

# B5.6 Nonlinear Dynamics, Bifurcations and Chaos

## Sheet 0 — HT 2025

Revision of some Prelims and Part A material, we will build on.

Please solve before the course starts. Do not hand in.

Solutions will be given in our first lecture in Week 1 (Wednesday, 22/01/2025).

1. Let matrix  $M \in \mathbb{R}^{3 \times 3}$  and vector  $\mathbf{x}_0 \in \mathbb{R}^3$  be given as

$$M = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_0 = \begin{pmatrix} 8 \\ 1 \\ 3 \end{pmatrix}$$

- (a) Let  $\mathbf{x}_k \in \mathbb{R}^3$ ,  $k = 0, 1, 2, \dots$ , be a sequence defined iteratively by  $\mathbf{x}_{k+1} = M\mathbf{x}_k$ . Find  $\mathbf{x}_k$ .

- (b) Find the solution of the system of ordinary differential equations (ODEs)

$$\frac{d\mathbf{x}}{dt} = M\mathbf{x} \quad \text{with the initial condition} \quad \mathbf{x}(0) = \mathbf{x}_0.$$

2. Let matrix  $M \in \mathbb{R}^{2 \times 2}$  be given as

$$M = \begin{pmatrix} 1 & 2 \\ -1 & \mu \end{pmatrix}.$$

- (a) Given  $\mathbf{x}_0 \in \mathbb{R}^2$ , consider sequence  $\mathbf{x}_k \in \mathbb{R}^2$ ,  $k = 0, 1, 2, \dots$ , defined iteratively by  $\mathbf{x}_{k+1} = M\mathbf{x}_k$ . Show that

$$\lim_{k \rightarrow \infty} \|\mathbf{x}_k\| = 0, \quad \text{for } \mu \in (-2, -1),$$

and

$$\lim_{k \rightarrow \infty} \|\mathbf{x}_k\| = \infty, \quad \text{for } \mu \in (-1, \infty).$$

- (b) Find and classify the critical point of the planar ODE system

$$\frac{d\mathbf{x}}{dt} = M\mathbf{x}$$

for each value of the parameter  $\mu \in \mathbb{R}$ .

3. Let  $\mu \in [0, 4]$  be a parameter and  $x_0 = 0.7$ . Define the sequence  $x_k \in \mathbb{R}$ ,  $k = 0, 1, 2, \dots$ , iteratively by

$$x_{k+1} = \mu x_k (1 - x_k).$$

Write a computer code which plots the first 200 values of this sequence for the following values of parameter  $\mu$ :

$$\mu = 0.5, \quad \mu = 2, \quad \mu = 3.2, \quad \mu = 3.5, \quad \mu = 3.55, \quad \mu = 3.8, \quad \mu = 4.$$

4. Let  $X$  be a continuous random variable on interval  $[0, 1]$  with the probability density function  $p: (0, 1) \rightarrow (0, \infty)$  given by

$$p(x) = \frac{1}{\pi \sqrt{x(1-x)}}.$$

Let function  $F: [0, 1] \rightarrow [0, 1]$  be defined by  $F(x) = 4x(1-x)$ .

Find the probability density function of random variable  $F(X)$ .

5. Let  $\mu \in \mathbb{R}$  be a parameter. Consider a planar autonomous system of ODEs given by:

$$\begin{aligned} \frac{dx}{dt} &= x - \mu y + y^2(1-x) - x^3 \\ \frac{dy}{dt} &= \mu x - xy(1+x) + y - y^3 \end{aligned} \quad (*)$$

- (a) Let  $\mu \in (-1, 1)$ . Show that the ODE system  $(*)$  has three critical points given by

$$[0, 0], \quad \left[ \sqrt{1-\mu^2}, \mu \right] \quad \left[ -\sqrt{1-\mu^2}, \mu \right]$$

Classify the critical points and sketch the phase plane.

- (b) Write a computer code which calculates five solutions of ODEs  $(*)$  for the initial conditions

$$[x_1(0), y_1(0)] = [-2, -2], \quad [x_2(0), y_2(0)] = [-2, 2], \quad [x_3(0), y_3(0)] = [2, -2],$$

$$[x_4(0), y_4(0)] = [2, 2] \quad \text{and} \quad [x_5(0), y_5(0)] = [0.1, 0.1].$$

Plot trajectories  $[x_j(t), y_j(t)]$ ,  $j = 1, 2, 3, 4, 5$ , in the phase plane for

$$\mu = \frac{9}{10} \quad \text{and} \quad \mu = 2.$$

- (c) Transform the ODE system  $(*)$  to polar coordinates, *i.e.* write a system of ODEs for variables  $r(t)$  and  $\theta(t)$ , where

$$x(t) = r(t) \cos \theta(t) \quad \text{and} \quad y(t) = r(t) \sin \theta(t).$$

- (d) Show that the ODE system  $(*)$  has a periodic solution (limit cycle) for all values of parameter  $\mu$  satisfying  $|\mu| > 1$ .