \}$ is the stock price before the jump, i.e. S_{t^-}.

3. We could also choose other traded assets written on S to hedge the different jump sizes. We could also decide not to hedge the diffusion risk and use S to hedge a jump of specific size.

1. To obtain a PDE for this problem, assume that $\mathbb{E}[d\Pi] = r\,\Pi\,dt$. Under which specific assumptions (relevant to this problem) might one assume this?
2. One may assume that the jumps are not correlated to the market portfolio hence their is no compensation for bearing jump risk. This is equivalent to assuming that $\mathbb{E}[Y] - 1 = k$. Thus,

$$\mathbb{E}[d\Pi] = \left(V_t + \frac{1}{2}\,\sigma^2\,S^2\,V_{SS} + \mathbb{E}\left[V(J\,S) - V(S)\right]\,\lambda - S\,V_S\,\lambda\,\mathbb{E}[J-1] \right)\,dt\,,$$

equate it to $r\,(V - S\,V_S)\,dt$ and write the PIDE

$$V_t + \frac{\sigma^2}{2}\,S^2\,V_{SS} + (r - \lambda\,k)\,S\,V_S - r\,V + \lambda\,\mathbb{E}\left[V(S\,J,t) - V(S,t)\right] = 0\,.$$

Note if there are no jumps (i.e., $\lambda = 0$) we obtain the Black–Scholes PDE.