B5.6: Nonlinear Dynamics, Bifurcations and Chaos

- Lecturer: Radek Erban
- Lectures: Wednesdays 11am (lecture room L1), Thursdays 2pm (lecture room L2)
- Prerequisites: This course builds on ten Prelims and Part A courses. Students taking this course should have mastered the material in Part A courses on Differential Equations and Complex Analysis, and Prelims courses covering Probability, Computational Mathematics, Introductory Calculus, Multivariable Calculus, Fourier Series and PDEs, Geometry, Dynamics and Constructive Mathematics.
- Problem Sheet 0: You were asked to solve it before our first lecture.
  The solutions to Problem Sheet 0 will be provided in our first lecture (today).
- Classes: The course is accompanied by four Problem Sheets (labelled 1, 2, 3, and 4), which will be discussed in your classes, and by Problem Sheet 0.

  Three classes, covering Problem Sheets 1, 2 and 3, are scheduled in Hilary Term in Weeks 4, 6 and 8. Your last class will be in Trinity Term and will cover Problem Sheet 4, your vacation work.

MSc Students: A2 Mathematical Methods II

- The first 8 lectures of this course is part of the core syllabus for the MSc in Mathematical Modelling and Scientific Computing A2 Mathematical Methods.
- Course synopsis is divided into two parts **Lectures 1-8** and **Lectures 9-16**
- **Lectures 1-8:**
  Discrete-time (maps) and continuous-time (differential equations) dynamical systems. Notion of flows, stability of fixed points, Lyapunov function, invariant manifolds, stable manifold theorem, notion of hyperbolicity, center manifold. Chemical reaction networks. Stable, unstable and center subspaces. Poincaré-Bendixson theorem. Periodic solutions, stable and unstable limit cycles. Introduction to bifurcation theory, covering saddle-node, transcritical, supercritical pitchfork and subcritical pitchfork bifurcations. Extended center manifold. Logistic map. Periodic points of maps. Stability of N-cycles. Period-doubling bifurcation. Sharkovsky's theorem. Invariant distribution.

Part B examination, MSc examination and exam preparation

- Past Part B and MSc exam papers are available here:

  www.maths.ox.ac.uk/members/students/undergraduate-courses/examinations-assessments/past-papers

  www.maths.ox.ac.uk/members/students/postgraduate-courses/msc-mmsc/past-papers

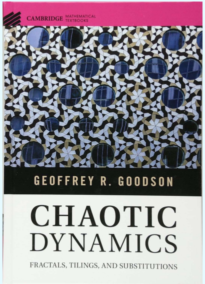
  Please note that this course was called B5.6 *Nonlinear Systems* in some previous years. It was renamed to B5.6 *Nonlinear Dynamics, Bifurcations and Chaos* to give a clearer sense of what the course covers.

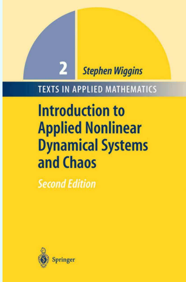
- First Notice to Candidates: "It must be stressed that to preserve the independence of the examiners, candidates are not allowed to make contact directly about matters relating to the content or marking of papers."

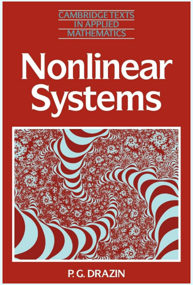
- 2024 Examiner's Report: "Most of the candidates demonstrated good understanding of the bookwork material. The exam was well-balanced with all three questions having the similar level of difficulty. While some candidates submitted some incomplete solutions, they often achieved at least 40% of raw marks. Other candidates also made successful attempts at more advanced parts of each question. In fact, each question received one complete solution (getting the perfect raw mark of 25), illustrating the solvability of each question under the exam conditions."

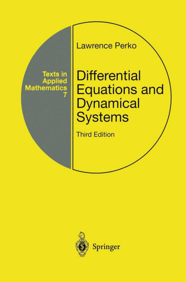
Reading List and Lecture Notes

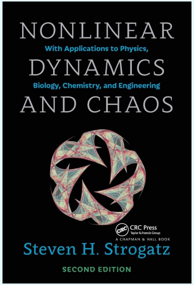
The B5.6 course material can be introduced with different levels of mathematical rigour, ranging from the 'definition-theorem-proof approach' to an example-based course covering dynamical systems appearing in applications. There are 6 books in the Reading List:

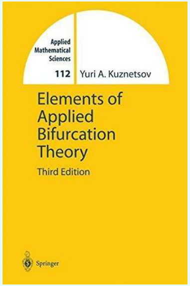












Students interested in building further theory with more proofs could like [Wiggins] or [Perko], or [Kuznetsov] (for bifurcations), or [Goodson] (for maps), while [Drazin] or [Strogatz] could be more appreciated by students interested in applications.

My slides will be uploaded to the course website after each lecture.

Introduction: main questions of course B5.6

Discrete-time dynamical system: Let $\mathbf{F} : \Omega \times \Theta \to \Omega$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}_k \in \Omega$ be defined iteratively by

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Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu}) \quad \text{with the initial condition} \quad \mathbf{x}(0) = \mathbf{x}_0$$

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We want to find $\mathbf{x}$ as a function of t and sketch the phase plane or phase space.
What is the behaviour of $\mathbf{x}(t)$ as $t \to \infty$?
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Linear example (Question 1(a) on Problem Sheet 0):

$$\mathbf{x}_{k+1} = M\mathbf{x}_k \qquad \text{for} \qquad M = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{x}_0 = \begin{pmatrix} 8 \\ 1 \\ 3 \end{pmatrix}$$

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Closed form formula for solutions [Prelims Probability and Calculus courses]:

$$\mathbf{x}_k = 3^k \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} + (-2)^k \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} + 2^k \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$$

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Linear example (Question 2(a) on Problem Sheet 0): $\mathbf{x}_{k+1} = M\mathbf{x}_k$ for $M = \begin{pmatrix} 1 & 2 \\ -1 & \mu \end{pmatrix}$

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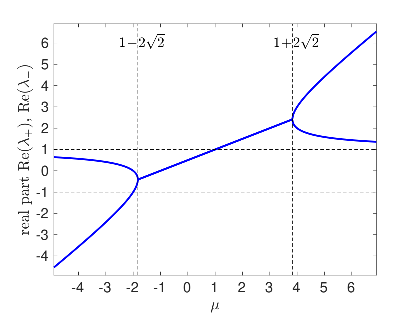

eigenvalues of M are

$$\lambda_\pm = \frac{1 + \mu \pm \sqrt{\left(\mu - 1 + 2\sqrt{2}\right)\left(\mu - 1 - 2\sqrt{2}\right)}}{2}$$

general solution $\lambda_+^k \mathbf{v}_+ + \lambda_-^k \mathbf{v}_-$

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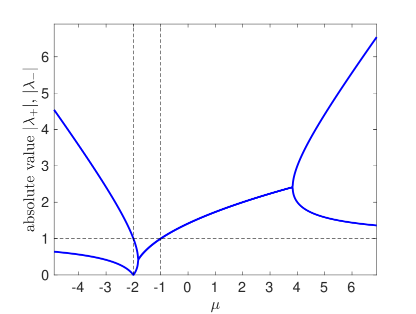

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$$\lim_{k\to\infty} \|\mathbf{x}_k\| = \infty \text{ for } \mu \in (-1, \infty)$$

$$\lim_{k\to\infty} \|\mathbf{x}_k\| = 0 \text{ for } \mu \in (-2, -1)$$

Nonlinear example: logistic map

Discrete-time dynamical system: Let $F : \Omega \times \Theta \to \Omega$, where $\Omega = [0, 1]$, $\Theta = [0, 4]$ and $F(x, \mu) = \mu\, x\,(1 - x)$. Let $x_0 = 0.7 \in \Omega$, $\mu \in \Theta$ and $x_k \in [0, 1]$, $k = 0, 1, 2, \ldots,$ be defined iteratively by $x_{k+1} = F(x_k; \mu)$, *i.e.*

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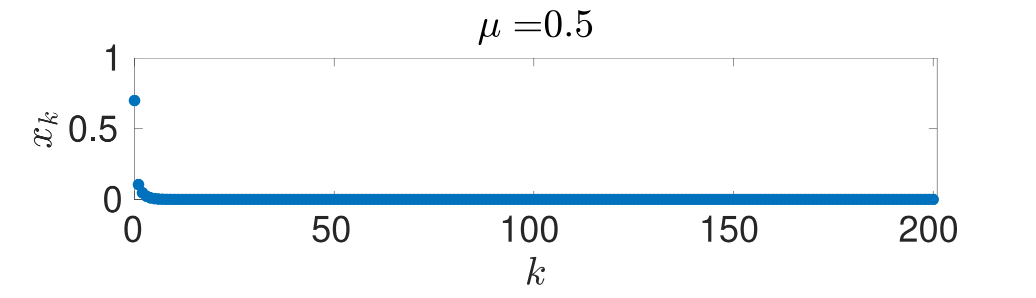

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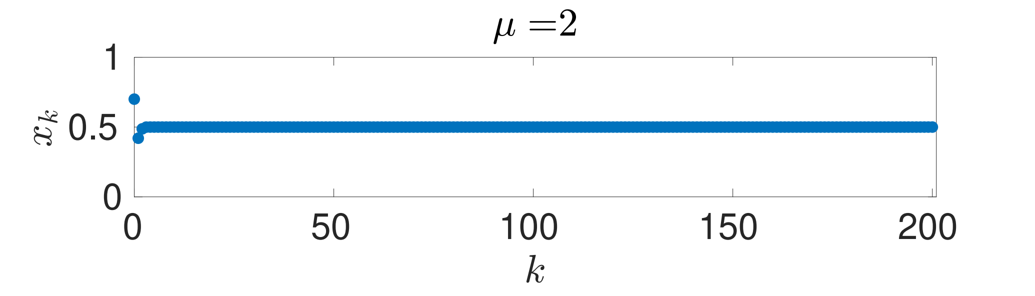

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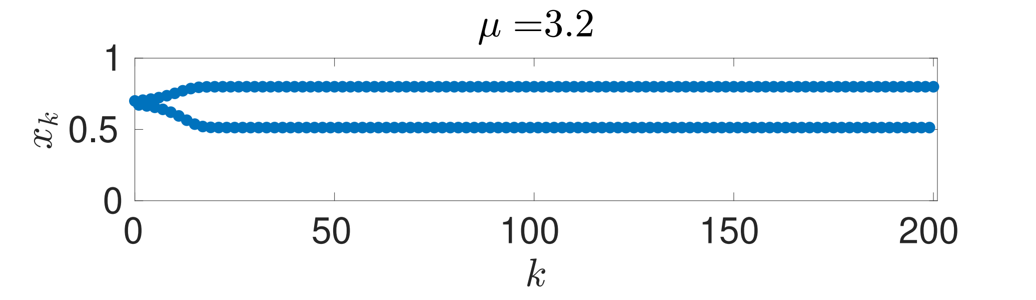

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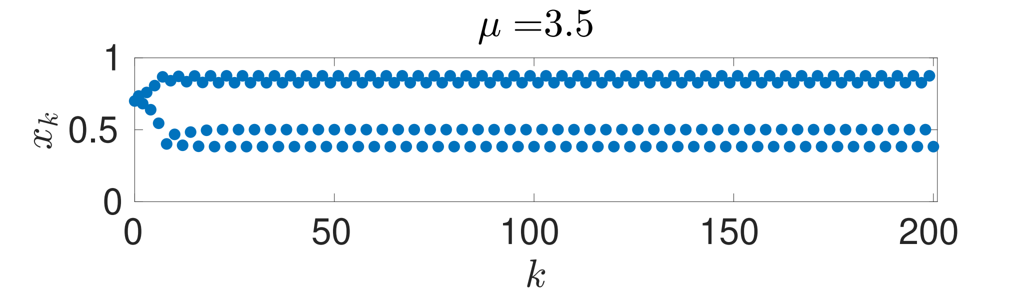

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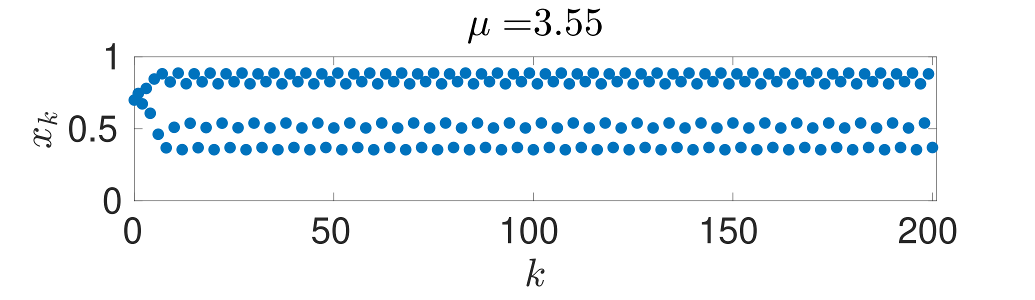

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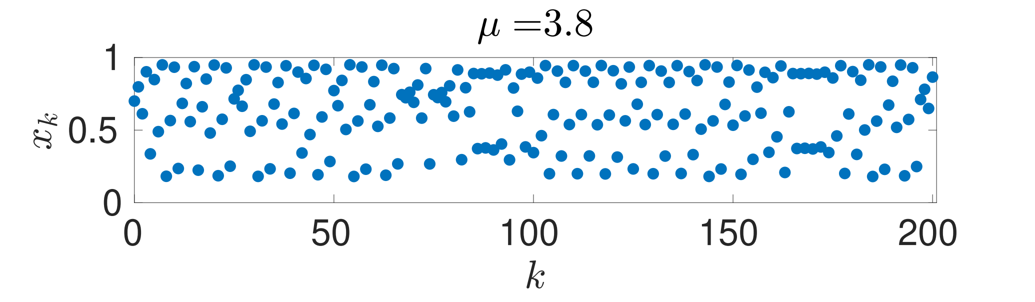

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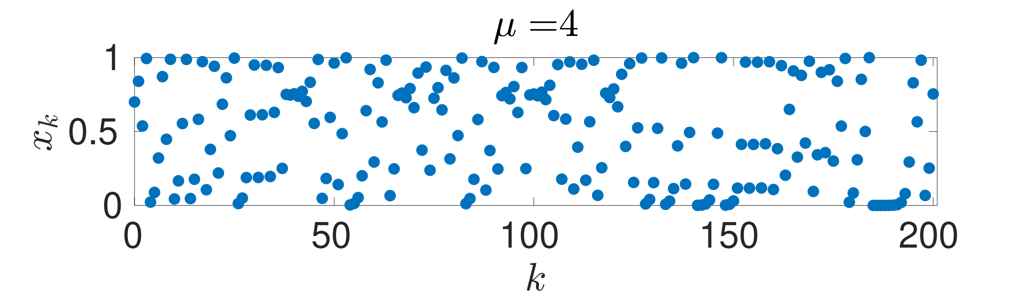

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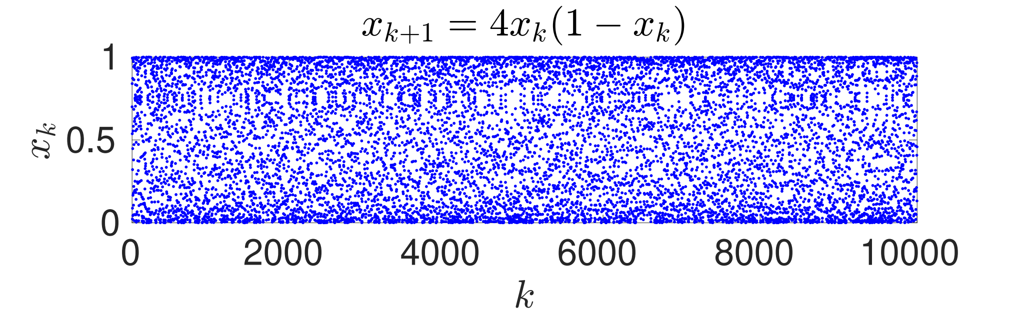

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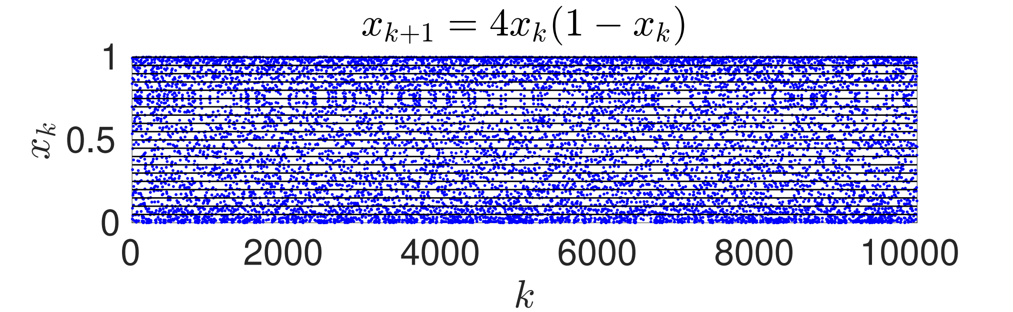
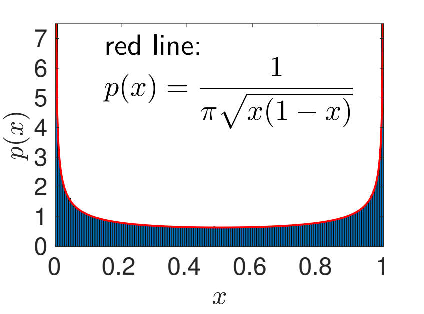


Nonlinear example: logistic map

Discrete-time dynamical system: Let $F : \Omega \times \Theta \to \Omega$, where $\Omega = [0,1]$, $\Theta = [0,4]$ and $F(x,\mu) = \mu\, x\,(1-x)$. Let $x_0 = 0.7 \in \Omega$, $\mu \in \Theta$ and $x_k \in [0,1]$, $k = 0,1,2,\ldots,$ be defined iteratively by $x_{k+1} = F(x_k;\mu)$, *i.e.*

$$x_{k+1} = \mu\, x_k\,(1-x_k)$$

Histogram of values x_k, for $k = 0,1,2,\ldots,10^6$ (blue bars): $x_{k+1} = 4\,x_k\,(1-x_k)$



Problem Sheet 0 Question 4:

Let X_k be a continuous random variable on interval $[0,1]$ with the probability density function $p(x)$. Then the random variable $X_{k+1} = F(X_k) = 4\,X_k\,(1-X_k)$ has the same probability density function $p(x)$.

[Prelims Probability and Calculus]

Prelims Probability and Calculus: Problem Sheet 0 Question 4

Let X be a continuous random variable on interval $[0,1]$ with the probability density function $p:[0,1]\to[0,\infty)$ given by $p(x)=1/(\pi\sqrt{x(1-x)})$. Let $F:[0,1]\to[0,1]$ be defined by $F(x)=4\,x\,(1-x)$. Then the cummulative distribution function of $F(X)$ is

$$\begin{aligned}\mathbb{P}(F(X)<x) &= \mathbb{P}\left(X<\frac{1}{2}\left(1-\sqrt{1-x}\right)\right)+\mathbb{P}\left(X>\frac{1}{2}\left(1+\sqrt{1-x}\right)\right)\\ &= \int_0^{\frac{1}{2}\left(1-\sqrt{1-x}\right)} p(z)\,\mathsf{d}z+\int_{\frac{1}{2}\left(1+\sqrt{1-x}\right)}^{1} p(z)\,\mathsf{d}z\end{aligned}$$

Prelims Probability and Calculus: Problem Sheet 0 Question 4

Let X be a continuous random variable on interval $[0,1]$ with the probability density function $p:[0,1]\to[0,\infty)$ given by $p(x)=1/(\pi\sqrt{x(1-x)})$. Let $F:[0,1]\to[0,1]$ be defined by $F(x)=4\,x\,(1-x)$. Then the cummulative distribution function of $F(X)$ is

$$
\begin{aligned}
\mathbb{P}(F(X)<x) &= \mathbb{P}\left(X<\frac{1}{2}\left(1-\sqrt{1-x}\right)\right)+\mathbb{P}\left(X>\frac{1}{2}\left(1+\sqrt{1-x}\right)\right) \\
&= \int_0^{\frac{1}{2}\left(1-\sqrt{1-x}\right)} p(z)\,\mathsf{d}z+\int_{\frac{1}{2}\left(1+\sqrt{1-x}\right)}^{1} p(z)\,\mathsf{d}z \\
&= 1+\frac{2}{\pi}\left(\sin^{-1}\sqrt{\frac{1}{2}\left(1-\sqrt{1-x}\right)}-\sin^{-1}\sqrt{\frac{1}{2}\left(1+\sqrt{1-x}\right)}\right) \\
&= 1-\frac{2}{\pi}\sin^{-1}\left(\sqrt{1-x}\right) \qquad \text{for} \quad x\in[0,1]
\end{aligned}
$$

Prelims Probability and Calculus: Problem Sheet 0 Question 4

Let X be a continuous random variable on interval $[0,1]$ with the probability density function $p:[0,1]\to[0,\infty)$ given by $p(x)=1/(\pi\sqrt{x(1-x)})$. Let $F:[0,1]\to[0,1]$ be defined by $F(x)=4\,x\,(1-x)$. Then the cummulative distribution function of $F(X)$ is

$$
\begin{aligned}
\mathbb{P}(F(X)<x) &= \mathbb{P}\left(X<\frac{1}{2}\left(1-\sqrt{1-x}\right)\right)+\mathbb{P}\left(X>\frac{1}{2}\left(1+\sqrt{1-x}\right)\right)\\
&= \int_0^{\frac{1}{2}\left(1-\sqrt{1-x}\right)} p(z)\,\mathsf{d}z+\int_{\frac{1}{2}\left(1+\sqrt{1-x}\right)}^{1} p(z)\,\mathsf{d}z\\
&= 1+\frac{2}{\pi}\left(\sin^{-1}\sqrt{\frac{1}{2}\left(1-\sqrt{1-x}\right)}-\sin^{-1}\sqrt{\frac{1}{2}\left(1+\sqrt{1-x}\right)}\right)\\
&= 1-\frac{2}{\pi}\sin^{-1}\left(\sqrt{1-x}\right) \qquad \text{for} \quad x\in[0,1]
\end{aligned}
$$

Consequently, the probability density function of $F(X)$ is:

$$
\frac{\mathsf{d}}{\mathsf{d}x}\,\mathbb{P}(F(X)<x)=-\frac{2}{\pi}\,\frac{\mathsf{d}}{\mathsf{d}x}\,\sin^{-1}\left(\sqrt{1-x}\right)=\frac{1}{\pi\sqrt{x(1-x)}}=p(x)
$$

B5.6 covers nonlinear dynamics (linear systems were in Prelims/Part A)

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$.
Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu}) \quad \text{with the initial condition} \quad \mathbf{x}(0) = \mathbf{x}_0$$

We want to find $\mathbf{x}$ as a function of t and sketch the phase plane or phase space.
What is the behaviour of $\mathbf{x}(t)$ as $t \to \infty$?
How do our answers depend on the initial value $\mathbf{x}_0$?
How does the behaviour of $\mathbf{x}(t)$ depend on parameters $\boldsymbol{\mu}$?

B5.6 covers nonlinear dynamics (linear systems were in Prelims/Part A)

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu}) \quad \text{with the initial condition} \quad \mathbf{x}(0) = \mathbf{x}_0$$

Linear example (Question 1(b) on Problem Sheet 0):

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x} \qquad \text{for} \qquad M = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{x}(0) = \begin{pmatrix} 8 \\ 1 \\ 3 \end{pmatrix}$$

B5.6 covers nonlinear dynamics (linear systems were in Prelims/Part A)

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

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Closed form solution formula [Prelims Calculus and Part A Differential Equations courses]:

$$\mathbf{x}(t) = e^{3t} \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} + e^{-2t} \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} + e^{2t} \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$$

B5.6 covers nonlinear dynamics (linear systems were in Prelims/Part A)

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$.
Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu}) \quad \text{with the initial condition} \quad \mathbf{x}(0) = \mathbf{x}_0$$

Linear example (Question 2(b) on Problem Sheet 0): $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$ for $M = \begin{pmatrix} 1 & 2 \\ -1 & \mu \end{pmatrix}$

B5.6 covers nonlinear dynamics (linear systems were in Prelims/Part A)

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$.
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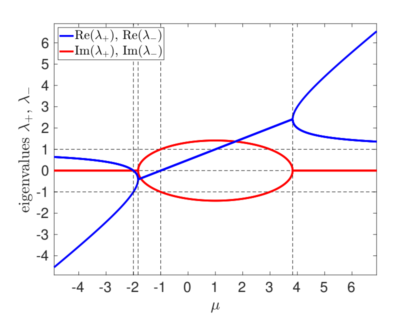

eigenvalues of M are

$$\lambda_\pm = \frac{1 + \mu \pm \sqrt{\left(\mu - 1 + 2\sqrt{2}\right)\left(\mu - 1 - 2\sqrt{2}\right)}}{2}$$

$[0, 0]$ is the only critical point

[Part A Differential Equations 1]

B5.6 covers nonlinear dynamics (linear systems were in Prelims/Part A)

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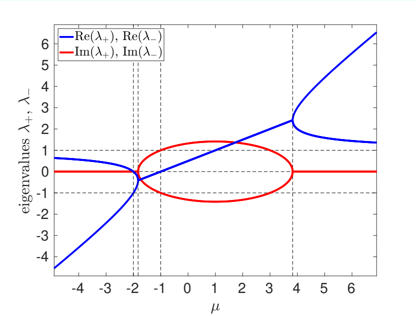

[Part A Differential Equations 1]

eigenvalues of M are

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$[0,0]$ is the only critical point which is

- saddle for $\mu < -2$
- stable node for $-2 < \mu < 1 - 2\sqrt{2}$
- stable spiral for $1 - 2\sqrt{2} < -1$
- unstable spiral for $-1 < \mu < 1 + 2\sqrt{2}$
- unstable node for $\mu > 1 + 2\sqrt{2}$

B5.6 covers nonlinear dynamics (linear systems were in Prelims/Part A)

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$.
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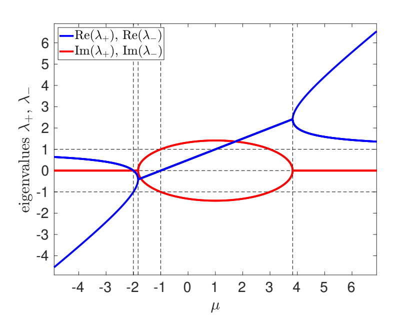
$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu}) \quad \text{with the initial condition} \quad \mathbf{x}(0) = \mathbf{x}_0$$

Linear example (Question 2(b) on Problem Sheet 0): $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$ for $M = \begin{pmatrix} 1 & 2 \\ -1 & \mu \end{pmatrix}$



[Part A Differential Equations 1]

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- saddle for $\mu < -2$
- stable node for $-2 < \mu < 1 - 2\sqrt{2}$
- stable spiral for $1 - 2\sqrt{2} < -1$
- unstable spiral for $-1 < \mu < 1 + 2\sqrt{2}$
- unstable node for $\mu > 1 + 2\sqrt{2}$

center for $\mu = -1$, stable/unstable inflected node for $\mu = 1 \pm 2\sqrt{2}$

Nonlinear example: Problem Sheet 0 Question 5

Let $\mu \in \mathbb{R}$ be a parameter. Consider a planar autonomous ODE system given by:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= x - \mu\, y + y^2(1-x) - x^3 \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \mu\, x - x\, y\,(1+x) + y - y^3\end{aligned}$$

Nonlinear example: Problem Sheet 0 Question 5

Let $\mu \in (-1, 1)$ be a parameter. Consider a planar autonomous ODE system given by:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - \mu\, y + y^2(1-x) - x^3$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \mu\, x - x\, y\,(1+x) + y - y^3$$

Part A Differential Equations 1: linearized system next to the critical point $[x_c, y_c]$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix} = M \begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix} \quad \text{where} \quad M = \begin{pmatrix} 1 - y_c^2 - 3x_c^2 & -\mu + 2\, y_c\,(1 - x_c) \\ \mu - y_c - 2\, x_c\, y_c & -x_c\,(1 + x_c) + 1 - 3\, y_c^2 \end{pmatrix}$$

$[0, 0]$: unstable spiral $\quad M = \begin{pmatrix} 1 & -\mu \\ \mu & 1 \end{pmatrix}$ $\quad$ eigenvalues: $1 \pm \mu i$

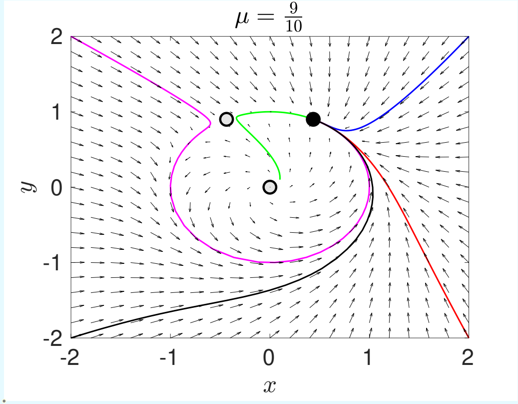
$\left[\sqrt{1-\mu^2}, \mu\right]$: stable node $\quad M = \begin{pmatrix} -2 + 2\mu^2 & \mu - 2\mu\sqrt{1-\mu^2} \\ -2\mu\sqrt{1-\mu^2} & -2\mu^2 - \sqrt{1-\mu^2} \end{pmatrix}$ eigenvalues: $-2, -\sqrt{1-\mu^2}$

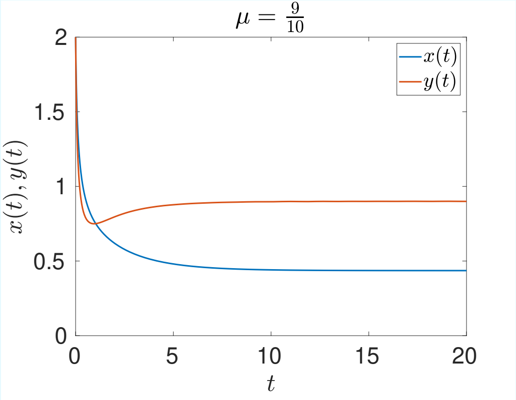
$\left[-\sqrt{1-\mu^2}, \mu\right]$: saddle $\quad M = \begin{pmatrix} -2 + 2\mu^2 & \mu + 2\mu\sqrt{1-\mu^2} \\ 2\mu\sqrt{1-\mu^2} & -2\mu^2 + \sqrt{1-\mu^2} \end{pmatrix}$ eigenvalues: $-2,\ \sqrt{1-\mu^2}$

Nonlinear example: Problem Sheet 0 Question 5

Let $\mu \in (-1, 1)$ be a parameter. Consider a planar autonomous ODE system given by:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x - \mu\, y + y^2(1 - x) - x^3 \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \mu\, x - x\, y\,(1 + x) + y - y^3 \end{aligned}$$

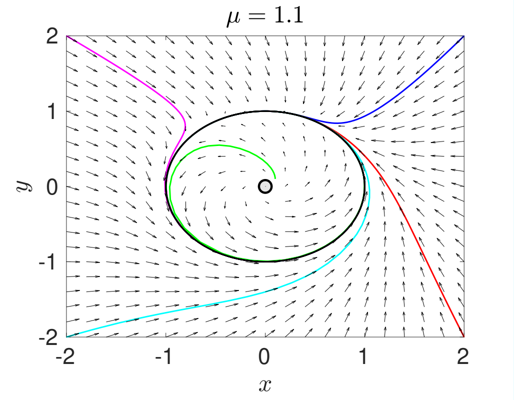


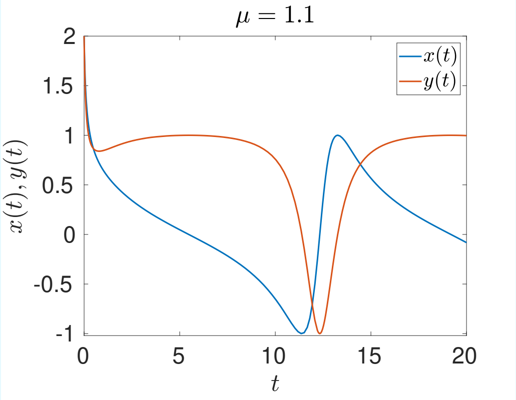


Nonlinear example: Problem Sheet 0 Question 5

Let $\mu > 1$ be a parameter. Consider a planar autonomous ODE system given by:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - \mu\, y + y^2(1-x) - x^3$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \mu\, x - x\, y\,(1+x) + y - y^3$$





Nonlinear example: Problem Sheet 0 Question 5

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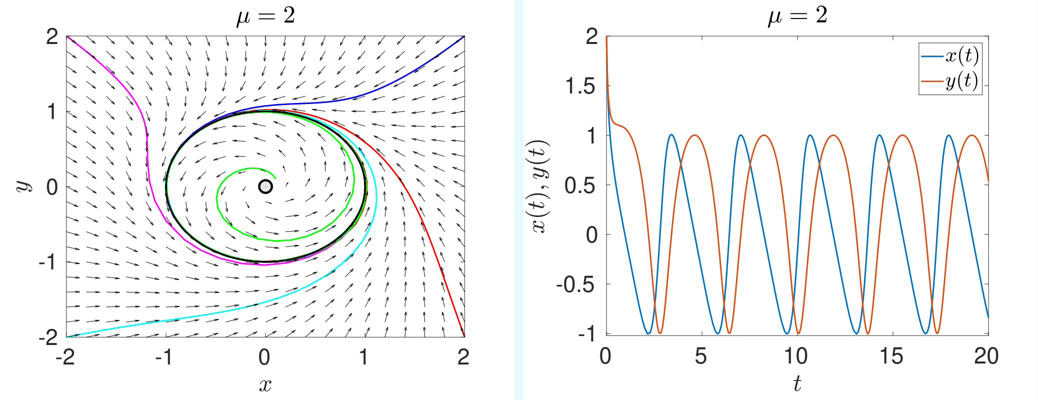

Nonlinear example: Problem Sheet 0 Question 5

Let $\mu \in \mathbb{R}$ be a parameter. Consider a planar autonomous ODE system given by:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x - \mu\, y + y^2(1-x) - x^3 \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \mu\, x - x\, y\,(1+x) + y - y^3 \end{aligned}$$

Prelims Calculus: We transform the ODEs to polar coordinates by using variables $r(t)$ and $\theta(t)$, where $x(t) = r(t)\cos\theta(t)$ and $y(t) = r(t)\sin\theta(t)$. We obtain

$$\frac{\mathrm{d}r}{\mathrm{d}t} = r(1-r^2)$$

We conclude that $r(t) \to 1$ as $t \to \infty$ for any initial condition satisfying $r(0) > 0$.

Nonlinear example: Problem Sheet 0 Question 5

Let $\mu \in \mathbb{R}$ be a parameter. Consider a planar autonomous ODE system given by:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= x - \mu\, y + y^2(1-x) - x^3\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \mu\, x - x\, y\,(1+x) + y - y^3\end{aligned}$$

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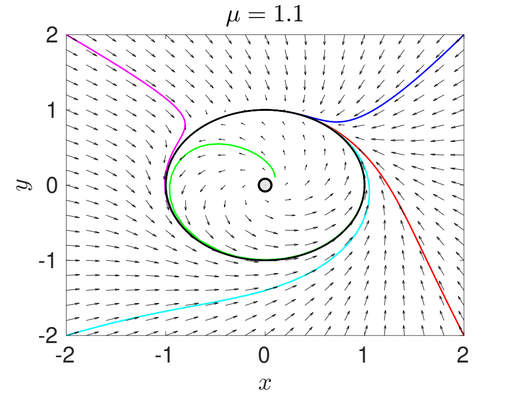
$$\frac{\mathrm{d}r}{\mathrm{d}t} = r(1-r^2)$$

We conclude that $r(t) \to 1$ as $t \to \infty$ for any initial condition satisfying $r(0) > 0$.

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \mu - y = \mu - r\sin(\theta)$$

If $\mu > 1$, then $\mathrm{d}\theta/\mathrm{d}t > \mu - 1 > 0$.



Nonlinear example: Problem Sheet 0 Question 5

Let $\mu \in \mathbb{R}$ be a parameter. Consider a planar autonomous ODE system given by:

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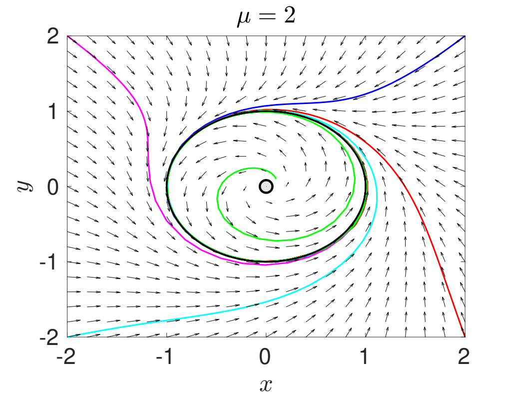
If $\mu > 1$, then $\mathrm{d}\theta/\mathrm{d}t > \mu - 1 > 0$.



Nonlinear example: Problem Sheet 0 Question 5

Let $\mu \in \mathbb{R}$ be a parameter. Consider a planar autonomous ODE system given by:

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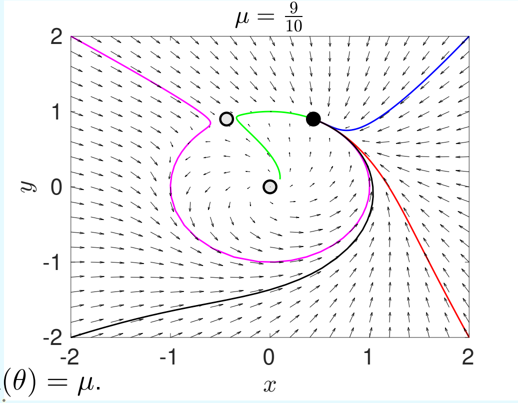
$$\frac{\mathrm{d}r}{\mathrm{d}t} = r(1-r^2)$$

We conclude that $r(t) \to 1$ as $t \to \infty$ for any initial condition satisfying $r(0) > 0$.

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \mu - y = \mu - r\sin(\theta)$$



If $|\mu| < 1$, then $\mathrm{d}\theta/\mathrm{d}t = 0$ for $r = 1$ and $\sin(\theta) = \mu$.

ODEs and Chaos

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

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ODEs and Chaos

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- If $n = 2$ and $\mathbf{x}(t)$ is bounded, then $\mathbf{x}(t)$ can either be (i) equal to a critical point or to a periodic solution; or (ii) it will converge to a critical point or to a periodic solution.

ODEs and Chaos

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- If $n = 2$ and $\mathbf{x}(t)$ is bounded, then $\mathbf{x}(t)$ can either be (i) equal to a critical point or to a periodic solution; or (ii) it will converge to a critical point or to a periodic solution.
- If $n \geq 3$, then ODEs can also have chaotic solutions (our logistic map example shows that discrete-time dynamical systems can be chaotic even for $n = 1$).

ODEs and Chaos

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

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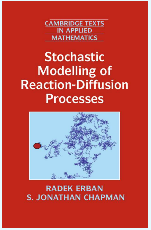
- If $n = 2$ and $\mathbf{x}(t)$ is bounded, then $\mathbf{x}(t)$ can either be (i) equal to a critical point or to a periodic solution; or (ii) it will converge to a critical point or to a periodic solution.
- If $n \geq 3$, then ODEs can also have chaotic solutions (our logistic map example shows that discrete-time dynamical systems can be chaotic even for $n = 1$).
- In course B5.6, we will focus on chaotic solutions of ODEs for $n = 3$, but chaos is common for $n \gg 3$. Examples are discussed in course B5.1 Stochastic Modelling of Biological Processes. [video of molecular dynamics simulation of ions in water]



ODEs and Chaos

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

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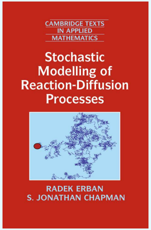

- B5.6: we will consider relatively simple ODEs (small n, polynomials): (1) good for developing general theory; (2) there are also interesting applications

Chemical reaction networks

Consider a well-stirred (well-mixed) chemical system with n chemical species X_1, X_2, ..., X_n which are subject to ℓ chemical reactions.

Let $x_i(t)$ be the concentration of the chemical species X_i, $i = 1, 2, \ldots, n$.

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Let $x_i(t)$ be the concentration of the chemical species X_i, $i = 1, 2, \ldots, n$.

The time evolution of concentration $x_1(t)$ is given by the ODE $\displaystyle \frac{\mathrm{d}x_1}{\mathrm{d}t} = \sum_{j=1}^{\ell} c_j \, r_j,$

where r_j is the rate of the j th reaction and c_j is the change in the number of molecules of X_1 corresponding to the occurrence of one j-th reaction, i.e. it is the difference between the number (stoichiometric coefficient) in front of X_1 on the right hand side of the reaction and the corresponding stoichiometric coefficient on the left hand side. The rate $r_j \equiv r_j(t)$ is computed as a product of the rate constant and the concentrations of the reactants (mass action kinetics).

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Example: system of $n = 3$ chemical species which are subject to $\ell = 5$ reactions:

$X_1 + X_2 \xrightarrow{k_1} X_3$	$X_3 \xrightarrow{k_2} 2X_1$	$2X_3 \xrightarrow{k_3} \emptyset$	$\emptyset \xrightarrow{k_4} X_1$	$X_1 + 2X_3 \xrightarrow{k_5} X_2$
$c_1 = -1$	$c_2 = 2$	$c_3 = 0$	$c_4 = 1$	$c_5 = -1$
$r_1 = k_1 \, x_1 \, x_2$	$r_2 = k_2 \, x_3$	$r_3 = k_3 \, x_3^2$	$r_4 = k_4$	$r_5 = k_5 \, x_1 \, x_3^2$

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -k_1 \, x_1 \, x_2 \; + \; 2 \, k_2 \, x_3 \; + \; k_4 \; - \; k_5 \, x_1 \, x_3^2$$

Chemical reaction networks

Consider a well-stirred (well-mixed) chemical system with n chemical species X_1, X_2, $\ldots$, X_n which are subject to ℓ chemical reactions.

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The time evolution of concentration $x_1(t)$ is given by the ODE $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \displaystyle\sum_{j=1}^{\ell} c_j \, r_j$,

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$c_1 = -1$	$c_2 = 2$	$c_3 = 0$	$c_4 = 1$	$c_5 = -1$
$r_1 = k_1 \, x_1 \, x_2$	$r_2 = k_2 \, x_3$	$r_3 = k_3 \, x_3^2$	$r_4 = k_4$	$r_5 = k_5 \, x_1 \, x_3^2$

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -k_1 \, x_1 \, x_2 \, + \, 2 \, k_2 \, x_3 \, + \, k_4 \, - \, k_5 \, x_1 \, x_3^2$$

The units of $x_i(t)$ are usually moles (or number of molecules) per unit of volume, k_1 and k_3 have units of $[\mathrm{m}^3 \, \mathrm{sec}^{-1}]$, k_2 is in $[\mathrm{sec}^{-1}]$, k_4 is in $[\mathrm{m}^{-3} \, \mathrm{sec}^{-1}]$ and k_5 is in $[\mathrm{m}^6 \, \mathrm{sec}^{-1}]$, but we will assume that $x_i(t)$ and all parameters are dimensionless.

Chemical reaction networks

Consider a well-stirred (well-mixed) chemical system with n chemical species X_1, X_2, $\ldots$, X_n which are subject to ℓ chemical reactions.

Let $x_i(t)$ be the concentration of the chemical species X_i, $i = 1, 2, \ldots, n$.

The time evolution of concentration $x_2(t)$ is given by the ODE $\dfrac{\mathrm{d}x_2}{\mathrm{d}t} = \sum_{j=1}^{\ell} c_j \, r_j,$

Example: system of $n = 3$ chemical species which are subject to $\ell = 5$ reactions:

$X_1 + X_2 \xrightarrow{k_1} X_3$	$X_3 \xrightarrow{k_2} 2X_1$	$2X_3 \xrightarrow{k_3} \emptyset$	$\emptyset \xrightarrow{k_4} X_1$	$X_1 + 2X_3 \xrightarrow{k_5} X_2$
$c_1 = -1$	$c_2 = 0$	$c_3 = 0$	$c_4 = 0$	$c_5 = 1$
$r_1 = k_1 \, x_1 \, x_2$	$r_2 = k_2 \, x_3$	$r_3 = k_3 \, x_3^2$	$r_4 = k_4$	$r_5 = k_5 \, x_1 \, x_3^2$

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -k_1 \, x_1 \, x_2 \, + \, 2 \, k_2 \, x_3 \, + \, k_4 \, - \, k_5 \, x_1 \, x_3^2$$

similary for $x_2(t)$: $\dfrac{\mathrm{d}x_2}{\mathrm{d}t} = -k_1 \, x_1 \, x_2 \, + \, k_5 \, x_1 \, x_3^2$

Chemical reaction networks

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Let $x_i(t)$ be the concentration of the chemical species X_i, $i = 1, 2, \ldots, n$.

The time evolution of concentration $x_3(t)$ is given by the ODE $\dfrac{\mathrm{d}x_3}{\mathrm{d}t} = \displaystyle\sum_{j=1}^{\ell} c_j \, r_j,$

Example: system of $n = 3$ chemical species which are subject to $\ell = 5$ reactions:

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$$c_1 = 1 \qquad c_2 = -1 \qquad c_3 = -2 \qquad c_4 = 0 \qquad c_5 = -2$$

$$r_1 = k_1 \, x_1 \, x_2 \qquad r_2 = k_2 \, x_3 \qquad r_3 = k_3 \, x_3^2 \qquad r_4 = k_4 \qquad r_5 = k_5 \, x_1 \, x_3^2$$

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similary for $x_2(t)$: $$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -k_1 \, x_1 \, x_2 \; + \; k_5 \, x_1 \, x_3^2$$

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Chemical reaction networks

Consider a well-stirred (well-mixed) chemical system with n chemical species X_1, X_2, $\ldots$, X_n which are subject to ℓ chemical reactions.

Let $x_i(t)$ be the concentration of the chemical species X_i, $i = 1, 2, \ldots, n$.

The time evolution of concentration $x_1(t)$ is given by the ODE $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \displaystyle\sum_{j=1}^{\ell} c_j \, r_j,$

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other examples: Questions 3, 4 and 6 on Problem Sheet 1

Course B5.6: ODEs with relatively small n and simple right hand sides (often polynomials). They appear in applications as (i) models of (bio)chemical systems; or (ii) they can also be constructed in experiments (synthetic biology, DNA computing).

Polynomials can also approximate more complicated right hand sides of ODEs (stable manifold, center manifold, bifurcations). Let us go back to some theory.

B5.6: Nonlinear Dynamics, Bifurcations and Chaos (Lecture 2)

- summary of Lecture 1: we discussed
  Discrete-time (maps) and continuous-time (differential equations) dynamical systems.
  Chemical reaction networks. (Questions 3, 4 and 6 on Problem Sheet 1)

- today: we will continue in our discussion of Problem Sheet 1

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- course synopsis of **Lectures 1-8** (taken by both Part B and MSc students):
  Discrete-time (maps) and continuous-time (differential equations) dynamical systems. Notion of flows, stability of fixed points, Lyapunov function, invariant manifolds, stable manifold theorem, notion of hyperbolicity, center manifold. Chemical reaction networks. Stable, unstable and center subspaces. Poincaré-Bendixson theorem. Periodic solutions, stable and unstable limit cycles. Introduction to bifurcation theory, covering saddle-node, transcritical, supercritical pitchfork and subcritical pitchfork bifurcations. Extended center manifold. Logistic map. Periodic points of maps. Stability of N-cycles. Period-doubling bifurcation. Sharkovsky's theorem. Invariant distribution.

The flow defined by an ODE

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$.
Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu}) \quad \text{with the initial condition} \quad \mathbf{x}(0) = \mathbf{x}_0$$

Then we define the *flow* $\phi_t : \Omega \to \Omega$ by

$$\phi_t(\mathbf{x}_0) = \phi(t, \mathbf{x}_0) = \mathbf{x}(t)$$

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Example: Question 1(b) on Problem Sheet 0 for general initial condition $\mathbf{x}_0 \in \mathbb{R}^3$:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x} \qquad \text{for} \qquad M = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & 2 \end{pmatrix}$$

Then

$$\phi_t(\mathbf{x}_0) = \phi(t, \mathbf{x}_0) = \exp[Mt]\,\mathbf{x}_0 = \left(\sum_{j=0}^{\infty} \frac{M^j t^j}{j!} \right) \mathbf{x}_0$$

where we have used the definition of the matrix exponential: $\exp[A] = \displaystyle\sum_{j=0}^{\infty} \frac{A^j}{j!}$

The flow defined by an ODE

Considering the linear system of ODEs given by $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$ where matrix $M \in \mathbb{R}^{n\times n}$, the flow $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ is given by $\phi_t = \exp[Mt]$.

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In particular, the properties of the matrix exponential imply that the flow ϕ_t satisfies

(a) $\phi_0 = I$ (b) $\phi_s \circ \phi_t = \phi_{s+t}$ (c) $\phi_t \circ \phi_{-t} = \phi_{-t} \circ \phi_t = I$

where $I \in \mathbb{R}^{n\times n}$ is the identity matrix.

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For linear systems, the properties (a)–(c) mean:

(a) $\phi_0(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$;
(b) $\phi_s(\phi_t(\mathbf{x})) = \phi_{s+t}(\mathbf{x}))$ for all $s, t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$;
(c) $\phi_t(\phi_{-t}(\mathbf{x})) = \phi_{-t}(\phi_t(\mathbf{x})) = \mathbf{x}$ for all $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$.

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Nonlinear ODE system: $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$

Part A Differential Equations 1: Picard's existence theorem implies the global existence and uniqueness of solutions for $\mathbf{f} \in C^1(\mathbb{R}^n \times \mathbb{R}^m)$ which satisfies the global Lipschitz condition $|\mathbf{f}(\mathbf{x}; \boldsymbol{\mu}) - \mathbf{f}(\mathbf{y}; \boldsymbol{\mu})| \le C|\mathbf{x} - \mathbf{y}|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\boldsymbol{\mu} \in \mathbb{R}^m$.
Then $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ is defined for all $t \in \mathbb{R}$ and ϕ_t satisfies the properties (a)–(c).

The flow defined by an ODE

Considering the linear system of ODEs given by $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$ where matrix $M \in \mathbb{R}^{n\times n}$, the flow $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ is given by $\phi_t = \exp[Mt]$.

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where $I \in \mathbb{R}^{n\times n}$ is the identity matrix.

Note: Assuming the global Lipschitz condition could exclude some interesting ODEs. Our assumptions on $\Omega \subset \mathbb{R}^n$, $\Theta \subset \mathbb{R}^m$ and $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$ could be relaxed. In some cases, we would only get the local existence of solutions to the nonlinear ODE system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$$

ϕ_t would not be defined for all $t \in \mathbb{R}$ and ϕ_t would only satisfy properties (a)–(c) where it is defined.

Let us illustrate this with an example with $n = 1$.

The flow defined by an ODE: nonlinear example

Consider the ODE $\dfrac{\mathsf{d}x}{\mathsf{d}t} = x^2$ (it does not satisfy the global Lipschitz condition).

The flow defined by an ODE: nonlinear example

Consider the ODE $\dfrac{\mathrm{d}x}{\mathrm{d}t} = x^2$ (it does not satisfy the global Lipschitz condition).

Given the initial condition $x(0) = x_0 \in \mathbb{R}$, we can solve this ODE to obtain

$$x(t) = \frac{x_0}{1 - t\,x_0} \qquad \text{for} \quad t \in I(x_0),$$

where $I(x_0)$ is the *maximal interval of existence* given by $I(0) = \mathbb{R}$,

$$I(x) = \left(-\infty, \frac{1}{x}\right) \quad \text{for } x > 0, \qquad \text{and} \qquad I(x) = \left(\frac{1}{x}, \infty\right) \quad \text{for } x < 0.$$

The flow defined by an ODE: nonlinear example

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In particular, the flow ϕ_t is defined as the mapping $\phi : Q \to \mathbb{R}$, where

$$Q = \{(t,x) \,|\, x \in \mathbb{R} \text{ and } t \in I(x)\} \qquad \text{and} \qquad \phi_t(x) = \phi(t,x) = \frac{x}{1 - tx}.$$

Problem Sheet 1 Question 7: We can rescale time to get a topologically equivalent ODE system which has $I(x) = \mathbb{R}$.

In general, the time along trajectories can be rescaled without affecting the phase portrait. In what follows, we will assume that ϕ_t is defined for all $t \in \mathbb{R}$ and $\phi \in C^1(\mathbb{R} \times \Omega)$ for any considered parameter values $\boldsymbol{\mu} \in \Theta$.

Equilibrium points, flow, trajectory - summary

Given $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$, $\Theta \subset \mathbb{R}^m$, and $\boldsymbol{\mu} \in \Theta$, we consider ODE system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x};\boldsymbol{\mu})$$

- $\mathbf{x}_c$ is an *equilibrium point* or *critical point* or *fixed point* if $\mathbf{f}(\mathbf{x}_c;\boldsymbol{\mu}) = 0$
- the *flow* of the ODE is the map $\phi_t : \Omega \to \Omega$ such that $\phi_t(\mathbf{x}_0) = \phi(t, \mathbf{x}_0) = \mathbf{x}(t;\mathbf{x}_0)$, where $\mathbf{x}(t;\mathbf{x}_0) \in \Omega$ is the solution with the initial condition $\mathbf{x}(0) = \mathbf{x}_0 \in \Omega$
- an *orbit* or *trajectory* based on $\mathbf{x}_0$ is the curve $\Gamma_{\mathbf{x}_0} \subset \Omega$ defined by

$$\Gamma_{\mathbf{x}_0} = \left\{ \mathbf{x}(t;\mathbf{x}_0) \,\middle|\, t \in I(x_0) \right\} ,$$

  where $I(x_0)$ is the maximum interval of existence (WLOG we assume $I(x_0) = \mathbb{R}$)

- $S \subset \Omega$ is an *invariant set* if $\phi_t(S) \subset S$ for all $t \in \mathbb{R}$

Equilibrium points: stability

Given $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$, $\Theta \subset \mathbb{R}^m$, and $\boldsymbol{\mu} \in \Theta$, we consider ODE system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$$

- $\mathbf{x}_c$ is an *equilibrium point* or *critical point* or *fixed point* if $\mathbf{f}(\mathbf{x}_c; \boldsymbol{\mu}) = 0$
- $\mathbf{x}_c$ is *stable* if

  $\forall \varepsilon > 0\, \exists \delta > 0$ such that $\forall \mathbf{x}_0 \in B_\delta(\mathbf{x}_c)$ and $t \geq 0$ we have $\phi_t(\mathbf{x}) \in B_\varepsilon(\mathbf{x}_c)$

  where the open ball of radius r is defined by $B_r(\mathbf{x}_c) = \left\{ \mathbf{x} \in \mathbb{R}^n \,\middle|\, \|\mathbf{x} - \mathbf{x}_c\| < r \right\}$
- $\mathbf{x}_c$ is *asymptotically stable* if (i) it is stable; and

  (ii) $\exists \delta > 0$ such that $\phi_t(\mathbf{x}_0) \to \mathbf{x}_c$ for all $\mathbf{x}_0 \in B_\delta(\mathbf{x}_c)$

Equilibrium points: stability, Lyapunov function

Given $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$, $\Theta \subset \mathbb{R}^m$, and $\boldsymbol{\mu} \in \Theta$, we consider ODE system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$$

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  $\forall \varepsilon > 0\, \exists \delta > 0$ such that $\forall \mathbf{x}_0 \in B_\delta(\mathbf{x}_c)$ and $t \geq 0$ we have $\phi_t(\mathbf{x}) \in B_\varepsilon(\mathbf{x}_c)$

  where the open ball of radius r is defined by $B_r(\mathbf{x}_c) = \left\{ \mathbf{x} \in \mathbb{R}^n \,\middle|\, \|\mathbf{x} - \mathbf{x}_c\| < r \right\}$
- $\mathbf{x}_c$ is *asymptotically stable* if (i) it is stable; and

  (ii) $\exists \delta > 0$ such that $\phi_t(\mathbf{x}_0) \to \mathbf{x}_c$ for all $\mathbf{x}_0 \in B_\delta(\mathbf{x}_c)$
- Lyapunov function: $V \in C^1(A)$, where $A \subset \Omega \subset \mathbb{R}^n$ is open and $\mathbf{x}_c \in A$

  $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{x}_c$ and $V(\mathbf{x}_c) = 0$

  if $\mathrm{d}V/\mathrm{d}t \leq 0$ for all $\mathbf{x} \in A \setminus \{\mathbf{x}_c\}$, then $\mathbf{x}_c$ is stable

  if $\mathrm{d}V/\mathrm{d}t < 0$ for all $\mathbf{x} \in A \setminus \{\mathbf{x}_c\}$, then $\mathbf{x}_c$ is asymptotically stable

Problem Sheet 1 Question 5: proving stability by finding a suitable Lyapunov function

Equilibrium points: linearization

Given $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$, $\Theta \subset \mathbb{R}^m$, and $\boldsymbol{\mu} \in \Theta$, we consider ODE system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$$

- $\mathbf{x}_c$ is an *equilibrium point* or *critical point* or *fixed point* if $\mathbf{f}(\mathbf{x}_c; \boldsymbol{\mu}) = 0$
- linearization at $\mathbf{x}_c$ is given by

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$$

where M is the Jacobian matrix

$$M = D\mathbf{f}(\mathbf{x}_c) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}_c) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}_c) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}_c) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}_c) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}_c) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{x}_c) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\mathbf{x}_c) & \frac{\partial f_n}{\partial x_2}(\mathbf{x}_c) & \cdots & \frac{\partial f_n}{\partial x_n}(\mathbf{x}_c) \end{pmatrix}$$

- equilibrium point $\mathbf{x}_c$ is called

hyperbolic: if none of the eigenvalues of the matrix $D\mathbf{f}(\mathbf{x}_c)$ have zero real part
sink: if all of the eigenvalues of the matrix $D\mathbf{f}(\mathbf{x}_c)$ have negative real part
source: if all of the eigenvalues of the matrix $D\mathbf{f}(\mathbf{x}_c)$ have positive real part
saddle: if it is a hyperbolic equilibrium point and $D\mathbf{f}(\mathbf{x}_c)$ has at least one eigenvalue with a positive real part and at least one with a negative real part

Invariant manifolds

stable manifold theorem:

- the nonlinear system has locally similar behaviour close to a hyperbolic critical point
- it shows the existence of two invariant manifolds: stable manifold, unstable manifold

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We know what invariant means: set $S \subset \Omega$ is *invariant* if $\phi_t(S) \subset S$ for all $t \in \mathbb{R}$

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What is a manifold? Wiggins [page 29], Perko [page 107], Kuznetsov [page 598]

- linear settings: a linear vector subspace of $\mathbb{R}^n$
- nonlinear settings: a surface embedded in $\mathbb{R}^n$ which can be locally represented as a graph

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- linear settings: a linear vector subspace of $\mathbb{R}^n$
- nonlinear settings: a surface embedded in $\mathbb{R}^n$ which can be locally represented as a graph
- there is also the center manifold (invariant manifold that appears in the center manifold theorem), but we will start with the stable manifold theorem

Stable manifold theorem: linear systems

Consider the linear system $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$ where $M \in \mathbb{R}^{n\times n}$ and none of the eigenvalues of $M \in \mathbb{R}^{n\times n}$ have zero real part.

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Assume that M is diagonalizable (semi-simple) and denote its eigenvalues and eigenvectors by $\quad \lambda_j = a_j + \mathrm{i}\, b_j \quad$ and $\quad \mathbf{w}_j = \mathbf{u}_j + \mathrm{i}\, \mathbf{v}_j$,
where $a_j, b_j \in \mathbb{R}$, $\mathbf{u}_j, \mathbf{v}_j \in \mathbb{R}^n$, for $j = 1, 2, \ldots, n$. Then we define

stable subspace: $E^s = \mathrm{span}\big\{\mathbf{u}_j, \mathbf{v}_j \,\big|\, a_j < 0\big\}$

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Example (Question 1(b) on Problem Sheet 0): $\lambda_1 = -2$, $\lambda_2 = 2$ and $\lambda_3 = 3$

$$M = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & 2 \end{pmatrix} \qquad \mathbf{w}_1 = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}, \qquad \mathbf{w}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{w}_3 = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$$

Then we have $E^s = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \right\}$, $\qquad E^u = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \right\}$

Stable manifold theorem: linear systems

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Denote the eigenvalues and generalized eigenvectors of M by

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remarks: (1) if λ is an eigenvalue of matrix $M \in \mathbb{R}^{n\times n}$ of algebraic multiplicity $m \leq n$, then for $k = 1, 2, \ldots, m$, any nonzero solution $\mathbf{v}$ of $(A - \lambda I)^k \mathbf{v} = \mathbf{0}$ is called a generalized eigenvector of M

(2) if some eigenvalues of $M \in \mathbb{R}^{n\times n}$ have zero real part, we also define *center subspace*: $E^c = \operatorname{span}\{\mathbf{u}_j, \mathbf{v}_j \,|\, a_j = 0\}$

examples: Question 1 on Problem Sheet 1

Stable manifold theorem: linear systems

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Moreover, the solution is given by $\mathbf{x}(t) = \sum_{j=1}^{n} c_j\, e^{\lambda_j t}\mathbf{w}_j$ which implies:

if $\mathbf{x}(t) = \mathbf{x}_0 \in E^s$, then $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ and $\lim_{t\to-\infty} \|\mathbf{x}(t)\| = \infty$

if $\mathbf{x}(t) = \mathbf{x}_0 \in E^u$, then $\lim_{t\to\infty} \|\mathbf{x}(t)\| = \infty$ and $\lim_{t\to-\infty} \mathbf{x}(t) = \mathbf{0}$

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if $\mathbf{x}(t) = \mathbf{x}_0 \in E^u$, then $\displaystyle\lim_{t\to\infty} \|\,\mathbf{x}(t)\,\| = \infty$ and $\displaystyle\lim_{t\to-\infty} \mathbf{x}(t) = \mathbf{0}$

Question 2 on Problem Sheet 1: this is also true for non-diagonalizable matrix M

(the nonlinear system has locally similar behaviour close to a hyperbolic critical point)

Stable manifold theorem

Given C^1 vectory field $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$, $\Theta \subset \mathbb{R}^m$, and $\boldsymbol{\mu} \in \Theta$, we consider ODE system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$$

WLOG, assume that $\mathbf{0} \subset \Omega$ is the hyperbolic critical point, *i.e.* $\mathbf{f}(\mathbf{0}; \boldsymbol{\mu}) = \mathbf{0}$ and matrix $D\mathbf{f}(\mathbf{0})$ has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. In particular, our discussion of linear systems is applicable to the linear system $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$ with $M = D\mathbf{f}(\mathbf{0})$.

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Then there exists (local results):

- a k-dimensional differentiable manifold M^s_{loc} tangent to the stable subspace E^s of the linear system at $\mathbf{0}$ such that for all $t\geq 0$, we have $\phi_t(M^s_{\mathrm{loc}})\subset M^s_{\mathrm{loc}}$ and for all $\mathbf{x}_0\in M^s_{\mathrm{loc}}$, we have
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- an $(n-k)$-dimensional differentiable manifold M^u_{loc} tangent to the unstable subspace E^u of the linear system at $\mathbf{0}$ such that for all $t\leq 0$, we have $\phi_t(M^u_{\mathrm{loc}})\subset M^u_{\mathrm{loc}}$ and for all $\mathbf{x}_0\in M^u_{\mathrm{loc}}$, we have
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B5.6: Nonlinear Dynamics, Bifurcations and Chaos (Lecture 3)

- summary of Lecture 2: we discussed
  Notion of flows, stability of fixed points, Lyapunov function, invariant manifolds, stable manifold theorem, notion of hyperbolicity. Stable, unstable and center subspaces. (Questions 1, 2, 5 and 7 on Problem Sheet 1)
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- course synopsis of **Lectures 1-8** (taken by both Part B and MSc students):
  Discrete-time (maps) and continuous-time (differential equations) dynamical systems. Notion of flows, stability of fixed points, Lyapunov function, invariant manifolds, stable manifold theorem, notion of hyperbolicity, center manifold. Chemical reaction networks. Stable, unstable and center subspaces. Poincaré-Bendixson theorem. Periodic solutions, stable and unstable limit cycles. Introduction to bifurcation theory, covering saddle-node, transcritical, supercritical pitchfork and subcritical pitchfork bifurcations. Extended center manifold. Logistic map. Periodic points of maps. Stability of N-cycles. Period-doubling bifurcation. Sharkovsky's theorem. Invariant distribution.

Stable manifold theorem (last slide of Lecture 2)

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$$\lim_{t\to-\infty} \phi_t(\mathbf{x}_0) = \mathbf{0}$$

Stable manifold theorem: example

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -x_1 - x_2^2$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_2 + x_1^2$$

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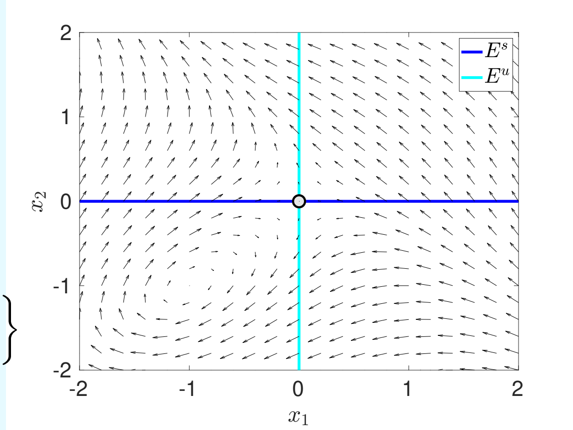
$\mathbf{0} = [0, 0]$ is a fixed point

$$D\mathbf{f}(\mathbf{0}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$E^s = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}, \quad E^u = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$



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M^s_{loc} is of the form $x_2 = c_1 x_1^2 + \mathcal{O}(x_1^3)$

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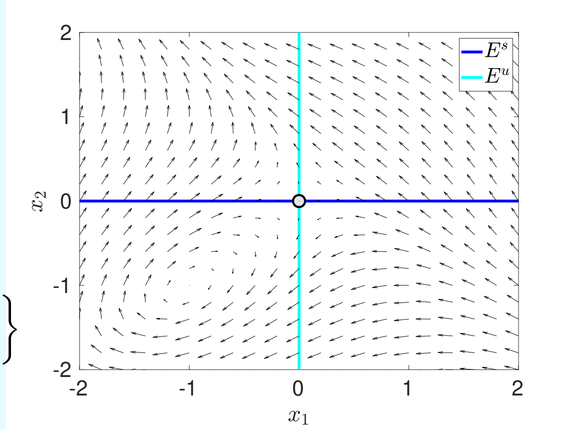

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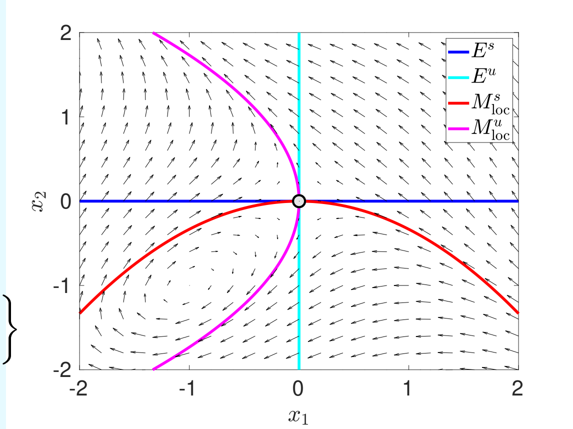

M^s_{loc} is of the form $x_2 = c_1 x_1^2 + \mathcal{O}(x_1^3)$

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differentiating these approximations, we get $c_1 = c_2 = -\dfrac{1}{3}$, *i.e.*

M^s_{loc} is of the form $x_2 = -\frac{x_1^2}{3} + \mathcal{O}(x_1^3)$ and M^u_{loc} is of the form $x_1 = -\frac{x_2^2}{3} + \mathcal{O}(x_2^3)$

Global stable and unstable manifolds

global stable and unstable manifolds: $M^s = \bigcup_{t \le 0} \phi_t(M^s_{\mathrm{loc}})$ and $M^u = \bigcup_{t \ge 0} \phi_t(M^u_{\mathrm{loc}})$

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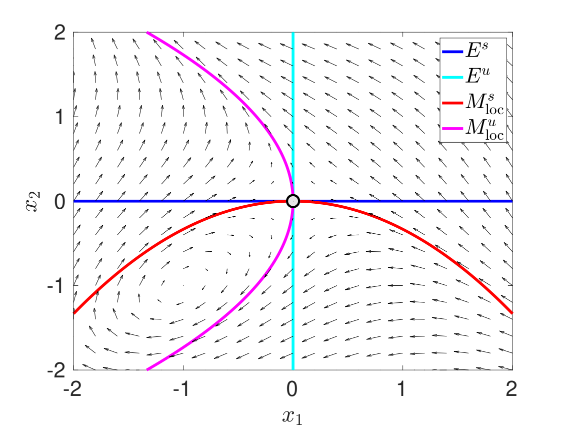

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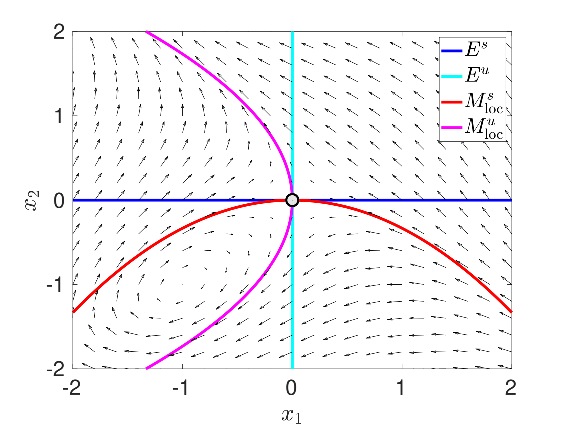
example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = -x_1 - x_2^2$

$\dfrac{\mathrm{d}x_2}{\mathrm{d}t} = x_2 + x_1^2$

observe that $A = 3x_1x_2 + x_1^3 + x_2^3$ is time independent:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 3(x_2 + x_1^2)\frac{\mathrm{d}x_1}{\mathrm{d}t} + 3(x_1 + x_2^2)\frac{\mathrm{d}x_2}{\mathrm{d}t}$$
$$= 3(x_2 + x_1^2)(-x_1 - x_2^2)$$
$$+ 3(x_1 + x_2^2)(x_2 + x_1^2) = 0$$



consequently, both stable and unstable manifolds satisfy $A = 3x_1x_2 + x_1^3 + x_2^3 = 0$

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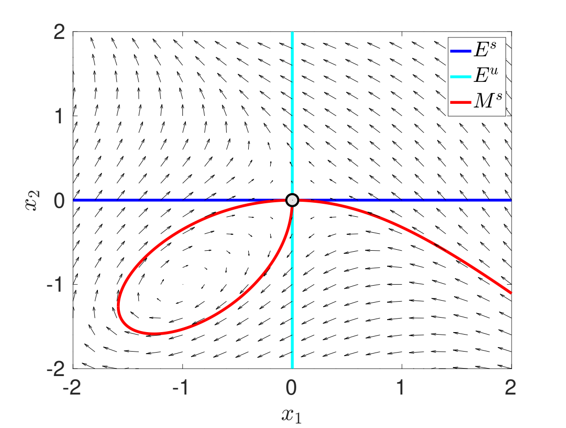
observe that $A = 3x_1x_2 + x_1^3 + x_2^3$ is time independent:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 3(x_2 + x_1^2)\frac{\mathrm{d}x_1}{\mathrm{d}t} + 3(x_1 + x_2^2)\frac{\mathrm{d}x_2}{\mathrm{d}t} = 3(x_2 + x_1^2)(-x_1 - x_2^2) + 3(x_1 + x_2^2)(x_2 + x_1^2) = 0$$



consequently, both stable and unstable manifolds satisfy $A = 3x_1x_2 + x_1^3 + x_2^3 = 0$

Global stable and unstable manifolds

global stable and unstable manifolds: $M^s = \bigcup_{t \leq 0} \phi_t(M^s_{\text{loc}})$ and $M^u = \bigcup_{t \geq 0} \phi_t(M^u_{\text{loc}})$

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -x_1 - x_2^2$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_2 + x_1^2$$

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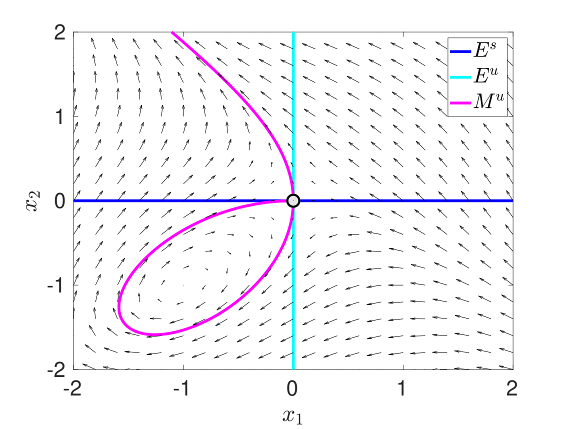

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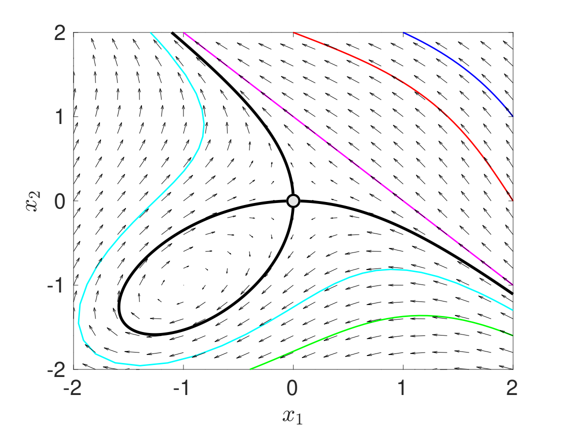

consequently, both stable and unstable manifolds satisfy $A = 3x_1x_2 + x_1^3 + x_2^3 = 0$

Periodic solutions (closed orbits)

[Part A Differential Equations 1]: The existence of suitable A is one possible approach to prove the existence of periodic solutions (closed orbits) in planar systems.

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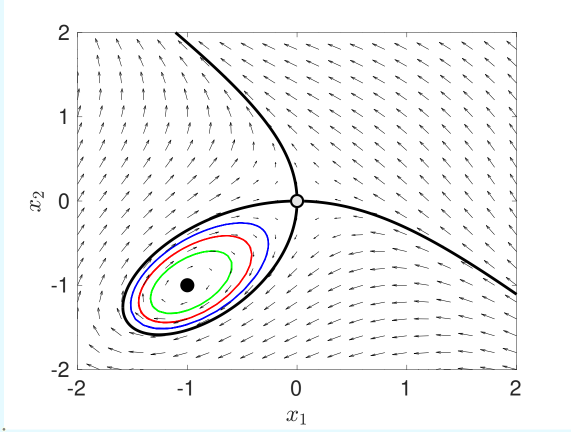

periodic solutions around point $[-1, -1]$ satisfy $A = 3x_1x_2 + x_1^3 + x_2^3 = c$ for $c \in (0, 1)$

Periodic solutions (closed orbits)

[Part A Differential Equations 1]: The existence of suitable A is one possible approach to prove the existence of periodic solutions (closed orbits) in planar systems.

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1 - x_1 x_2$$
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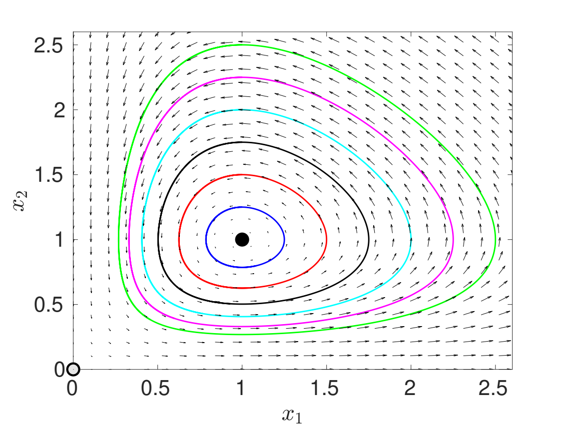
$A = \log(x_1) - x_1 + \log(x_2) - x_2$
is time independent

Lotka-Volterra predator-prey equations



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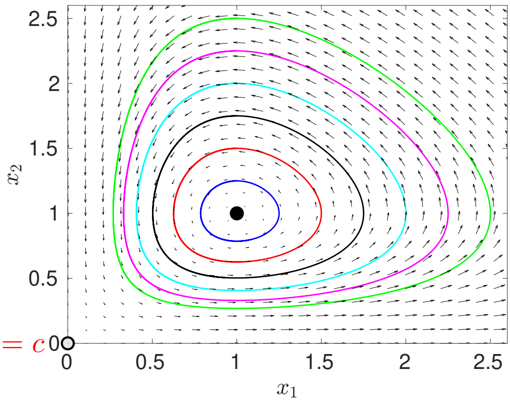

Lotka-Volterra predator-prey equations

periodic solutions around point $[1, 1]$
satisfy $A = \log(x_1) - x_1 + \log(x_2) - x_2 = c$

[Part A Differential Equations 1]:
see pages 39-41 of your lecture notes from last year

Periodic solutions (closed orbits)

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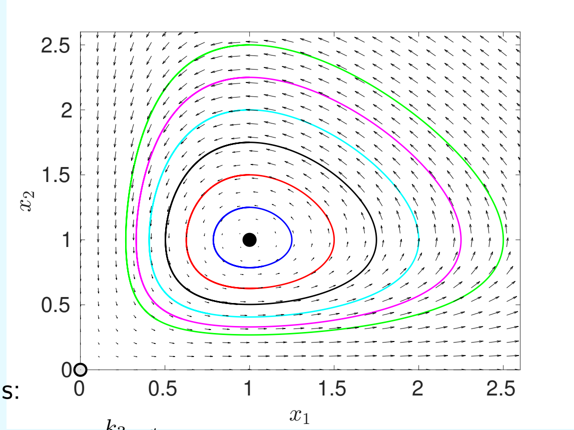

Note: Lotka-Volterra ODE system also describes a system of $n = 2$ chemical species X_1 and X_2 which are subject to the following $\ell = 3$ chemical reactions:

$$X_1 + X_2 \xrightarrow{k_1} 2X_2 \qquad X_1 \xrightarrow{k_2} 2X_1 \qquad X_2 \xrightarrow{k_3} \emptyset$$

where the values of the rate constants are: $k_1 = k_2 = k_3 = 1$

Poincaré-Bendixson theorem ($n = 2$)

Given C^1 vectory field $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^2$, where $\Omega \subset \mathbb{R}^2$, $\Theta \subset \mathbb{R}^m$, and $\boldsymbol{\mu} \in \Theta$, we consider the planar ODE system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$$

Suppose that $R \subset \Omega$ is compact (*i.e.* closed and bounded) and

- R does not contain any fixed points
- there exists $\mathbf{x}_0 \in R$ such that $\phi_t(\mathbf{x}_0) \in R$ for all $t \geq 0$, *i.e.* the trajectory is confined in R for $t \geq 0$

Poincaré-Bendixson theorem: Then either $\Gamma_{\mathbf{x}_0}$ is a closed orbit, or $\phi_t(\mathbf{x}_0)$ spirals toward a closed orbit as $t \to \infty$. In either case, R contains a closed orbit.

Poincaré-Bendixson theorem ($n = 2$)

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application of the Poincaré-Bendixson theorem

- we need to find a *trapping region*: compact connected subset of Ω such that the vector field $\mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$ points 'inward' everywhere on the boundary

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- we need to find a *trapping region*: compact connected subset of Ω such that the vector field $\mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$ points 'inward' everywhere on the boundary
- we need to show that any fixed point in the trapping region is unstable, and remove its small neighbourhood to construct R

Poincaré-Bendixson theorem ($n = 2$)

Given C^1 vectory field $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^2$, where $\Omega \subset \mathbb{R}^2$, $\Theta \subset \mathbb{R}^m$, and $\boldsymbol{\mu} \in \Theta$, we consider the planar ODE system

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application of the Poincaré-Bendixson theorem [Question 6 on Problem Sheet 1]

- we need to find a *trapping region*: compact connected subset of Ω such that the vector field $\mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$ points 'inward' everywhere on the boundary [Question 6(c)]
- we need to show that any fixed point in the trapping region is unstable, and remove its small neighbourhood to construct R [Question 6(d)]

Poincaré-Bendixson theorem: example

$$\frac{\mathsf{d}x_1}{\mathsf{d}t} = x_1 + x_2 - x_1^3$$
$$\frac{\mathsf{d}x_2}{\mathsf{d}t} = -x_1 - x_2^3$$

Poincaré-Bendixson theorem: example

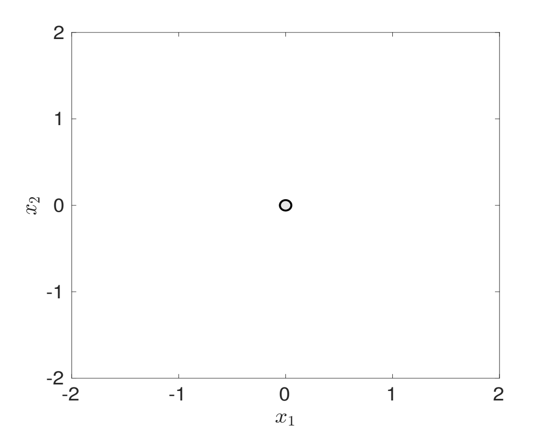
$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1 + x_2 - x_1^3 \; = \; f_1(x_1, x_2)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -x_1 - x_2^3 \; = \; f_2(x_1, x_2)$$

solving $f_1(x_1, x_2) = f_2(x_1, x_2) = 0$, we get $\mathbf{0} = [0, 0]$ as the only fixed point

$$D\mathbf{f}(\mathbf{0}) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

eigenvalues $\lambda_\pm = (1 \pm i\sqrt{3})/2$

$\mathbf{0} = [0, 0]$ is unstable (spiral)



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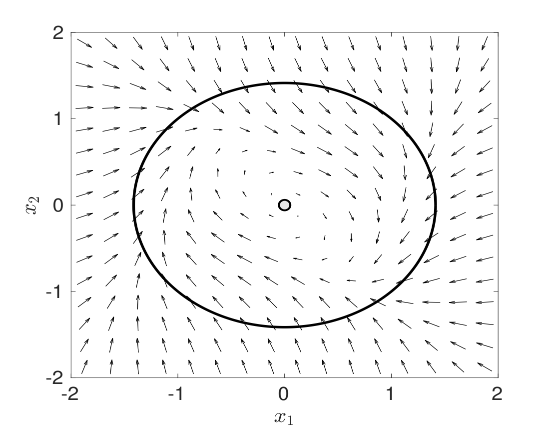

trapping region: circle with boundary $\mathbf{z}(s) = [\sqrt{2}\cos(s), \sqrt{2}\sin(s)]$ for $s \in [0, 2\pi]$
and inward pointing normal $\mathbf{n}(s) = [-\cos(s), -\sin(s)]$

we have $\mathbf{n}(s) \cdot \mathbf{f}(\mathbf{z}(s)) = \sqrt{2}(2(\cos^4(s) + \sin^4(s)) - \cos^2(s)) > 0$

Poincaré-Bendixson theorem: example

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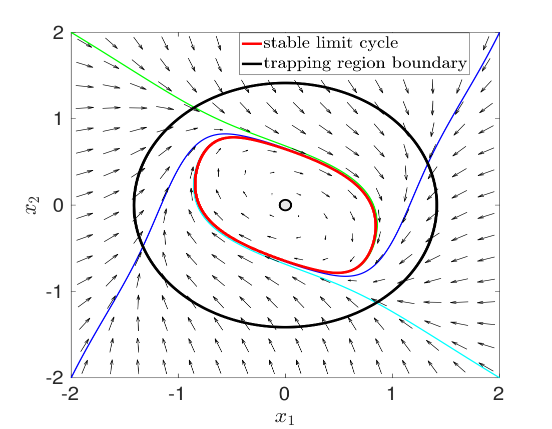

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$\Longrightarrow$ the Poincaré-Bendixson theorem implies the existence of a closed orbit

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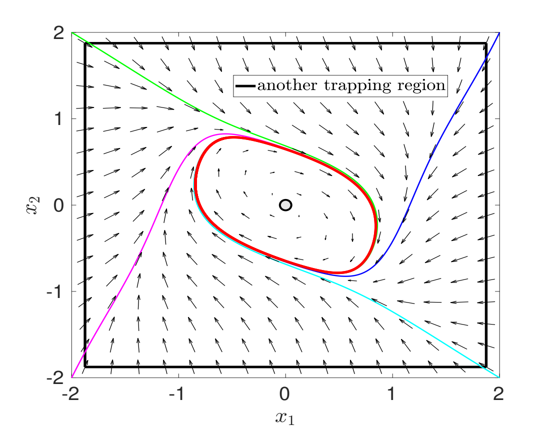

note: trapping region could also be chosen as a square in this example

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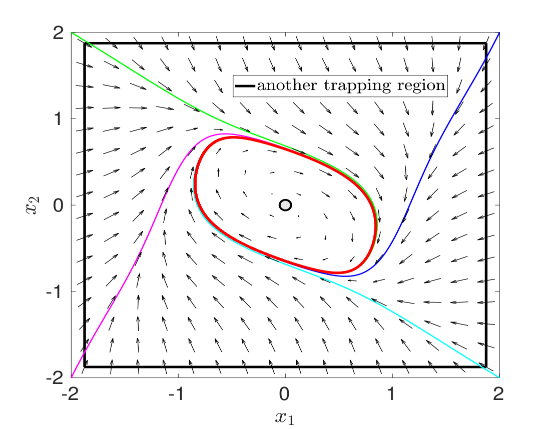

note: trapping region could also be chosen as a square in this example

[Question 6 on Problem Sheet 1]: you could choose a trapping region as a polygon (specify its vertices, parameterize its edges and verify that the vector field $\mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$ points inward everywhere on its boundary)

Center manifold

Given C^r vectory field $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$, $\Theta \subset \mathbb{R}^m$, and $\boldsymbol{\mu} \in \Theta$, we consider ODE system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$$

Assume that $\mathbf{x}_c \subset \Omega$ is the critical point, *i.e.* $\mathbf{f}(\mathbf{x}_c; \boldsymbol{\mu}) = \mathbf{0}$ and matrix $D\mathbf{f}(\mathbf{x}_c)$ has $k > 0$ eigenvalues with zero real part and $n - k$ eigenvalues with non-zero real part.

Then there exists a k-dimensional C^r-manifold M^c_{loc} tangent to center subspace E^s of the linear system at $\mathbf{x}_c$ such that for all $t \geq 0$, we have $\phi_t(M^c_{\mathrm{loc}}) \subset M^c_{\mathrm{loc}}$.

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- If the unstable manifold is non-empty, then the fixed point $\mathbf{x}_c$ is unstable.

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- If the unstable manifold is non-empty, then the fixed point $\mathbf{x}_c$ is unstable.
- Suppose the unstable manifold is empty and the system has both a non-empty stable and center manifold. Then the stability of the fixed point $\mathbf{x}_c$ is governed by the dynamics on the center manifold.

B5.6: Nonlinear Dynamics, Bifurcations and Chaos (Lecture 4)

- summary of Lecture 3: we discussed
  Invariant manifolds, stable manifold theorem. Poincaré-Bendixson theorem. Periodic solutions, stable and unstable limit cycles. (Question 6 on Problem Sheet 1)
- today: we will conclude our discussion of Problem Sheet 1

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- course synopsis of **Lectures 1-8** (taken by both Part B and MSc students):
  Discrete-time (maps) and continuous-time (differential equations) dynamical systems. Notion of flows, stability of fixed points, Lyapunov function, invariant manifolds, stable manifold theorem, notion of hyperbolicity, center manifold. Chemical reaction networks. Stable, unstable and center subspaces. Poincaré-Bendixson theorem. Periodic solutions, stable and unstable limit cycles. Introduction to bifurcation theory, covering saddle-node, transcritical, supercritical pitchfork and subcritical pitchfork bifurcations. Extended center manifold. Logistic map. Periodic points of maps. Stability of N-cycles. Period-doubling bifurcation. Sharkovsky's theorem. Invariant distribution.

Stable manifold theorem (last slide of Lecture 2)

Given C^1 vectory field $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$, $\Theta \subset \mathbb{R}^m$, and $\boldsymbol{\mu} \in \Theta$, we consider ODE system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$$

WLOG, assume that $\mathbf{0} \subset \Omega$ is the hyperbolic critical point, *i.e.* $\mathbf{f}(\mathbf{0}; \boldsymbol{\mu}) = \mathbf{0}$ and matrix $D\mathbf{f}(\mathbf{0})$ has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. In particular, our discussion of linear systems is applicable to the linear system $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$ with $M = D\mathbf{f}(\mathbf{0})$.

Then there exists (local results):

- a k-dimensional differentiable manifold M^s_{loc} tangent to the stable subspace E^s of the linear system at $\mathbf{0}$ such that for all $t \geq 0$, we have $\phi_t(M^s_{\mathrm{loc}}) \subset M^s_{\mathrm{loc}}$ and for all $\mathbf{x}_0 \in M^s_{\mathrm{loc}}$, we have
$$\lim_{t\to\infty} \phi_t(\mathbf{x}_0) = \mathbf{0}$$
- an $(n-k)$-dimensional differentiable manifold M^u_{loc} tangent to the unstable subspace E^u of the linear system at $\mathbf{0}$ such that for all $t \leq 0$, we have $\phi_t(M^u_{\mathrm{loc}}) \subset M^u_{\mathrm{loc}}$ and for all $\mathbf{x}_0 \in M^u_{\mathrm{loc}}$, we have
$$\lim_{t\to-\infty} \phi_t(\mathbf{x}_0) = \mathbf{0}$$

Examples of non-hyperbolic critical points $(n = 2)$

$$\frac{\mathsf{d}x_1}{\mathsf{d}t} = x_2 - 3x_1^3$$

$$\frac{\mathsf{d}x_2}{\mathsf{d}t} = -x_1$$

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$\mathbf{0} = [0, 0]$ is a critical point

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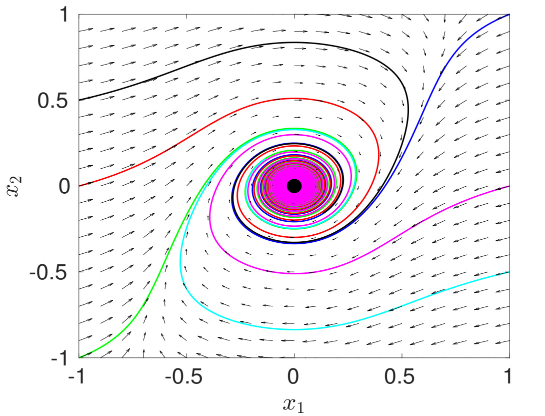
eigenvalues $\lambda_\pm = \pm i$

$\mathbf{0} = [0, 0]$ is a center



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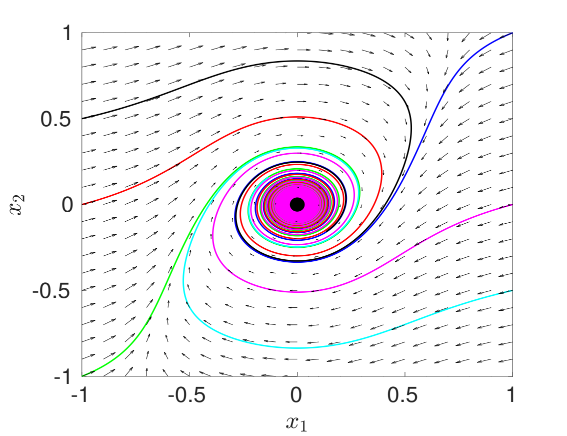
eigenvalues $\lambda_{\pm} = \pm i$

$\mathbf{0} = [0, 0]$ is a center



[Part A Differential Equations 1:] examples of ODEs where the critical point of the linearized system is a center (periodic solutions), and the non-linear system either has periodic solutions (Lotka-Volterra equations), or does not have periodic solutions the critical point can be stable or unstable (examples above)

Examples of non-hyperbolic critical points $(n = 2)$

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2 + 3x_1^3$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -x_1$$

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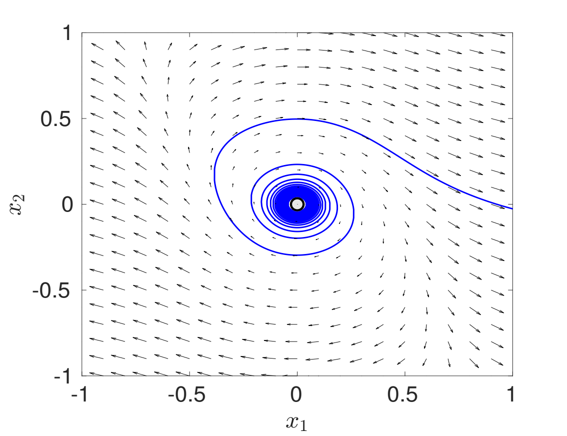

[Part A Differential Equations 1:] examples of ODEs where the critical point of the linearized system is a center (periodic solutions), and the non-linear system either has periodic solutions (Lotka-Volterra equations), or does not have periodic solutions the critical point can be stable or unstable (examples above)

Examples of non-hyperbolic critical points $(n = 2)$

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2$$
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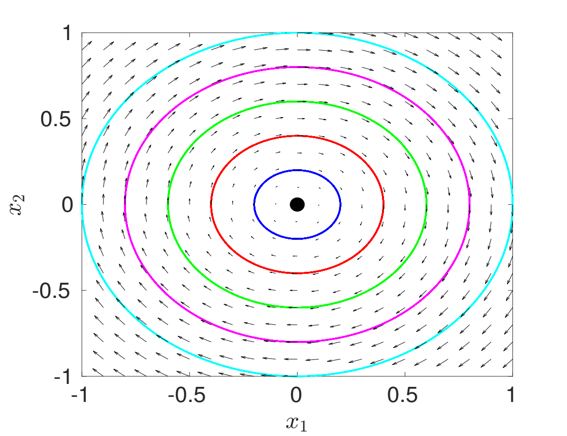

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Examples of non-hyperbolic critical points $(n = 2)$

$$\frac{\mathsf{d}x_1}{\mathsf{d}t} = -x_1^2$$
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eigenvalues $\lambda_{1,2} = 0, -1$

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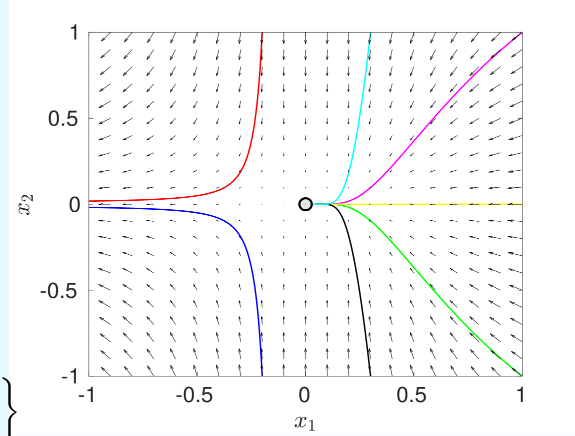

$\mathbf{0} = [0, 0]$ is a saddle-node

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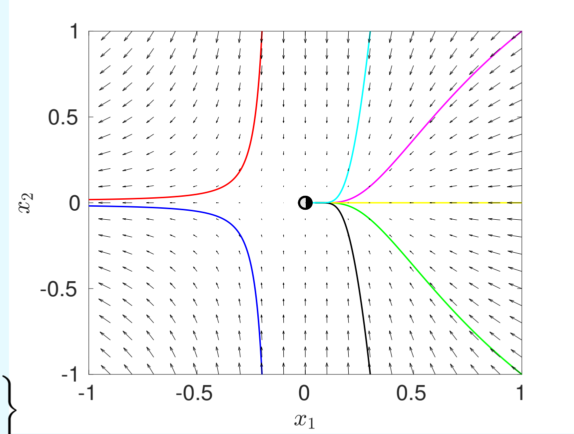

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Examples of non-hyperbolic critical points $(n = 2)$

$$\frac{\mathsf{d}x_1}{\mathsf{d}t} = x_2$$
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Examples of non-hyperbolic critical points $(n = 2)$

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2$$
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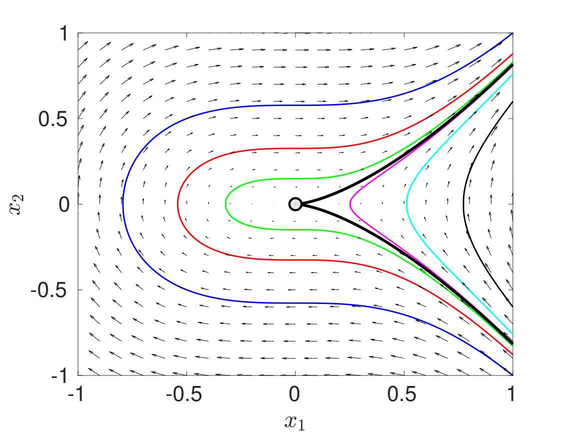
$\mathbf{0} = [0, 0]$ is a critical point

$$D\mathbf{f}(\mathbf{0}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

eigenvalue $\lambda = 0$

$\mathbf{0} = [0, 0]$ is a cusp



Center manifold (last slide of Lecture 3)

Given C^r vectory field $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$, $\Theta \subset \mathbb{R}^m$, and $\boldsymbol{\mu} \in \Theta$, we consider ODE system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$$

Assume that $\mathbf{x}_c \subset \Omega$ is the critical point, *i.e.* $\mathbf{f}(\mathbf{x}_c; \boldsymbol{\mu}) = \mathbf{0}$ and matrix $D\mathbf{f}(\mathbf{x}_c)$ has $k > 0$ eigenvalues with zero real part and $n - k$ eigenvalues with non-zero real part.

Then there exists a k-dimensional C^r-manifold M^c_{loc} tangent to center subspace E^s of the linear system at $\mathbf{x}_c$ such that for all $t \geq 0$, we have $\phi_t(M^c_{\mathrm{loc}}) \subset M^c_{\mathrm{loc}}$.

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- If the unstable manifold is non-empty, then the fixed point $\mathbf{x}_c$ is unstable.

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Then there exists a k-dimensional C^r-manifold M^c_{loc} tangent to center subspace E^s of the linear system at $\mathbf{x}_c$ such that for all $t \geq 0$, we have $\phi_t(M^c_{\mathrm{loc}}) \subset M^c_{\mathrm{loc}}$.

- If the unstable manifold is non-empty, then the fixed point $\mathbf{x}_c$ is unstable.
- Suppose the unstable manifold is empty and the system has both a non-empty stable and center manifold. Then the stability of the fixed point $\mathbf{x}_c$ is governed by the dynamics on the center manifold.

Reduction to the center manifold

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2(x_2 - x_1^3)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1^2 - x_2$$

Reduction to the center manifold

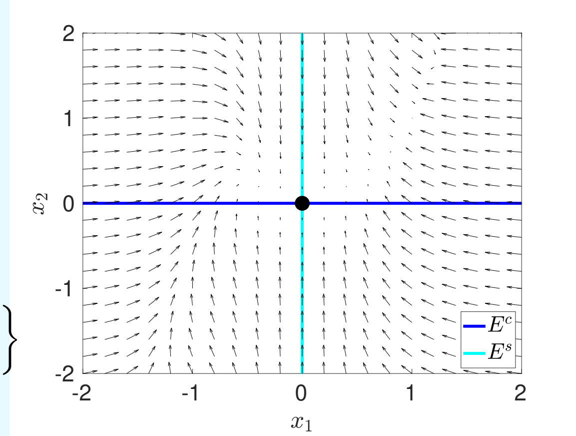
example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2(x_2 - x_1^3)$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1^2 - x_2$$

consider fixed point $\mathbf{x}_c = \mathbf{0} = [0, 0]$

$$D\mathbf{f}(\mathbf{0}) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

$$E^c = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}, \quad E^s = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$



Reduction to the center manifold

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2(x_2 - x_1^3)$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1^2 - x_2$$

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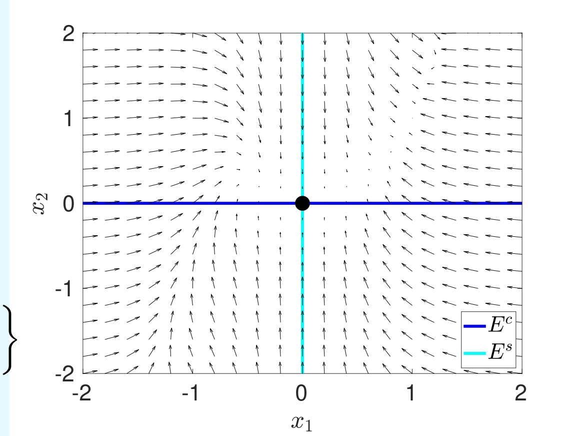


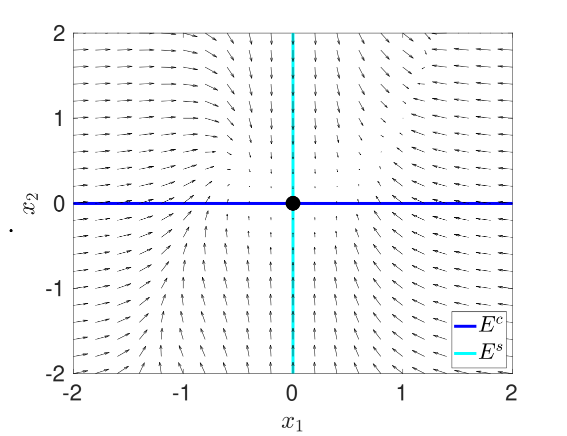
Warning: On the center linear subspace, we have $x_2 = 0$. Substituting $x_2 = 0$ into the first equation gives $\mathrm{d}x_1/\mathrm{d}t = -x_1^5$, but this does not mean that the origin is stable!

Reduction to the center manifold

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2(x_2 - x_1^3)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1^2 - x_2$$

M^c_{loc} is of the form
$$x_2 = h(x_1) = h_2\, x_1^2 + h_3\, x_1^3 + h_4\, x_1^4 + \ldots$$

- differentiating, we get
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = (2h_2x_1 + 3h_3x_1^2 + \ldots)\,\frac{\mathrm{d}x_1}{\mathrm{d}t}$$



Reduction to the center manifold

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2(x_2 - x_1^3)$

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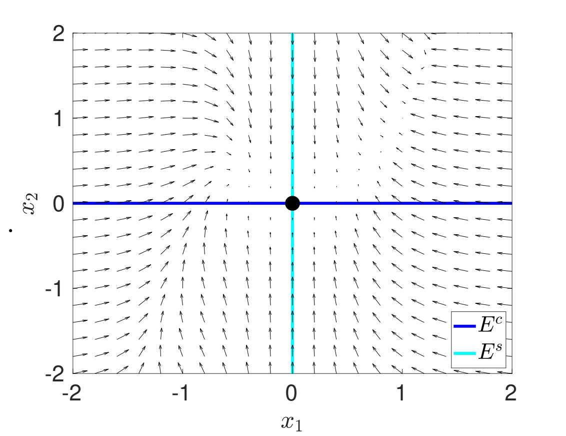

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- substituting

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2(h(x_1) - x_1^3), \quad \frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1^2 - h(x_1)$$

we have $x_1^2 - h(x_1) = (2h_2 x_1 + 3h_3 x_1^2 + \dots)\, x_1^2(h(x_1) - x_1^3)$

Reduction to the center manifold

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2(x_2 - x_1^3)$
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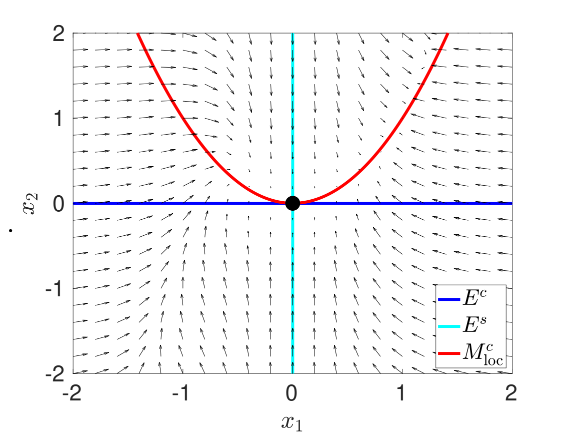

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we have $x_1^2 - h(x_1) = (2h_2x_1 + 3h_3x_1^2 + \dots)\, x_1^2(h(x_1) - x_1^3)$

- equating coefficients of powers of x_1 gives $h_2 = 1$, $h_3 = 0$ and $h_4 = 0$

Reduction to the center manifold

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2(x_2 - x_1^3)$$
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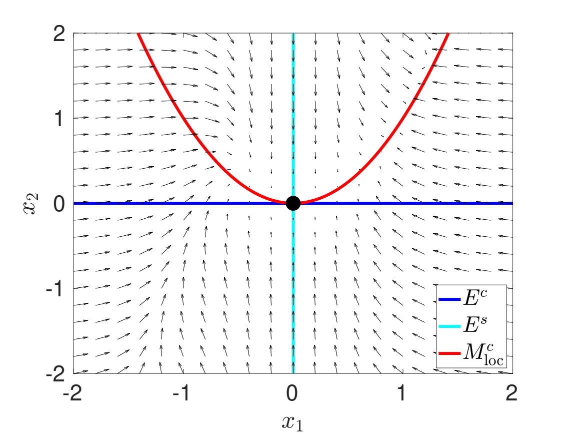
M^c_{loc} is of the form

$x_2 = h(x_1) = x_1^2 + \mathcal{O}(x_1^5)$



Reduction to the center manifold

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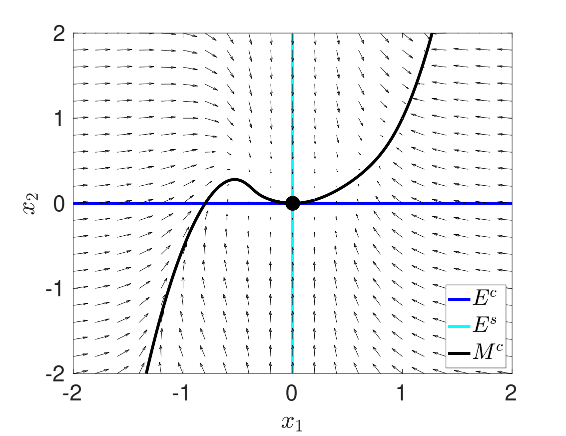
M^c_{loc} is of the form

$$x_2 = h(x_1) = x_1^2 + \mathcal{O}(x_1^5)$$

on the center manifold the dynamics is given by

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1^2(h(x_1) - x_1^3)\\ &= x_1^4 - x_1^5 + \mathcal{O}(x_1^7)\end{aligned}$$



Reduction to the center manifold

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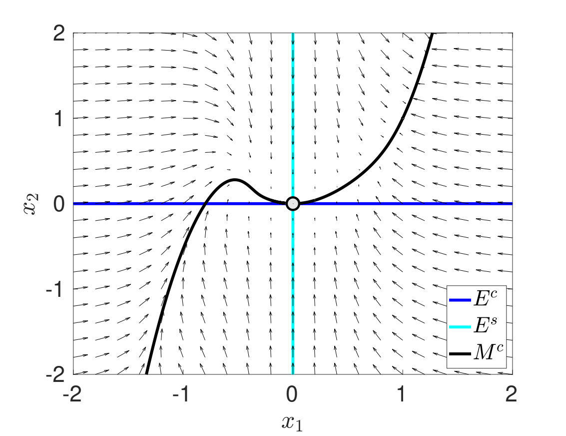
which implies that the origin is unstable



Reduction to the center manifold

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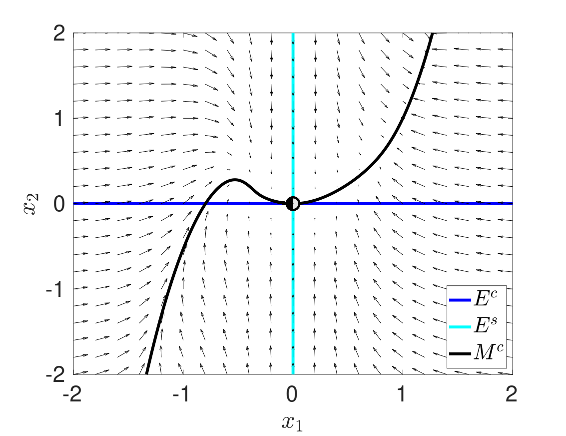

Reduction to the center manifold

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on the center manifold the dynamics is given by

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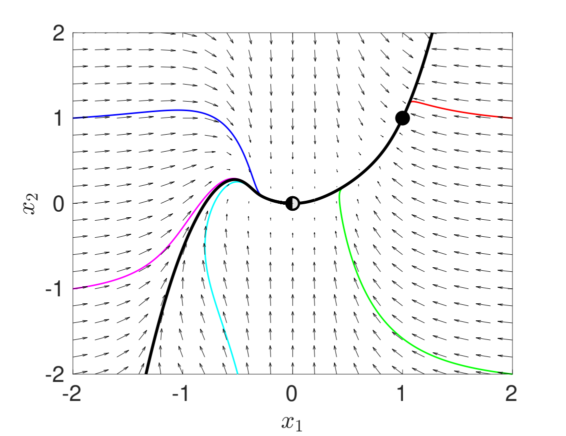

Reduction to the center manifold

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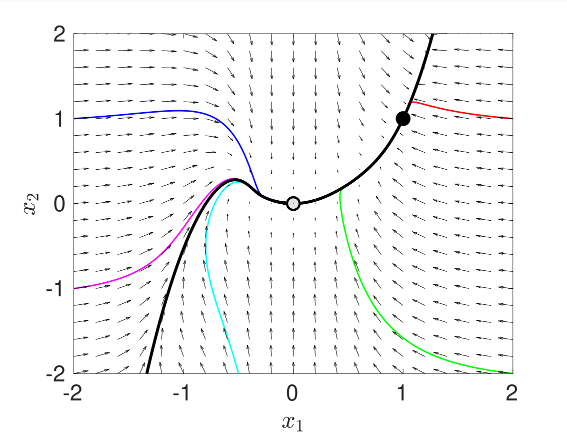
which implies that the origin is unstable

another example: Question 3 on Problem Sheet 1



Example 2: reduction to the center manifold

$$\frac{\mathsf{d}x_1}{\mathsf{d}t} = x_2 - x_1 + 2x_2(x_1 + x_2) - (x_1 + x_2)^6$$
$$\frac{\mathsf{d}x_2}{\mathsf{d}t} = x_1 - x_2 - 2x_1(x_1 + x_2) - (x_1 + x_2)^6$$

Example 2: reduction to the center manifold

$$\frac{\mathsf{d}x_1}{\mathsf{d}t} = x_2 - x_1 + 2x_2(x_1 + x_2) - (x_1 + x_2)^6$$
$$\frac{\mathsf{d}x_2}{\mathsf{d}t} = x_1 - x_2 - 2x_1(x_1 + x_2) - (x_1 + x_2)^6$$

consider fixed point $\mathbf{x}_c = \mathbf{0} = [0, 0]$

$$D\mathbf{f}(\mathbf{0}) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad E^c = \mathsf{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}, \quad E^s = \mathsf{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

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new variables: $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = -2y_1y_2 - 2y_1^6 \qquad \frac{\mathrm{d}y_2}{\mathrm{d}t} = -2y_2 + 2y_1^2$$

Example 2: reduction to the center manifold

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2 - x_1 + 2x_2(x_1 + x_2) - (x_1 + x_2)^6$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1 - x_2 - 2x_1(x_1 + x_2) - (x_1 + x_2)^6$$

consider fixed point $\mathbf{x}_c = \mathbf{0} = [0, 0]$

$$D\mathbf{f}(\mathbf{0}) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad E^c = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \quad E^s = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

new variables: $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = -2y_1y_2 - 2y_1^6 \qquad \frac{\mathrm{d}y_2}{\mathrm{d}t} = -2y_2 + 2y_1^2$$

Warning: On the center linear subspace, we have $y_2 = 0$. Substituting $y_2 = 0$ into the first equation gives $\mathrm{d}y_1/\mathrm{d}t = -2y_1^6$, but this does not mean that the origin is unstable! In fact, the center manifold can be calculated as $y_2 = h(y_1) = y_1^2 + \mathcal{O}(y_1^4)$, which implies that the the dynamics on the center manifold is

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = -2y_1h(y_1) - 2y_1^6 = -2y_1^3 + \mathcal{O}(y_1^5)$$ which implies that the origin is stable.

B5.6: Nonlinear Dynamics, Bifurcations and Chaos (Lecture 5)

- summary of Lecture 4: we concluded our discussion of Problem Sheet 1
- today: we will start with our discussion of Problem Sheet 2 (covered in Lectures 5–8)
- course synopsis of **Lectures 1-8** (taken by both Part B and MSc students):
  Discrete-time (maps) and continuous-time (differential equations) dynamical systems. Notion of flows, stability of fixed points, Lyapunov function, invariant manifolds, stable manifold theorem, notion of hyperbolicity, center manifold. Chemical reaction networks. Stable, unstable and center subspaces. Poincaré-Bendixson theorem. Periodic solutions, stable and unstable limit cycles. Introduction to bifurcation theory, covering saddle-node, transcritical, supercritical pitchfork and subcritical pitchfork bifurcations. Extended center manifold. Logistic map. Periodic points of maps. Stability of N-cycles. Period-doubling bifurcation. Sharkovsky's theorem. Invariant distribution.

Bifurcations

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu}) \quad \text{with the initial condition} \quad \mathbf{x}(0) = \mathbf{x}_0 \in \Omega$$

Bifurcations: The qualitative structure of the flow can change as parameters $\boldsymbol{\mu}$ are varied. For example, critical points (fixed points) can be created or destroyed, or limit cycles can be created or destroyed. The parameter values at which these qualitative changes in the dynamics occur are called bifurcation points.

Bifurcations

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Some bifurcations can occur for $n = 1$, so we start with them.

- saddle-node bifurcation
- transcritical bifurcation
- supercritical pitchfork bifurcation
- subcritical pitchfork bifurcation

Saddle-node bifurcation

example: $$\frac{\mathsf{d}x_1}{\mathsf{d}t} = \mu + x_1^2$$

$$f(x_1;\mu) = \mu + x_1^2$$

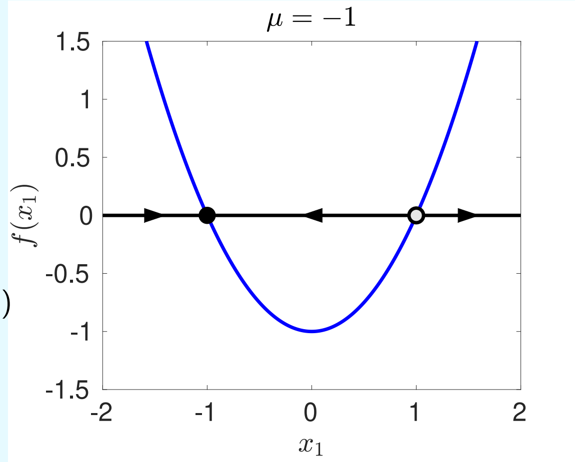
Saddle-node bifurcation

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \mu + x_1^2$

$$f(x_1;\mu) = \mu + x_1^2$$

$\mu < 0$

two fixed points at $x_1 = -\sqrt{-\mu}$ (stable) and $x_1 = \sqrt{-\mu}$ (unstable)



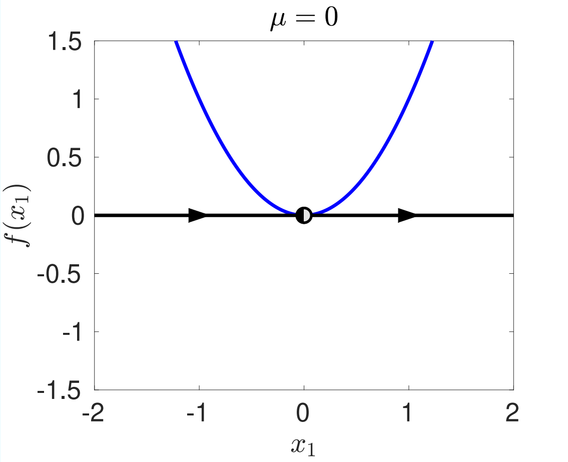
Saddle-node bifurcation

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \mu + x_1^2$

$$f(x_1;\mu) = \mu + x_1^2$$

as μ approaches zero from below, the two fixed points $-\sqrt{-\mu}$ and $\sqrt{-\mu}$ move toward each other

$\mu = 0$: the fixed points coalesce into a half-stable fixed point at $x_1 = 0$

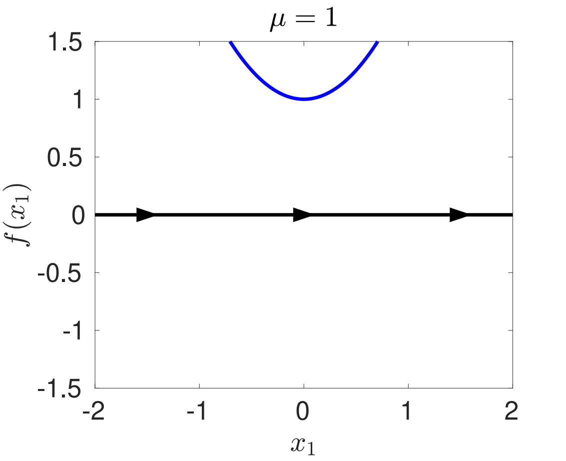


Saddle-node bifurcation

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \mu + x_1^2$

$$f(x_1; \mu) = \mu + x_1^2$$

$\mu > 0$: no fixed points



Saddle-node bifurcation

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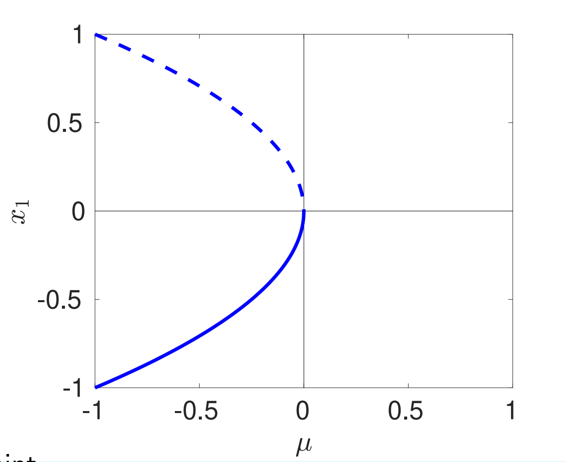


bifurcation diagram

saddle node bifurcation is a simple mechanism by which critical points can be created or destroyed

terminology:

critical point: fixed point, equilibrium point

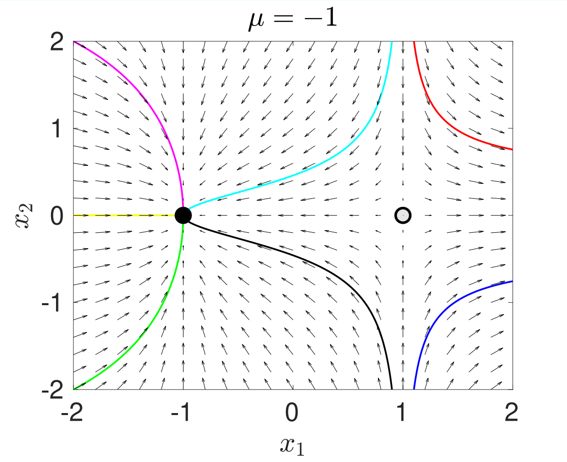
saddle-node bifurcation: fold bifurcation, turning point bifurcation

Saddle-node bifurcation

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu + x_1^2$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -x_2$$

$\mu < 0$

two fixed points at
$\mathbf{x} = [-\sqrt{-\mu}, 0]$ stable node
and $\mathbf{x} = [\sqrt{-\mu}, 0]$ saddle



Saddle-node bifurcation

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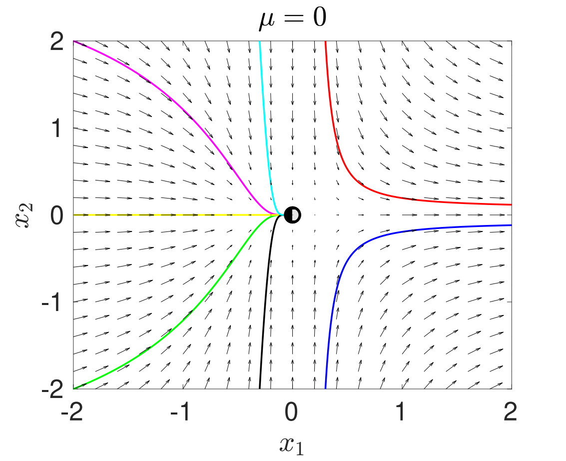

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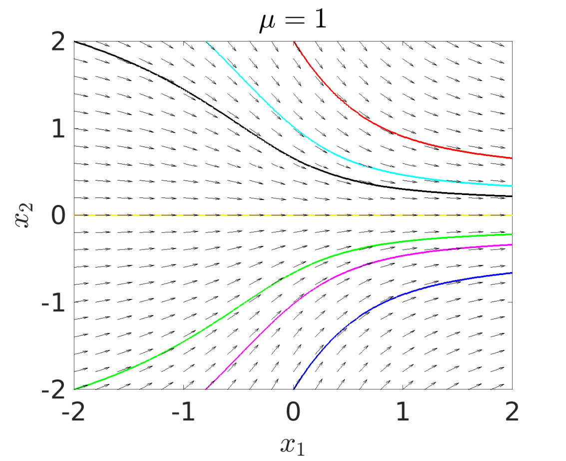

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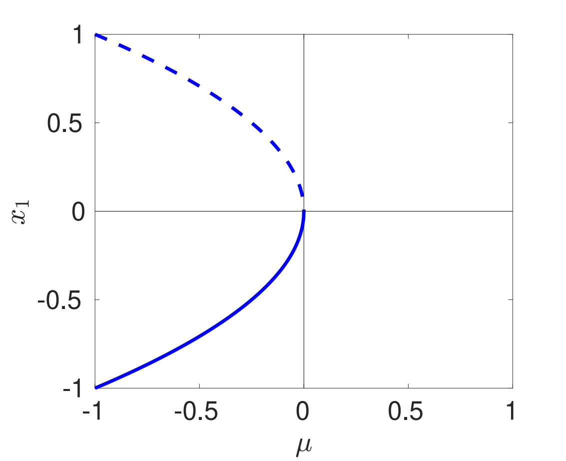

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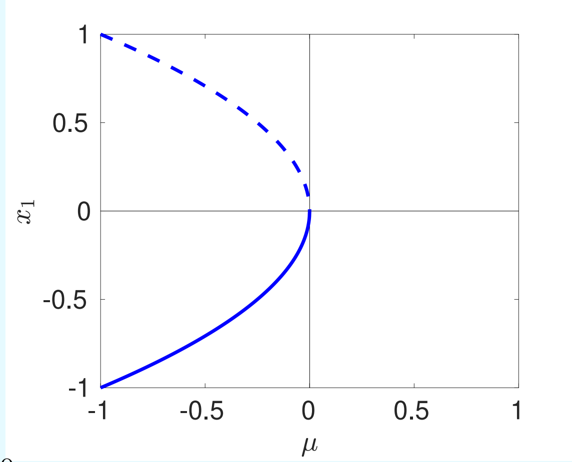

saddle node bifurcation is a general mechanism by which critical points can be created or destroyed

if it occurs at $x_1 = x_c$ and $\mu = \mu_c$, we have

$$f(x_c;\mu_c) = 0 \text{ and } \frac{\partial f}{\partial x_1}(x_c;\mu_c) = 0$$

Taylor expansion:

$$f(x_1;\mu) = (\mu - \mu_c)\frac{\partial f}{\partial \mu}(x_c;\mu_c) + (x_1 - x_c)^2\,\frac{1}{2}\,\frac{\partial^2 f}{\partial x_1^2}(x_c;\mu_c) + \ldots \quad \text{(normal form)}$$

Saddle-node bifurcation

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \mu - x_1^2$

$$f(x_1; \mu) = \mu - x_1^2$$

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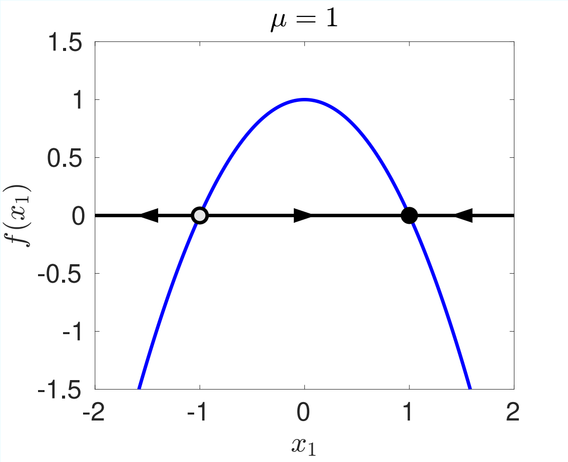
two fixed points at $x_1 = \sqrt{\mu}$ (stable) and $x_1 = -\sqrt{\mu}$ (unstable)



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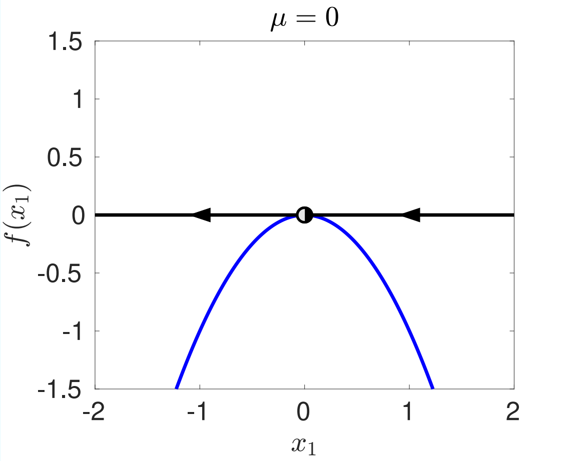

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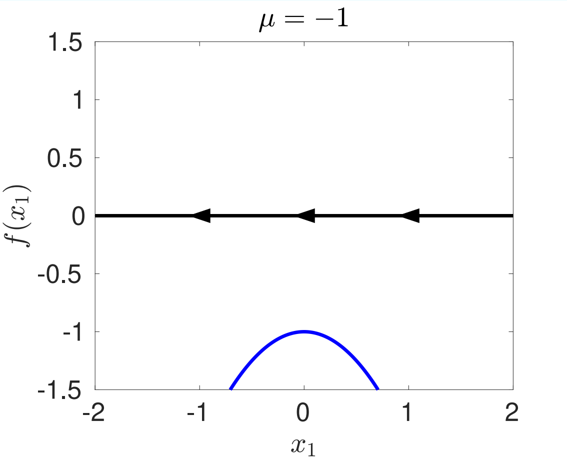

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two fixed points at
$\mathbf{x} = [-\sqrt{\mu}, 0]$ saddle
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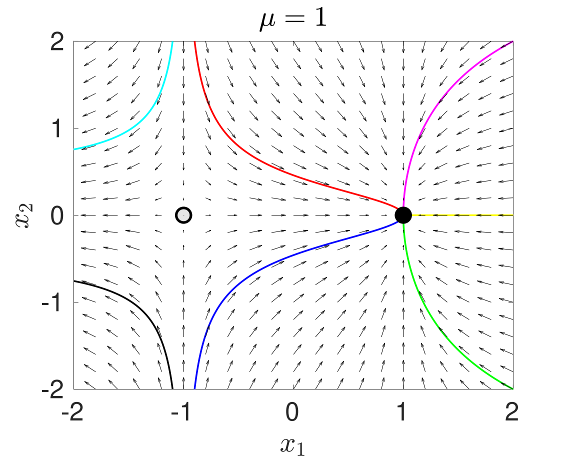

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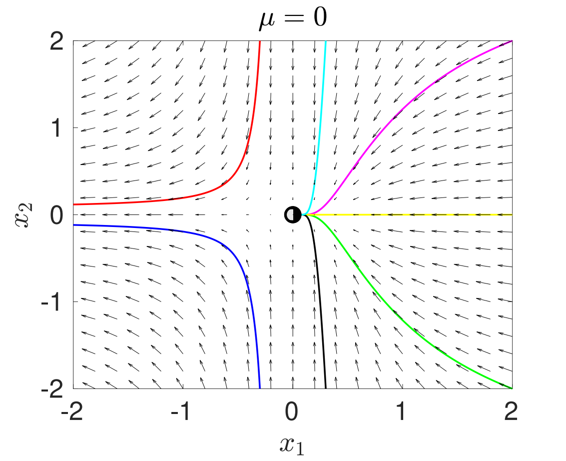

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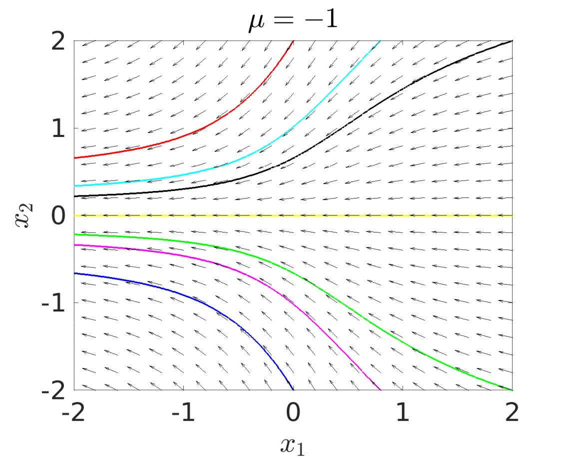

Saddle-node bifurcation

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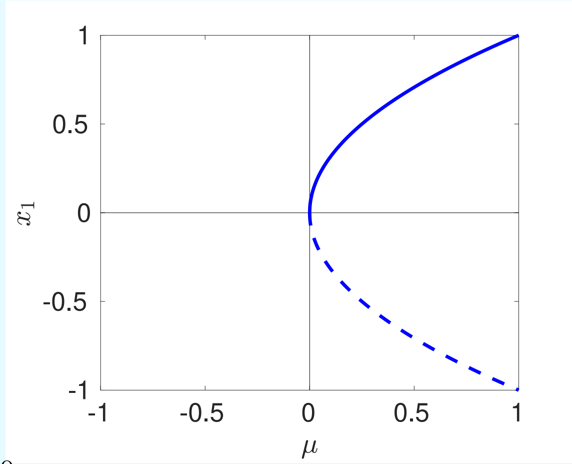
bifurcation diagram



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Transcritical bifurcation

example: $\dfrac{\mathsf{d}x_1}{\mathsf{d}t} = \mu x_1 - x_1^2$

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Transcritical bifurcation

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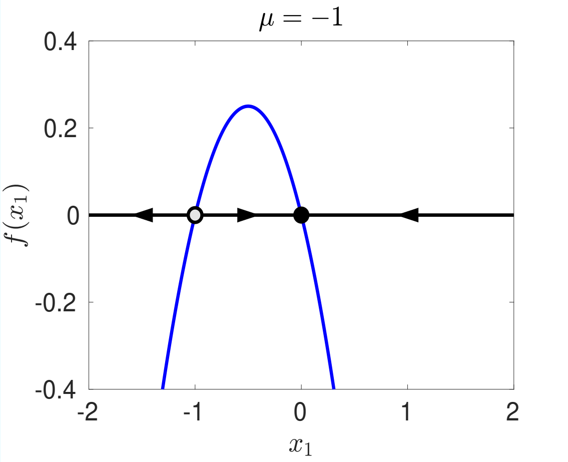
$\mu < 0$

two fixed points at $x_1 = \mu$ (unstable)
and $x_1 = 0$ (stable)



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as μ approaches zero,
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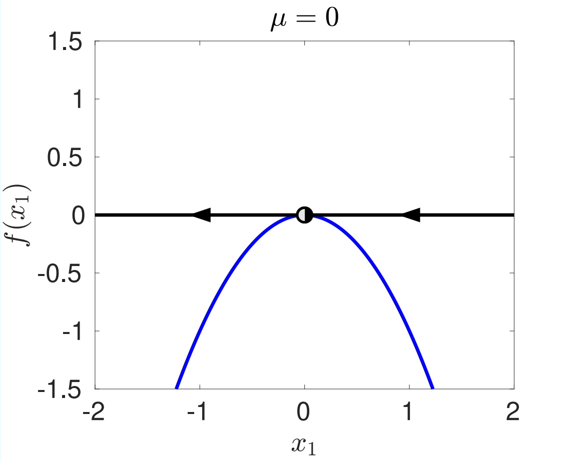

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two fixed points at $x_1 = \mu$ (stable)
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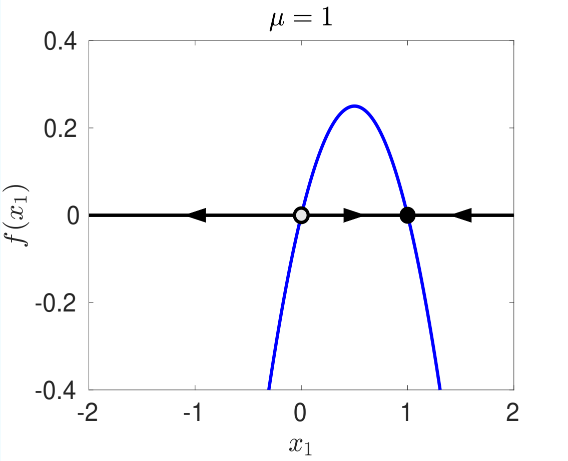

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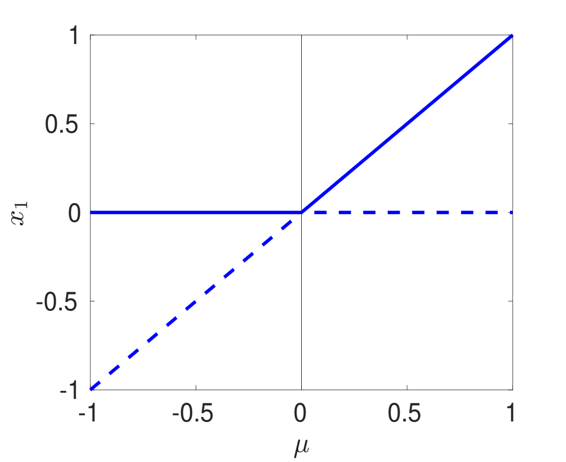
bifurcation diagram



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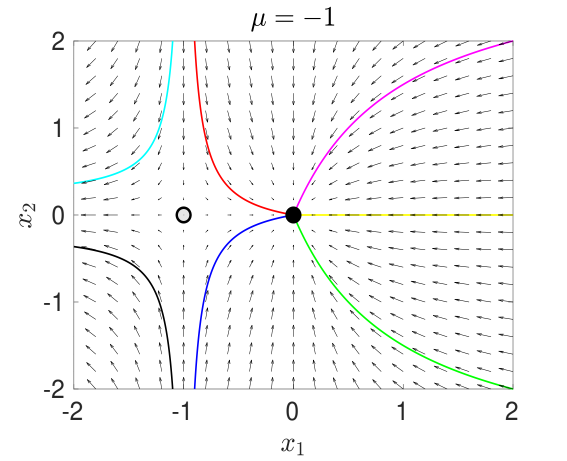
$\mu < 0$

two fixed points at

$\mathbf{x} = [\mu, 0]$ saddle (unstable)

and $\mathbf{x} = [0, 0]$ stable node



Transcritical bifurcation

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu x_1 - x_1^2$$
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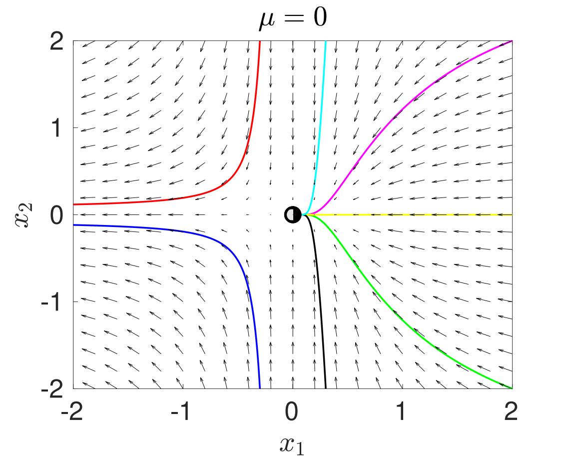

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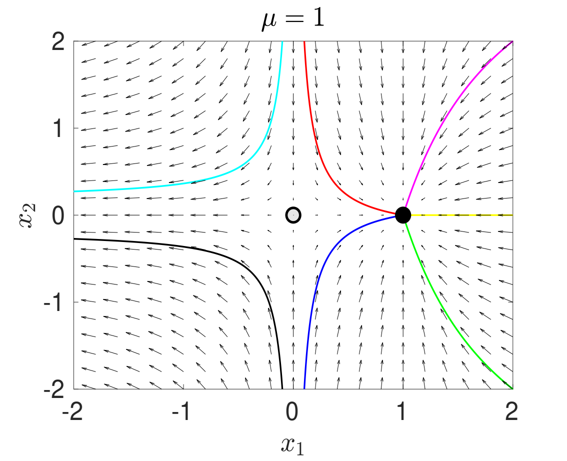

$\mu > 0$

two fixed points at

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and $\mathbf{x} = [0, 0]$ saddle (unstable)

Transcritical bifurcation

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu x_1 - x_1^2$$
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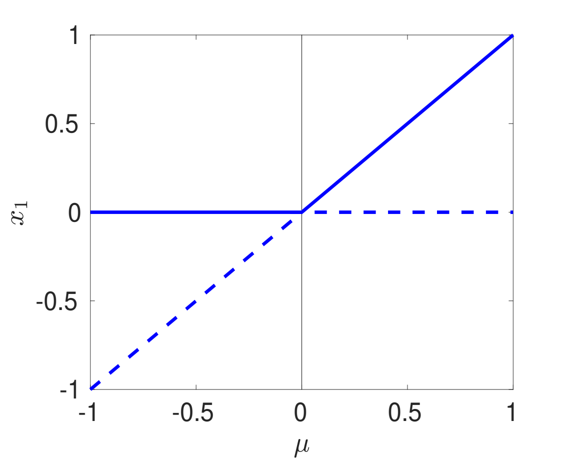
bifurcation diagram



Transcritical bifurcation

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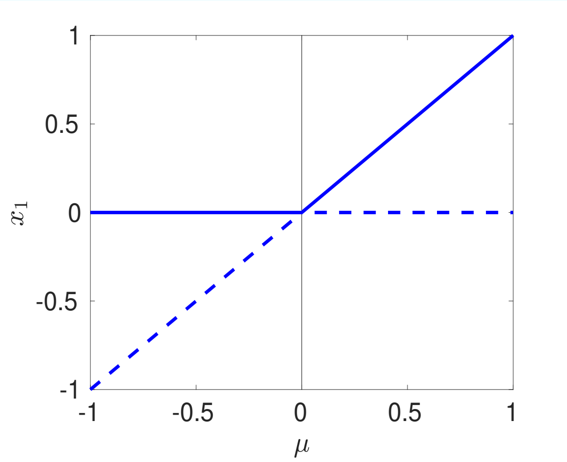

Other examples of ODE systems with bifurcations:

Questions 1, 2, 5 and 6 on Problem Sheet 2

Supercritical pitchfork bifurcation

example: $\dfrac{\mathsf{d}x_1}{\mathsf{d}t} = \mu x_1 - x_1^3$

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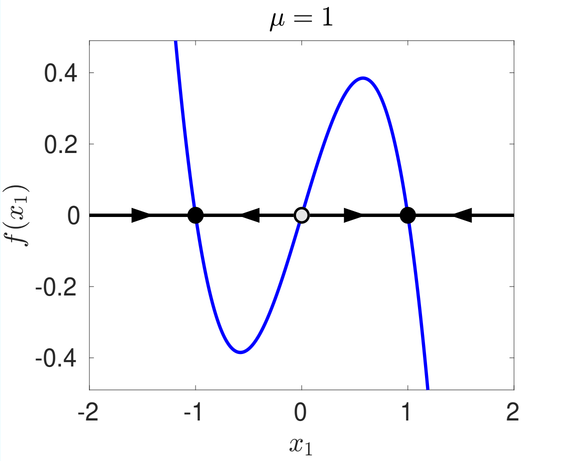
Supercritical pitchfork bifurcation

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \mu x_1 - x_1^3$

$$f(x_1;\mu) = \mu x_1 - x_1^3$$

$\mu > 0$

three fixed points at $x_1 = \pm\sqrt{\mu}$ (stable) and $x_1 = 0$ (unstable)



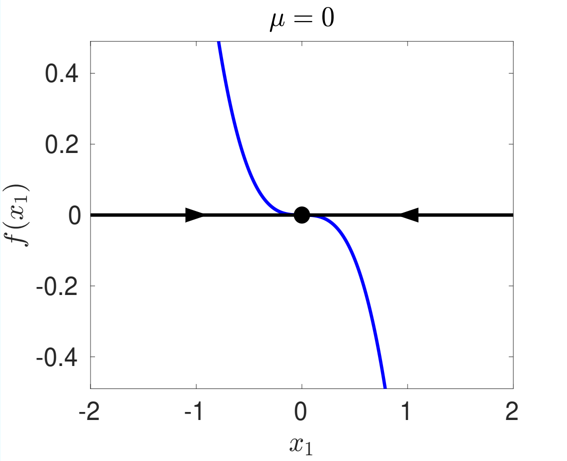
Supercritical pitchfork bifurcation

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \mu x_1 - x_1^3$

$$f(x_1;\mu) = \mu x_1 - x_1^3$$

as μ approaches zero from above, two fixed points $\sqrt{\mu}$ and $-\sqrt{\mu}$ move toward the third one

$\mu = 0$: the fixed points coalesce into a stable fixed point at $x_1 = 0$

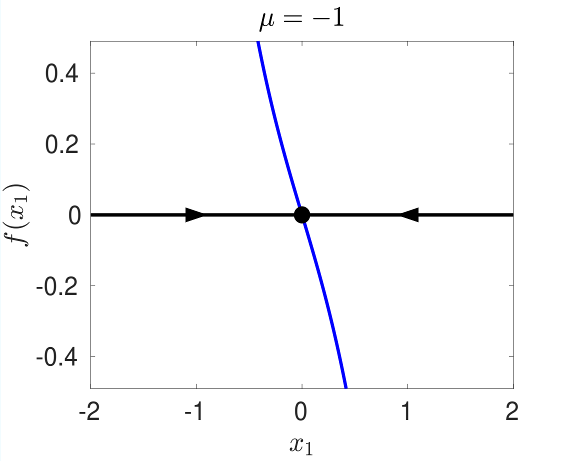


Supercritical pitchfork bifurcation

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \mu x_1 - x_1^3$

$$f(x_1;\mu) = \mu x_1 - x_1^3$$

$\mu < 0$: one stable fixed point at $x_1 = 0$



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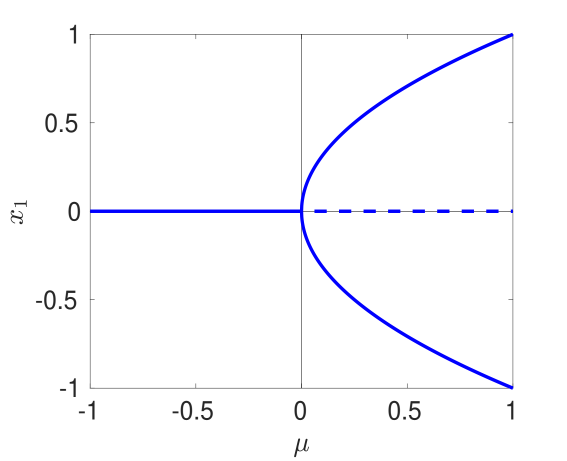

Supercritical pitchfork bifurcation

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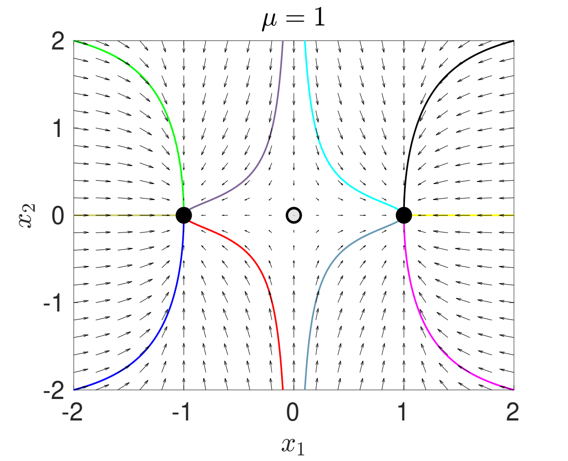
$\mu > 0$

three fixed points at

$\mathbf{x} = [-\sqrt{\mu}, 0]$ (stable node)

$\mathbf{x} = [0, 0]$ (saddle)

$\mathbf{x} = [\sqrt{\mu}, 0]$ (stable node)



Supercritical pitchfork bifurcation

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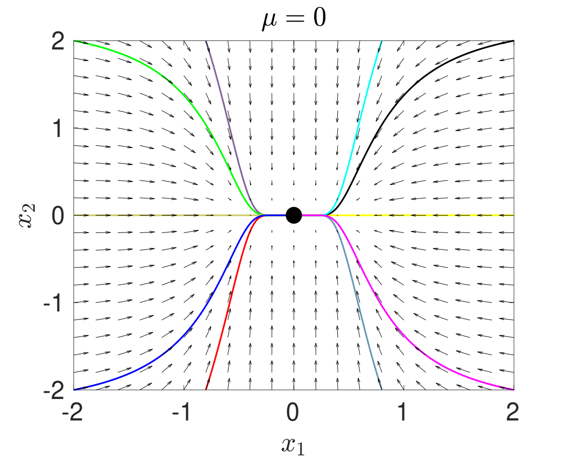

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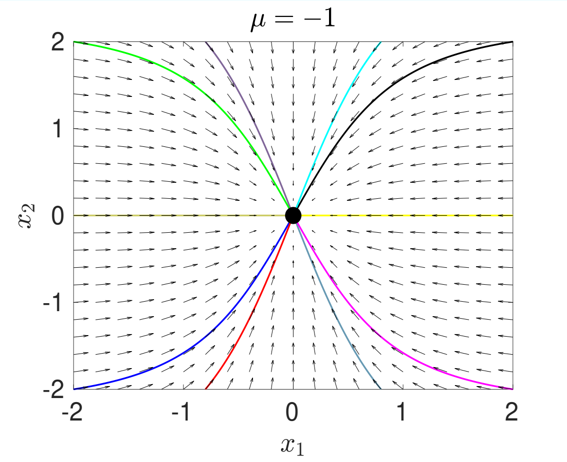

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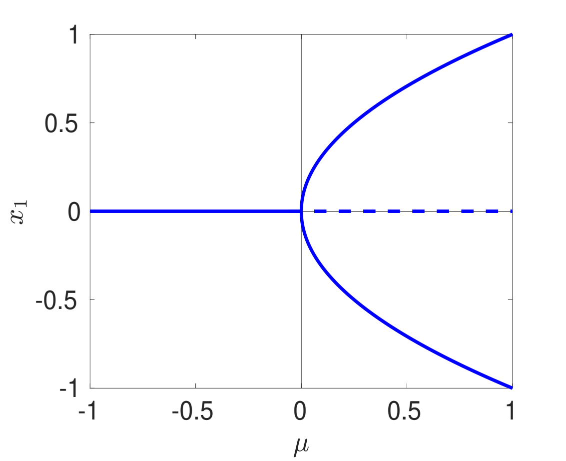
bifurcation diagram



Subcritical pitchfork bifurcation

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$$f(x_1;\mu) = \mu x_1 + x_1^3$$

Subcritical pitchfork bifurcation

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \mu x_1 + x_1^3$

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$\mu < 0$

three fixed points at
$x_1 = \pm\sqrt{-\mu}$ (unstable)
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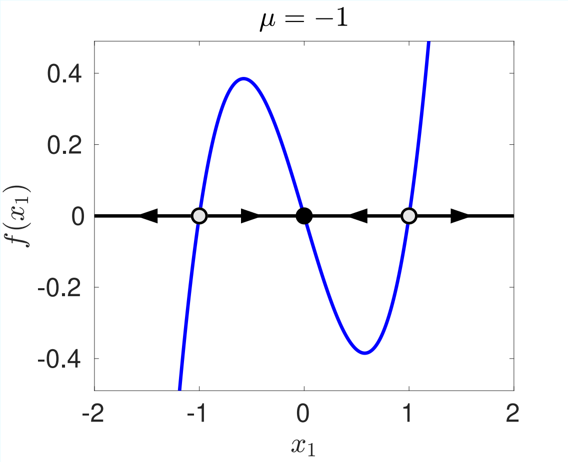

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$$f(x_1;\mu) = \mu x_1 + x_1^3$$

as μ approaches zero from below, two fixed points $-\sqrt{-\mu}$ and $\sqrt{-\mu}$ move toward the third one

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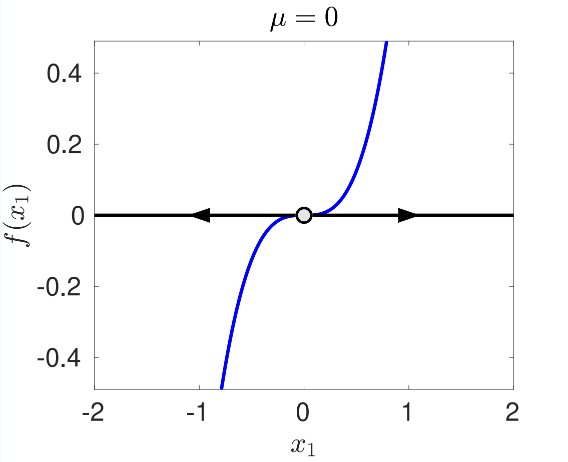
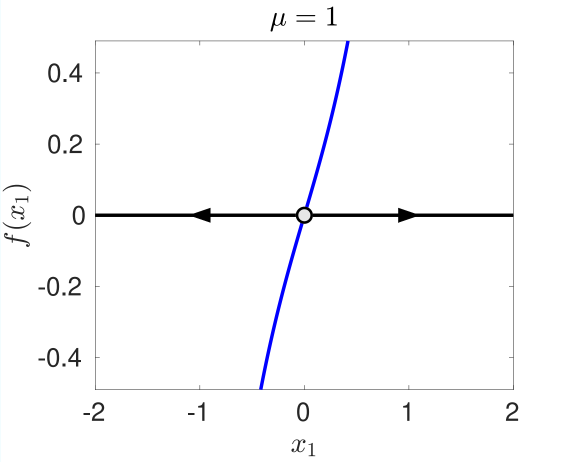


Subcritical pitchfork bifurcation

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \mu x_1 + x_1^3$

$$f(x_1;\mu) = \mu x_1 + x_1^3$$

$\mu > 0$:
one unstable fixed point at $x_1 = 0$



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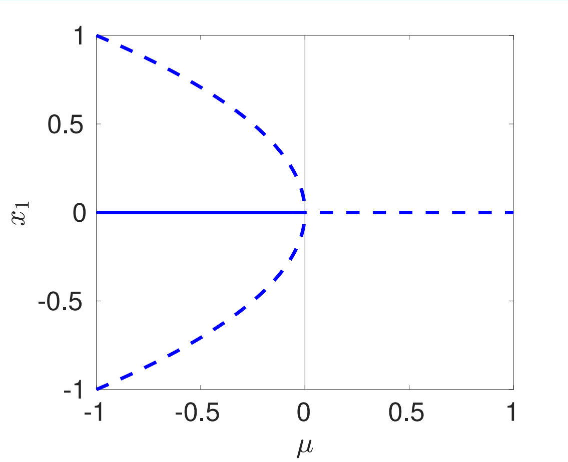
bifurcation diagram



Subcritical pitchfork bifurcation

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu x_1 + x_1^3$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -x_2$$

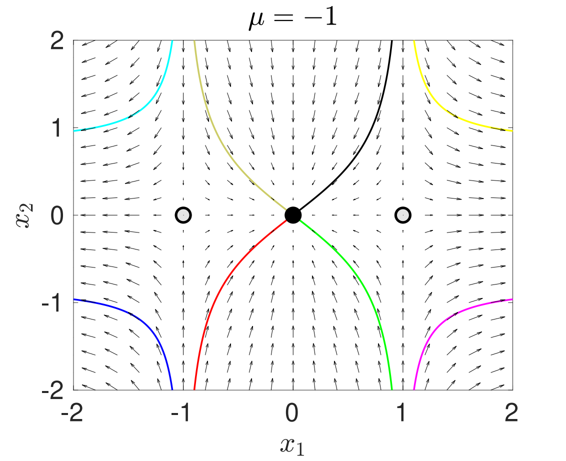
$\mu < 0$

three fixed points at

$\mathbf{x} = [-\sqrt{-\mu}, 0]$ (saddle)

$\mathbf{x} = [0, 0]$ (stable node)

$\mathbf{x} = [\sqrt{-\mu}, 0]$ (saddle)

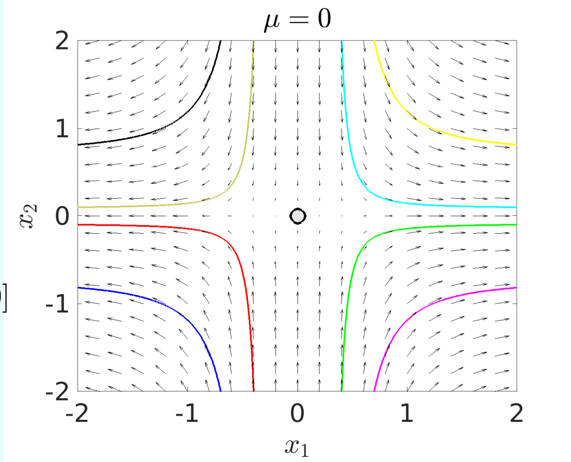


Subcritical pitchfork bifurcation

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu x_1 + x_1^3$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -x_2$$

as μ approaches zero from below,
two fixed points $[-\sqrt{-\mu}, 0]$ and $\sqrt{-\mu}, 0]$
move toward the third one

$\mu = 0$: the fixed points coalesce into
an unstable fixed point at $\mathbf{x} = [0, 0]$



Subcritical pitchfork bifurcation

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \mu x_1 + x_1^3$

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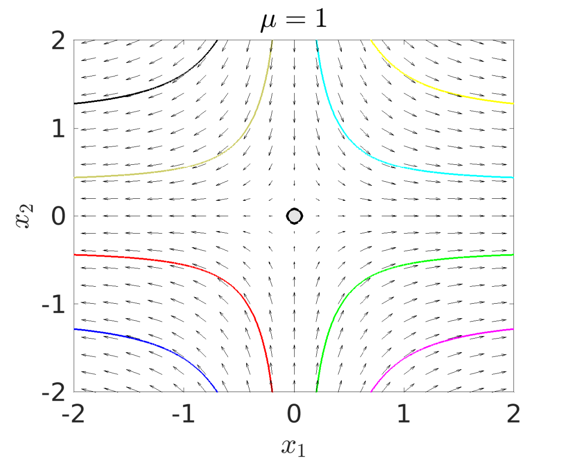

Subcritical pitchfork bifurcation

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu x_1 + x_1^3$$
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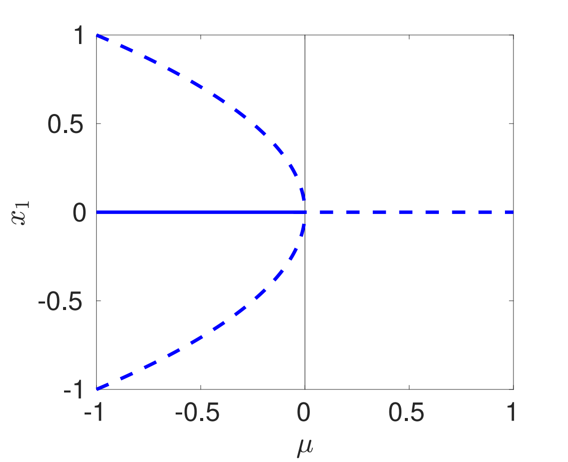
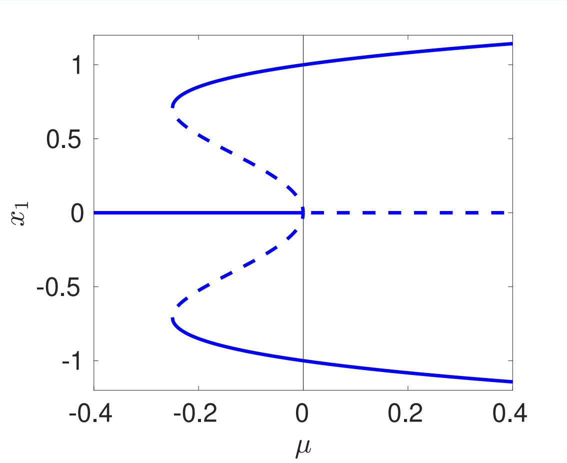
bifurcation diagram



Subcritical pitchfork bifurcation

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu x_1 + x_1^3 - x_1^5$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -x_2$$

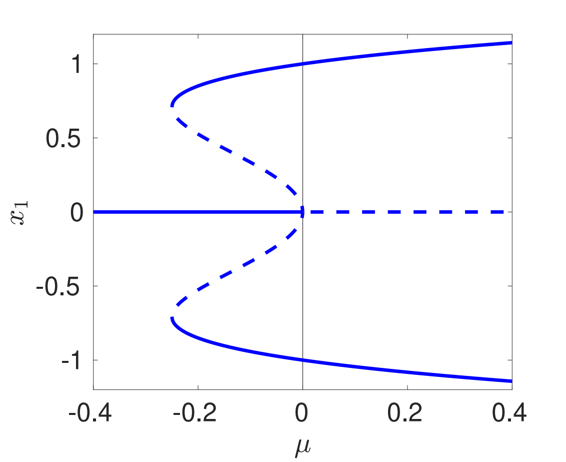
bifurcation diagram



Subcritical pitchfork bifurcation

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu x_1 + x_1^3 - x_1^5$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -x_2$$

bifurcation diagram



Other examples of ODEs with bifurcations:

Questions 1, 2, 5 and 6 on Problem Sheet 2

B5.6: Nonlinear Dynamics, Bifurcations and Chaos (Lecture 6)

- summary of Lecture 5: we discussed
  Introduction to bifurcation theory, covering saddle-node, transcritical, supercritical pitchfork and subcritical pitchfork bifurcations.
  (Questions 1, 2, 5 and 6 on Problem Sheet 2)
- today: we will continue in our discussion of Problem Sheet 2

B5.6: Nonlinear Dynamics, Bifurcations and Chaos (Lecture 6)

- summary of Lecture 5: we discussed
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  (Questions 1, 2, 5 and 6 on Problem Sheet 2)
- today: we will continue in our discussion of Problem Sheet 2
- course synopsis of **Lectures 1-8** (taken by both Part B and MSc students):
  Discrete-time (maps) and continuous-time (differential equations) dynamical systems. Notion of flows, stability of fixed points, Lyapunov function, invariant manifolds, stable manifold theorem, notion of hyperbolicity, center manifold. Chemical reaction networks. Stable, unstable and center subspaces. Poincaré-Bendixson theorem. Periodic solutions, stable and unstable limit cycles. Introduction to bifurcation theory, covering saddle-node, transcritical, supercritical pitchfork and subcritical pitchfork bifurcations. Extended center manifold. Logistic map. Periodic points of maps. Stability of N-cycles. Period-doubling bifurcation. Sharkovsky's theorem. Invariant distribution.

Extended center manifold

example: $$\frac{\mathsf{d}x_1}{\mathsf{d}t} = x_2^2 - x_1$$

$$\frac{\mathsf{d}x_2}{\mathsf{d}t} = \mu x_2 - x_1 x_2$$

Extended center manifold

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2^2 - x_1$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu x_2 - x_1 x_2$$

$[x_1, x_2] = [0, 0]$ is a critical point

linearized system $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$

for $M = \begin{pmatrix} -1 & 0 \\ 0 & \mu \end{pmatrix}$

Extended center manifold

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2^2 - x_1$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu x_2 - x_1 x_2$$
$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0$$

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the center manifold is given by

$$x_1 = h(x_2, \mu) = c_{20}\, x_2^2 + c_{11}\, \mu\, x_2 + c_{02}\, \mu^2 + \mathcal{O}(x_2^3, x_2^2\mu, x_2\mu^2, \mu^3)$$

Extended center manifold

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2^2 - x_1$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu x_2 - x_1 x_2$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0$$

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$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{\partial h}{\partial x_2}(x_2, \mu)\, \frac{\mathrm{d}x_2}{\mathrm{d}t} + \frac{\partial h}{\partial \mu}(x_2, \mu)\, \frac{\mathrm{d}\mu}{\mathrm{d}t}$$

Extended center manifold

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = x_2^2 - x_1$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu x_2 - x_1 x_2$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0$$

$[x_1, x_2, \mu] = [0, 0, 0]$ is a critical point

linearized system $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$

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the center manifold is given by

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$$x_2^2 - x_1 = \frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{\partial h}{\partial x_2}(x_2, \mu)\, \frac{\mathrm{d}x_2}{\mathrm{d}t} + \frac{\partial h}{\partial \mu}(x_2, \mu)\, \frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{\partial h}{\partial x_2}(x_2, \mu)\, (\mu x_2 - x_1 x_2)$$

Extended center manifold

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linearized system $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$

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$$x_2^2 - c_{20}\, x_2^2 - c_{11}\, \mu\, x_2 - c_{02}\, \mu^2 = (2\, c_{20}\, x_2 + c_{11}\, \mu)\, \mu x_2 + \mathcal{O}(x_2^3, x_2^2\mu, x_2\mu^2, \mu^3)$$

Extended center manifold

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2^2 - x_1$$
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$$c_{20} = 1, \qquad c_{11} = 0, \qquad c_{02} = 0$$

center manifold: $x_1 = x_2^2 + \mathcal{O}(x_2^3, x_2^2\mu, x_2\mu^2, \mu^3)$

the dynamics on the center manifold: $\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu x_2 - x_2^3 + \mathcal{O}(x_2^4, x_2^3\mu, x_2^2\mu^2, x_2\mu^3, \mu^4)$

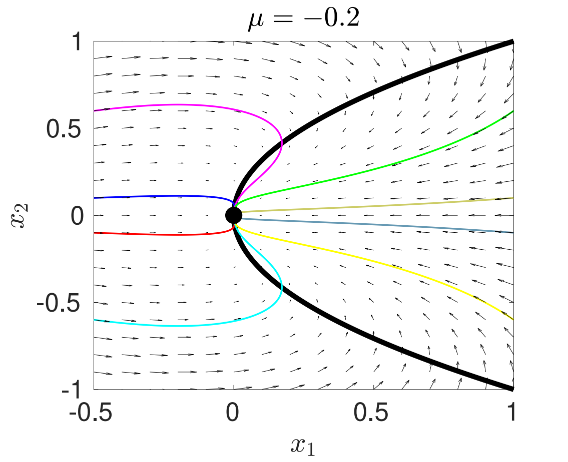
Extended center manifold

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2^2 - x_1$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu x_2 - x_1 x_2$$

center manifold:
$$x_1 = x_2^2 + \dots$$

dynamics on the center manifold:
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu x_2 - x_2^3 + \dots$$

supercritical pitchfork bifurcation



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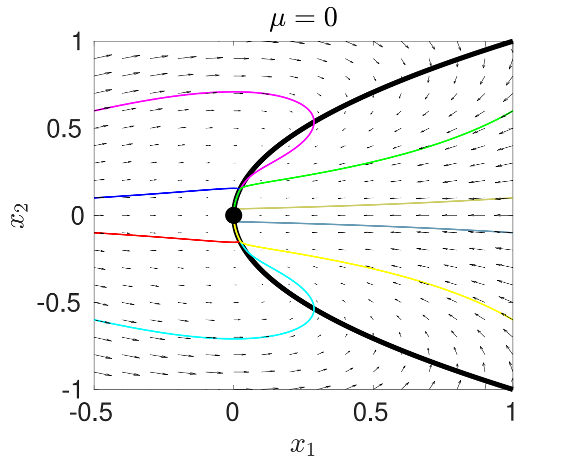

center manifold:

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dynamics on the center manifold:

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supercritical pitchfork bifurcation

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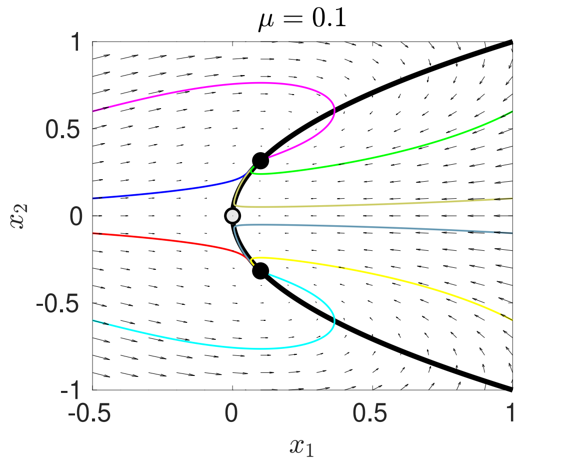

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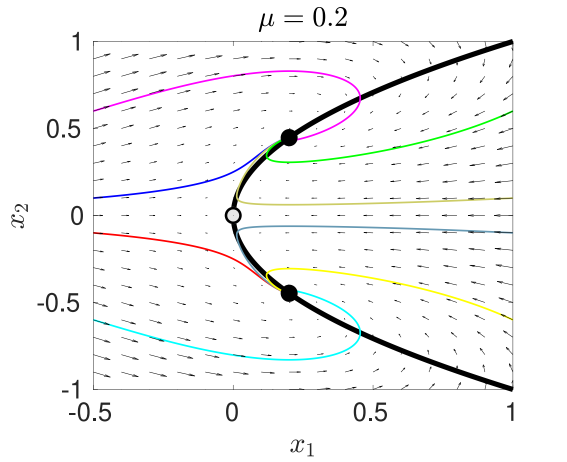
supercritical pitchfork bifurcation



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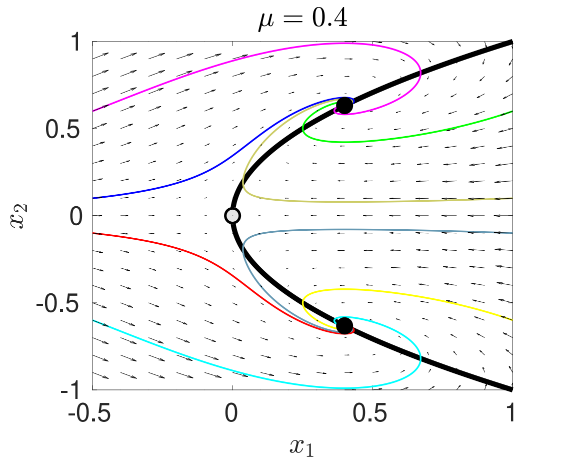
supercritical pitchfork bifurcation



Extended center manifold

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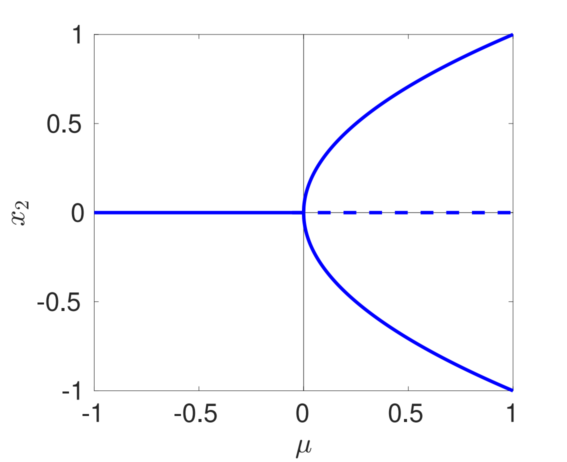

center manifold:
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supercritical pitchfork bifurcation

Other examples of the extended center manifold calculation in ODEs with bifurcations:
Questions 2, 5 and 6(e) on Problem Sheet 2

Bifurcations of continuous-time dynamical systems – summary

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x};\boldsymbol{\mu}) \quad \text{with the initial condition} \quad \mathbf{x}(0) = \mathbf{x}_0 \in \Omega$$

We have discussed bifurcations of fixed points, which can occur for $n \geq 1$ and $m \geq 1$ (so, they can be explained on examples with $n = 1$ and $m = 1$):

- saddle-node bifurcation
- transcritical bifurcation
- pitchfork bifurcation (supercritical, subcritical)

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We have discussed bifurcations of fixed points, which can occur for $n \geq 1$ and $m \geq 1$ (so, they can be explained on examples with $n = 1$ and $m = 1$):

- saddle-node bifurcation
- transcritical bifurcation
- pitchfork bifurcation (supercritical, subcritical)

We will discuss later in the course:

- bifurcations of limit cycles $(n > 1)$
- bifurcations with more than one parameter $(m > 1)$

Next, we will discuss bifurcations of discrete-time dynamical systems.

Fixed points

Discrete-time dynamical system: Let $\mathbf{F} : \Omega \times \Theta \to \Omega$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$.
Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}_k \in \Omega$ be defined iteratively by

$$\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k; \boldsymbol{\mu})$$

Fixed points

Discrete-time dynamical system: Let $\mathbf{F} : \Omega \times \Theta \to \Omega$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$.
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- $\boldsymbol{\alpha} \in \Omega$ is a *fixed point* if $\boldsymbol{\alpha} = \mathbf{F}(\boldsymbol{\alpha}; \boldsymbol{\mu})$

Fixed points

Discrete-time dynamical system: Let $\mathbf{F} : \Omega \times \Theta \to \Omega$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}_k \in \Omega$ be defined iteratively by

$$\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k; \boldsymbol{\mu})$$

- $\boldsymbol{\alpha} \in \Omega$ is a *fixed point* if $\boldsymbol{\alpha} = \mathbf{F}(\boldsymbol{\alpha}; \boldsymbol{\mu})$
- fixed point $\boldsymbol{\alpha}$ is *stable* if

  $$\forall \varepsilon > 0 \, \exists \delta > 0 \text{ such that } \forall \mathbf{x}_0 \in B_\delta(\boldsymbol{\alpha}) \text{ and } k \in \mathbb{N}_0 \text{ we have } \mathbf{x}_k \in B_\varepsilon(\boldsymbol{\alpha})$$

  where the open ball of radius r is defined by $B_r(\boldsymbol{\alpha}) = \left\{ \mathbf{x} \in \mathbb{R}^n \,\middle|\, \|\mathbf{x} - \boldsymbol{\alpha}\| < r \right\}$

Fixed points

Discrete-time dynamical system: Let $\mathbf{F} : \Omega \times \Theta \to \Omega$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}_k \in \Omega$ be defined iteratively by

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- fixed point $\boldsymbol{\alpha}$ is *asymptotically stable* if (i) it is stable; and
  (ii) $\exists \delta > 0$ such that $\forall \mathbf{x}_0 \in B_\delta(\boldsymbol{\alpha})$ we have $\lim\limits_{k\to\infty} \mathbf{x}_k = \boldsymbol{\alpha}$

Fixed points

Discrete-time dynamical system: Let $\mathbf{F} : \Omega \times \Theta \to \Omega$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}_k \in \Omega$ be defined iteratively by

$$\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k; \boldsymbol{\mu})$$

- $\boldsymbol{\alpha} \in \Omega$ is a *fixed point* if $\boldsymbol{\alpha} = \mathbf{F}(\boldsymbol{\alpha}; \boldsymbol{\mu})$
- fixed point $\boldsymbol{\alpha}$ is *stable* if

  $\forall \varepsilon > 0 \, \exists \delta > 0$ such that $\forall \mathbf{x}_0 \in B_\delta(\boldsymbol{\alpha})$ and $k \in \mathbb{N}_0$ we have $\mathbf{x}_k \in B_\varepsilon(\boldsymbol{\alpha})$

  where the open ball of radius r is defined by $B_r(\boldsymbol{\alpha}) = \left\{ \mathbf{x} \in \mathbb{R}^n \,\middle|\, \|\mathbf{x} - \boldsymbol{\alpha}\| < r \right\}$
- fixed point $\boldsymbol{\alpha}$ is *asymptotically stable* if (i) it is stable; and
  (ii) $\exists \delta > 0$ such that $\forall \mathbf{x}_0 \in B_\delta(\boldsymbol{\alpha})$ we have $\lim\limits_{k \to \infty} \mathbf{x}_k = \boldsymbol{\alpha}$
- Prelims Constructive Mathematics: we considered $n = 1$ where $x_{k+1} = F(x_k)$
  - $\alpha \in \mathbb{R}$ is a fixed point if $\alpha = F(\alpha)$
  - if $|F'(\alpha)| < 1$, then α is asymptotically stable

Example

$$x_{k+1} = 1 - 6\,x_k + 15\,x_k^2 - 10\,x_k^3$$

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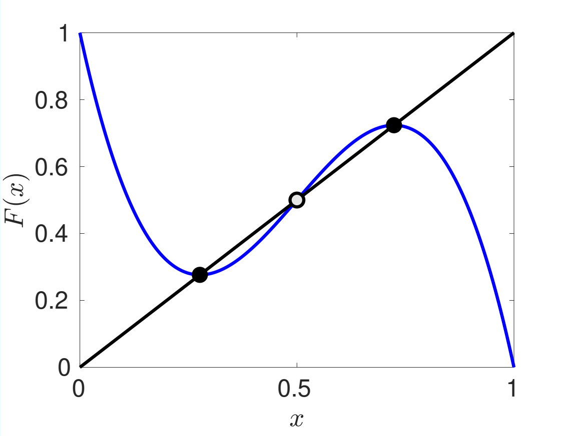

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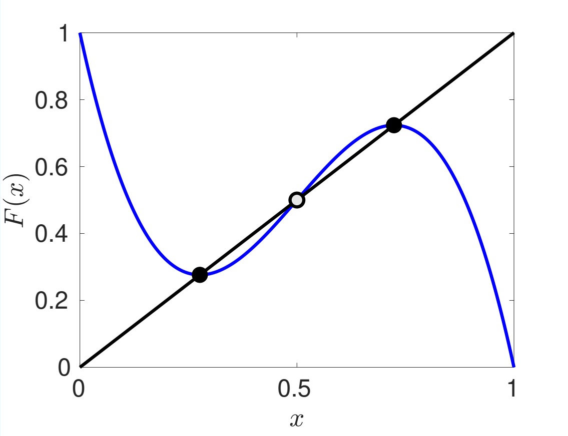

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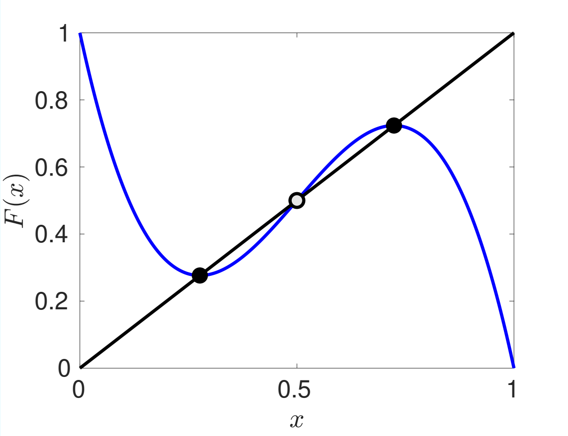

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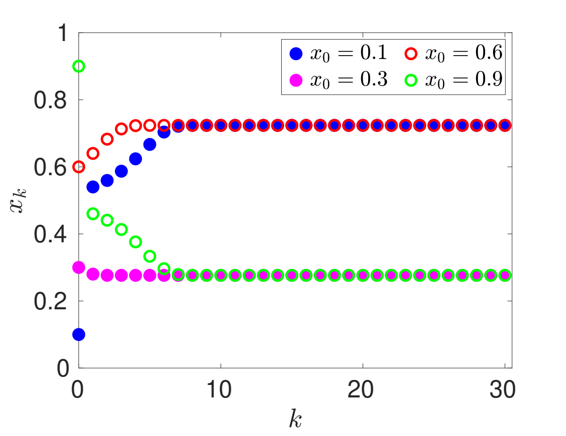

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Fixed points

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- to find periodic points and the corresponding N-cycles, we need to solve $\boldsymbol{\alpha} = \mathbf{F}_{\boldsymbol{\mu}}^{(N)}(\boldsymbol{\alpha})$ and we also need to exclude solutions with some lesser period

  Question 4 on Problem Sheet 2

Fixed points, periodic points, N-cycles, orbits and bifurcations

Discrete-time dynamical system: Let $\mathbf{F} : \Omega \times \Theta \to \Omega$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}_k \in \Omega$ be defined iteratively by

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- if $\mathbf{x}_0$ is a periodic point with period N, then its orbit is a finite set (N-cycle)
- if orbit is a finite set, then it is (eventually) periodic, i.e. there exists $j \in \mathbb{N}_0$ such that $\mathbf{F}_{\boldsymbol{\mu}}^{(j)}(\mathbf{x}_0)$ is a *periodic point* with *period* $N \in \mathbb{N}$

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i.e. there exists $j \in \mathbb{N}_0$ such that $\mathbf{F}_{\boldsymbol{\mu}}^{(j)}(\mathbf{x}_0)$ is a *periodic point* with *period* $N \in \mathbb{N}$
- if orbit is an infinite set, then it can approach a fixed point or an N-cycle, or it can be chaotic - we will illustrate this on examples with $n = 1$ and $m = 1$
- bifurcations: the qualitative behaviour of orbits can change as parameters $\boldsymbol{\mu}$ are varied (for example, fixed points or N-cycles can be created or destroyed, or their stability changes); the parameter values at which these qualitative changes in the dynamics occur are called bifurcation points

Example

$$x_{k+1} = (1 - x_k)(1 - 5\, x_k + \mu\, x_k^2)$$

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fixed points: $F(x;\mu) = x$

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our previous example: $\mu = 10$

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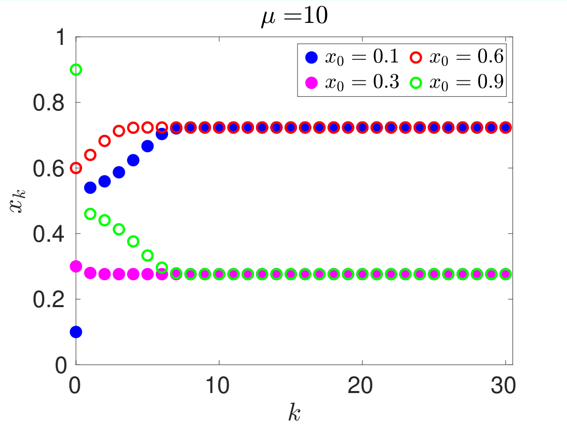

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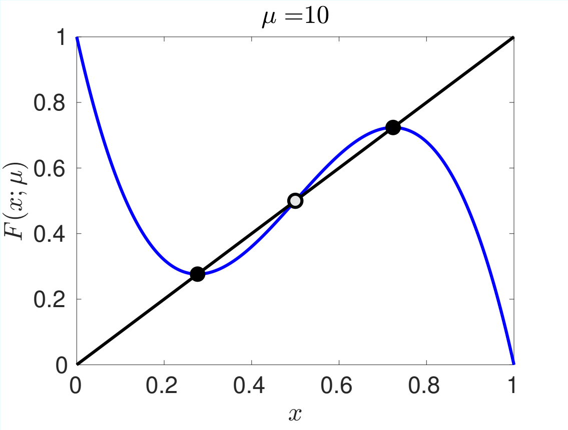

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If $\mu \in \Theta = [6.3, 11.8]$,
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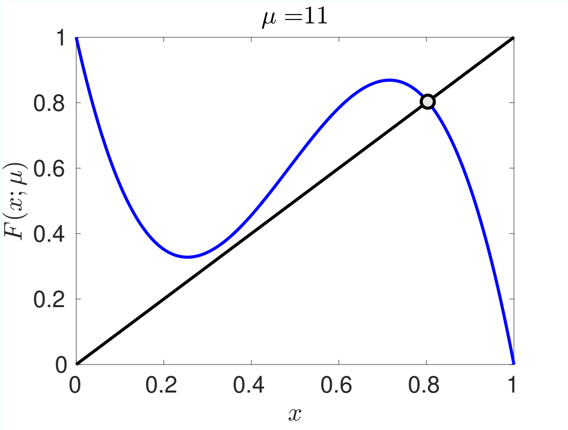
We study dynamics and bifurcations of $F : \Omega \times \Theta \to \Omega$, where $\Omega = [0,1]$.



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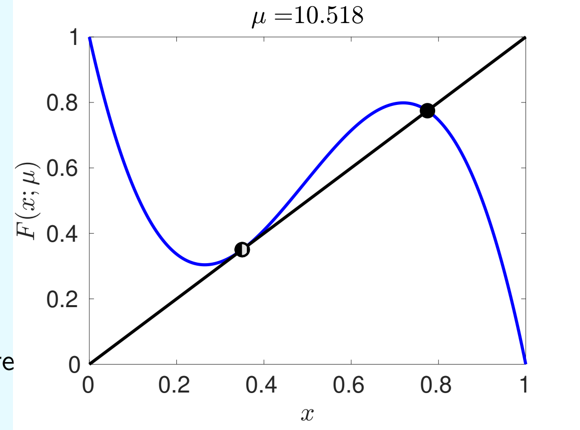
three fixed points for $\mu \in (\mu_1, \mu_2)$ where $\mu_1 = 9.7066\ldots$ and $\mu_2 = 10.518\ldots$

one fixed point for $\mu < \mu_1$ and $\mu > \mu_2$



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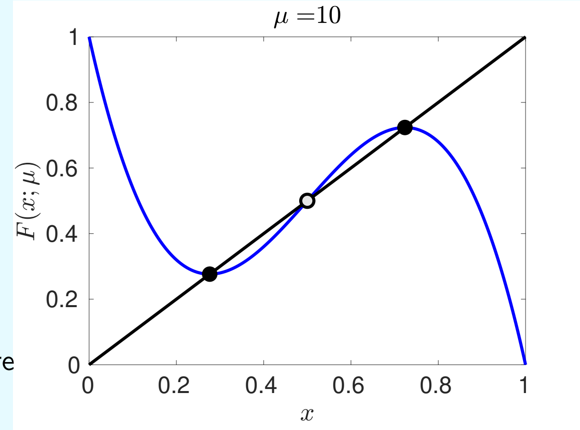

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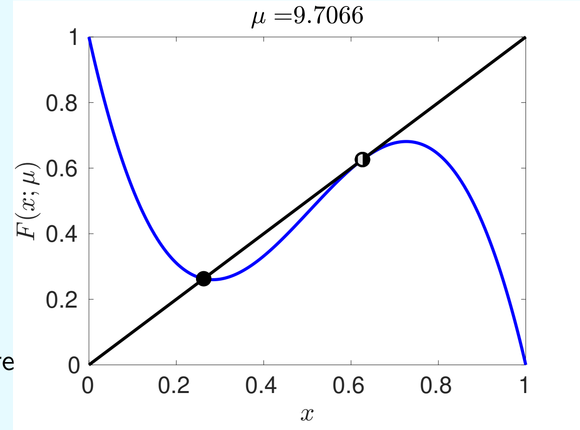

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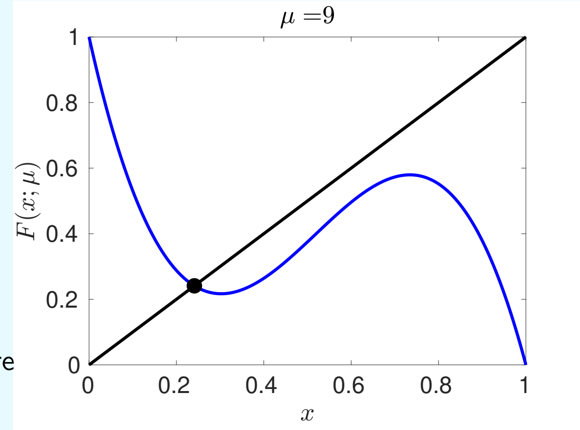

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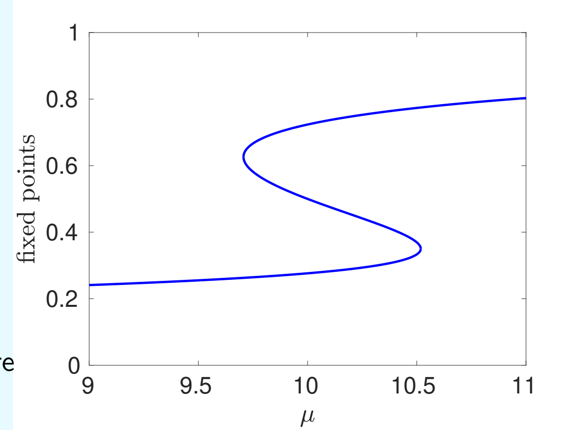

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$$\mu(x) = \frac{7x - 5x^2 - 1}{(1 - x)\,x^2}$$

μ_1 and μ_2 can be found by solving $0 = \mu'(x) = \dfrac{2 - 10x + 14x^2 - 5x^3}{(1-x)^2\,x^3} = 0$

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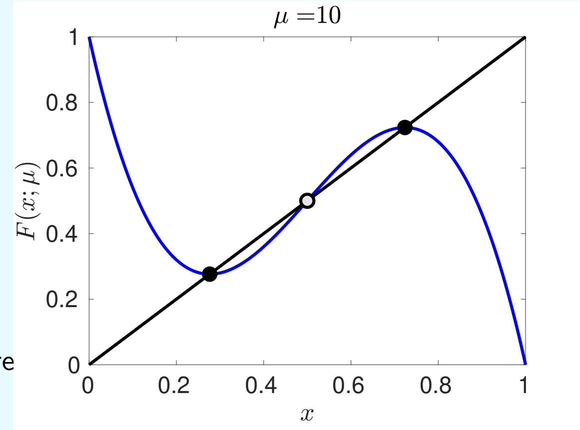

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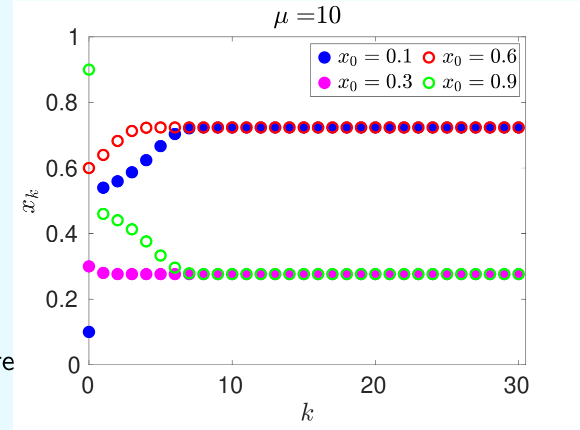

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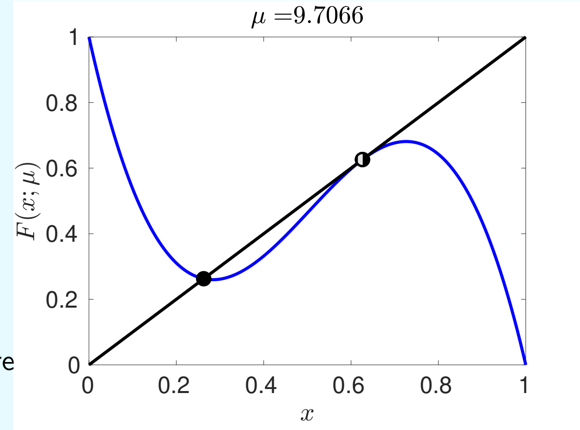

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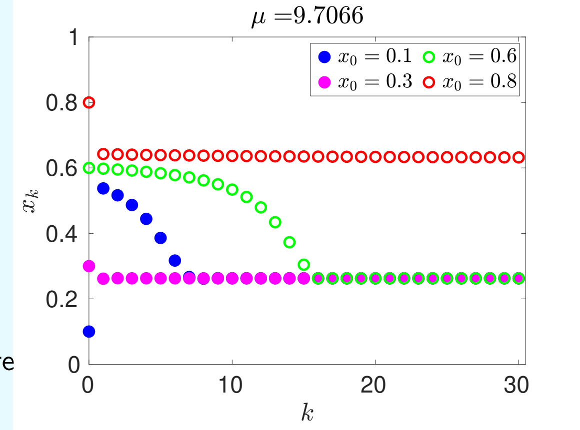

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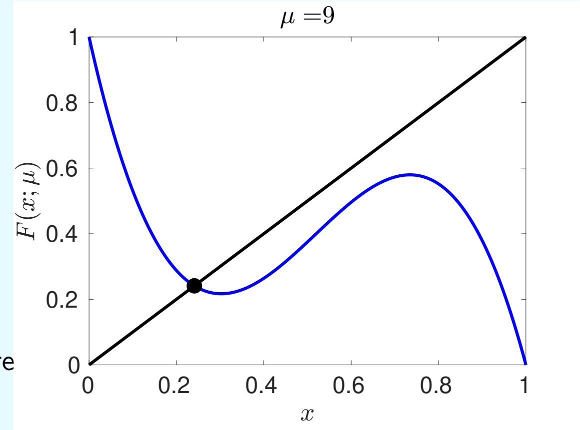

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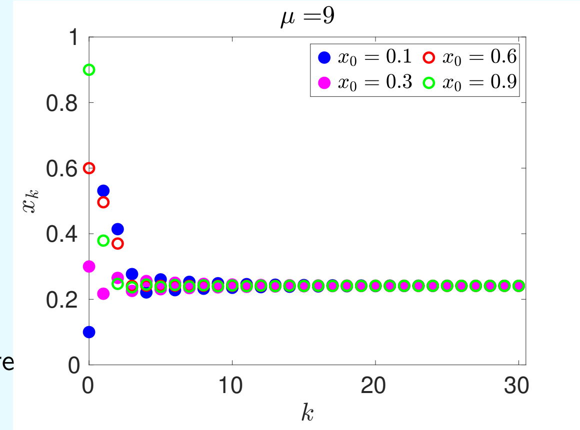

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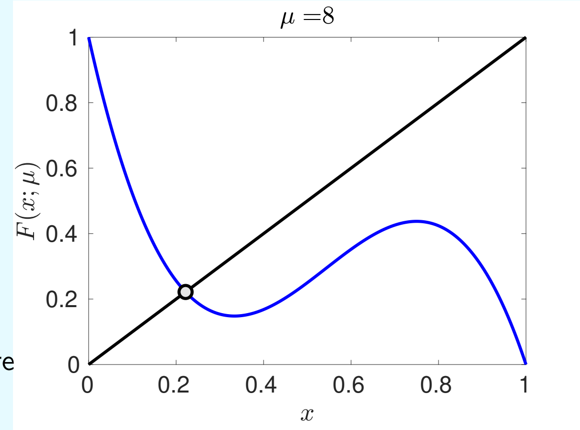

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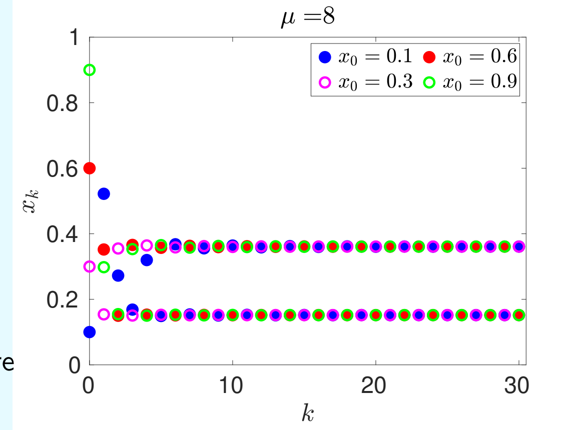

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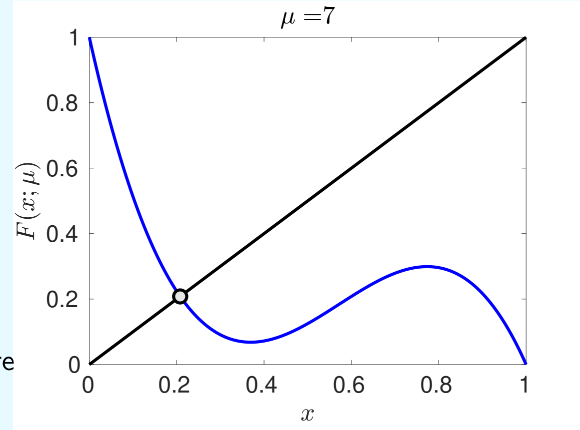

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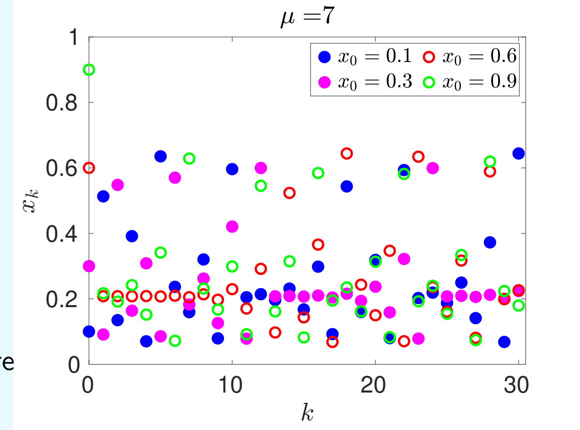

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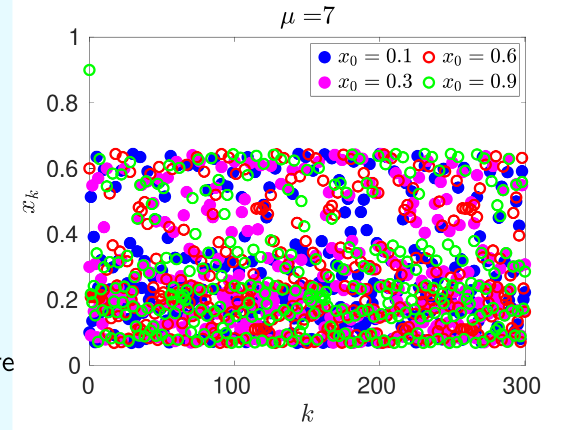

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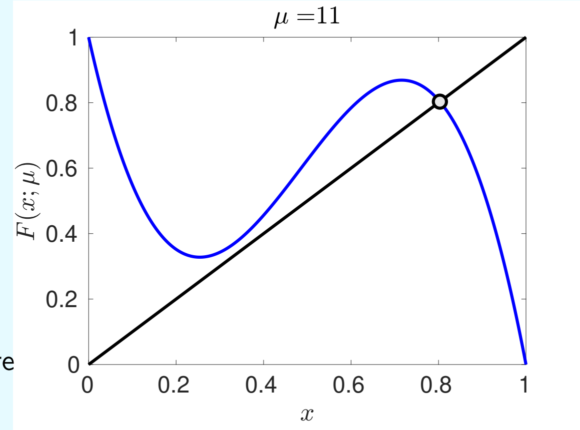

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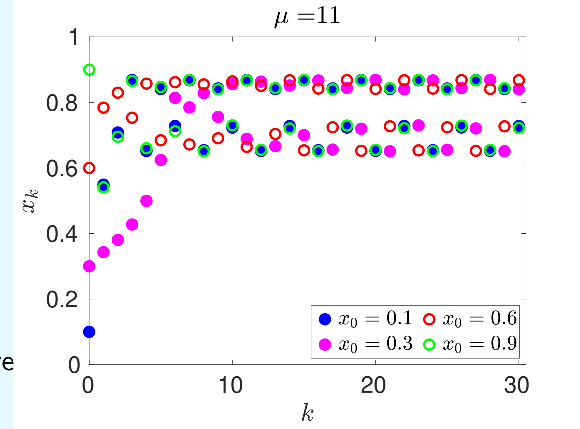

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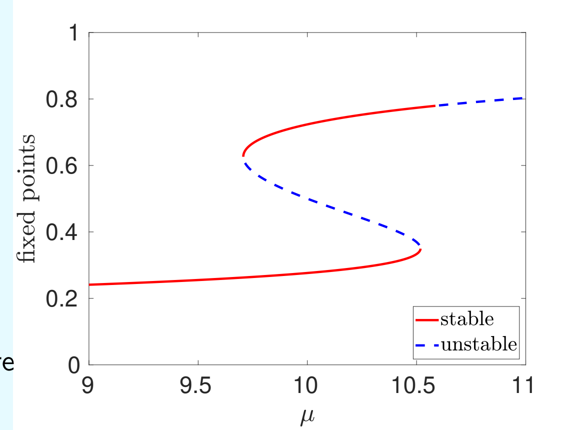

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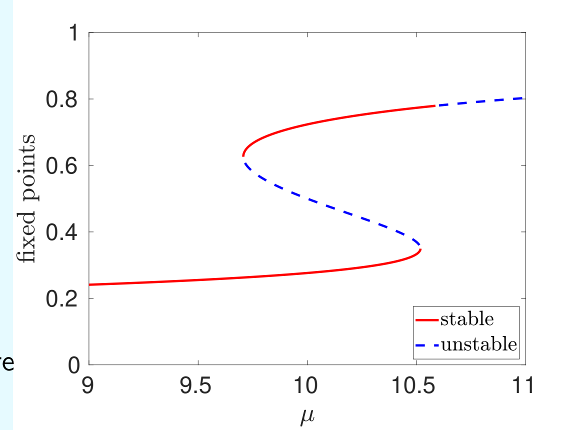

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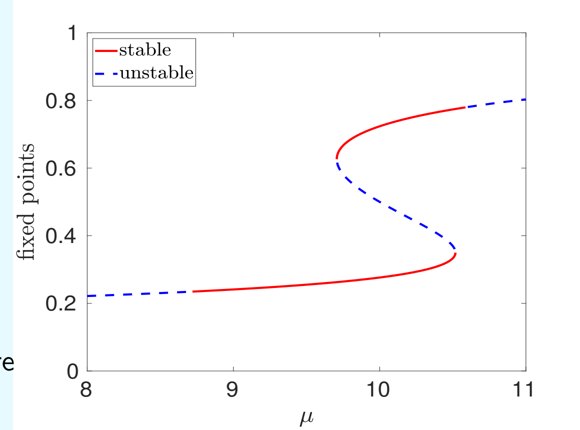

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B5.6: Nonlinear Dynamics, Bifurcations and Chaos (Lecture 7)

- summary of Lecture 6: we discussed
  Extended center manifold. Discrete-time (maps) dynamical systems. Fixed points. Periodic points of maps. N-cycles.
  (Questions 3, 4, 5 and 6 on Problem Sheet 2)
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- course synopsis of **Lectures 1-8** (taken by both Part B and MSc students):
  Discrete-time (maps) and continuous-time (differential equations) dynamical systems. Notion of flows, stability of fixed points, Lyapunov function, invariant manifolds, stable manifold theorem, notion of hyperbolicity, center manifold. Chemical reaction networks. Stable, unstable and center subspaces. Poincaré-Bendixson theorem. Periodic solutions, stable and unstable limit cycles. Introduction to bifurcation theory, covering saddle-node, transcritical, supercritical pitchfork and subcritical pitchfork bifurcations. Extended center manifold. Logistic map. Periodic points of maps. Stability of N-cycles. Period-doubling bifurcation. Sharkovsky's theorem. Invariant distribution.

Stability of N-cycles

Discrete-time dynamical system ($n = 1$, $m = 1$):
Let $F : \Omega \times \Theta \to \Omega$, where $\Omega \subset \mathbb{R}$ and $\Theta \subset \mathbb{R}$.
Let $x_0 \in \Omega$, $\mu \in \Theta$ and $x_k \in \Omega$ be defined iteratively by

$$x_{k+1} = F(x_k; \mu) = F_\mu(x_k)$$

- $\alpha \in \Omega$ is a *periodic point* with *period* $N \in \mathbb{N}$ if
  $$\alpha = F_\mu^{(N)}(\alpha) \text{ and } \alpha \neq F_\mu^{(k)}(\alpha) \text{ for } k = 1, 2, \ldots, N-1$$
  and the set $\left\{\alpha, F_\mu(\alpha), F_\mu^{(2)}(\alpha), \ldots, F_\mu^{(N-1)}(\alpha)\right\}$ is called an N-*cycle*

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- N-cycle is asymptotically stable if α is an asymptotically stable fixed point of $F_\mu^{(N)}$

Stability of N-cycles

Discrete-time dynamical system ($n = 1$, $m = 1$):
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Example: logistic map $x_{k+1} = \mu\, x_k\, (1 - x_k)$

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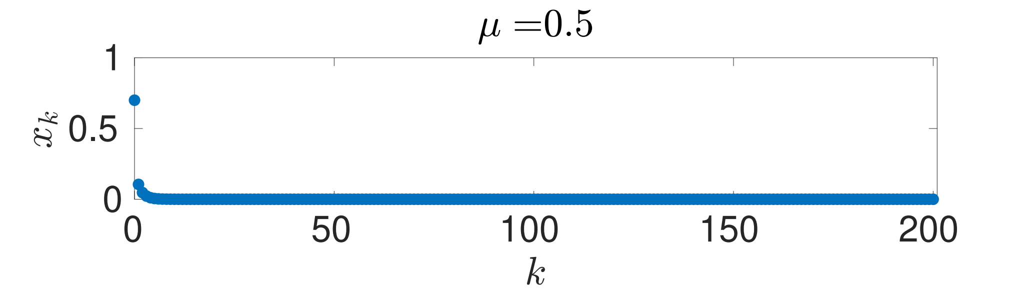

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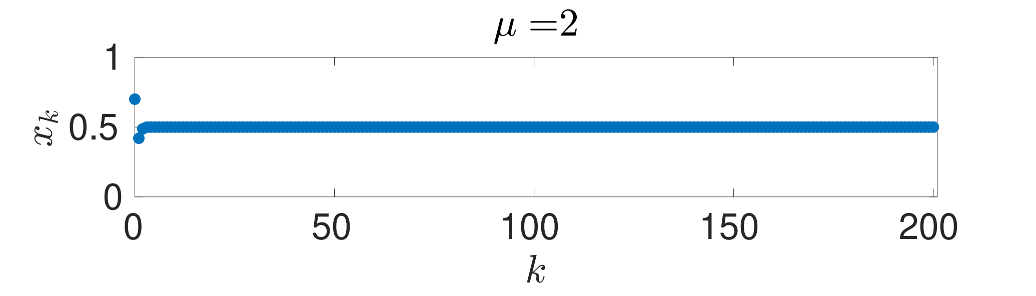

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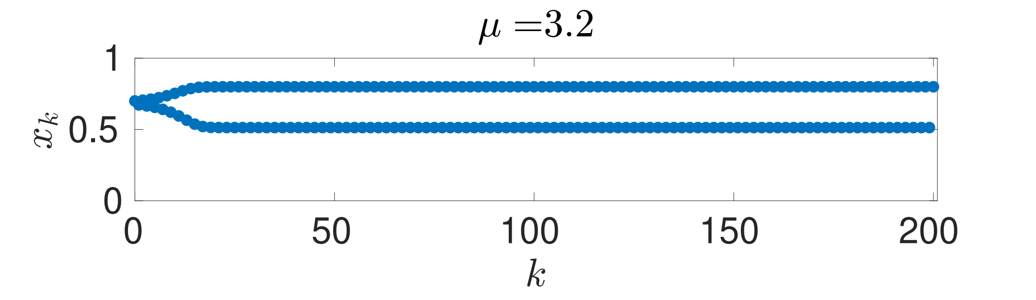

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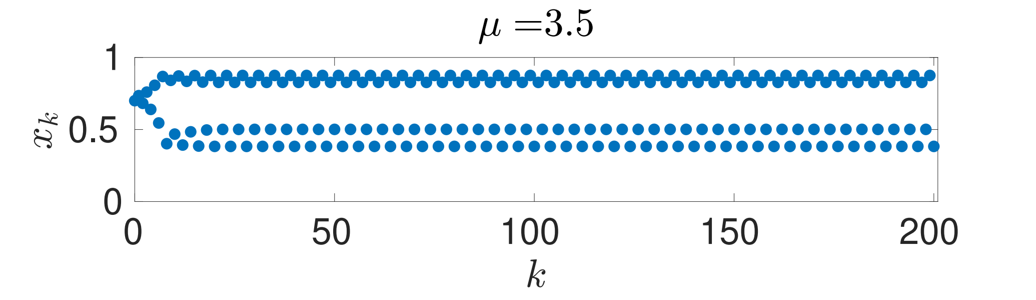

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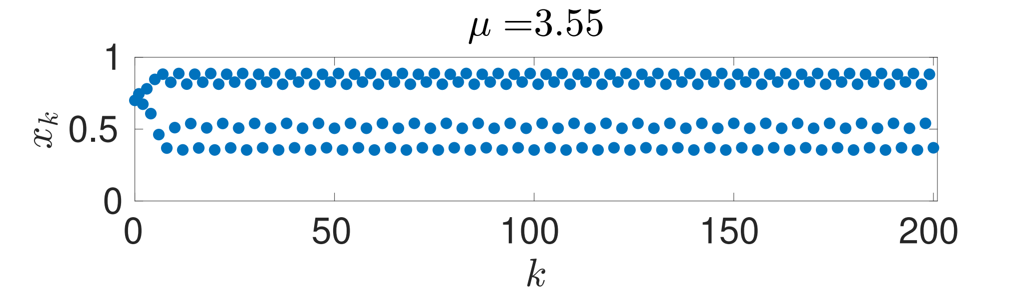

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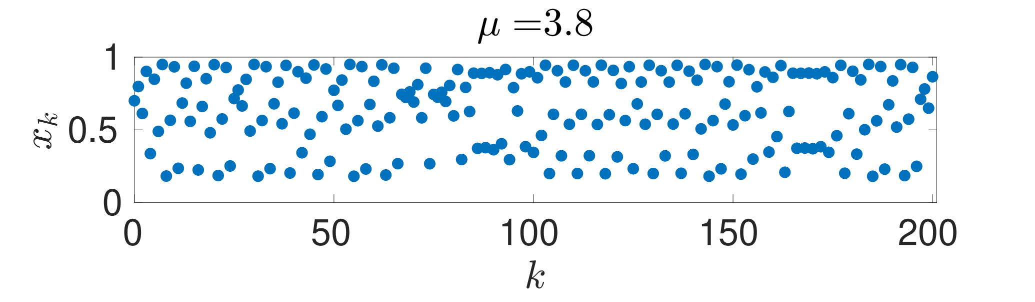

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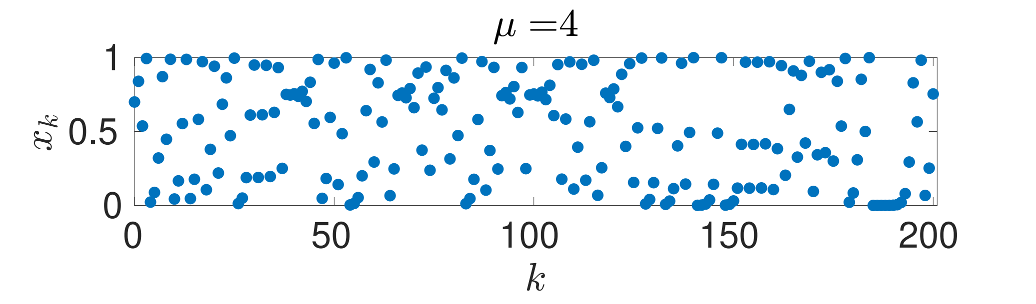

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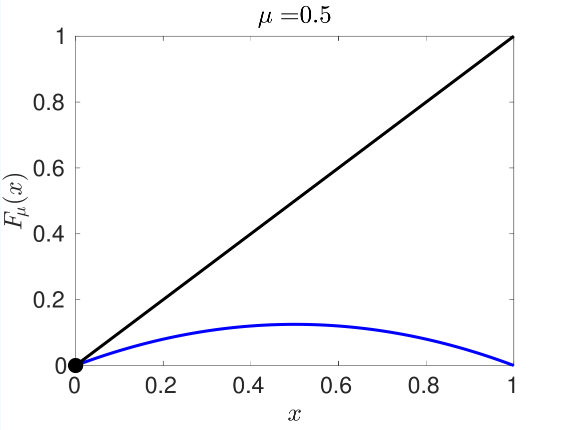
α_1 is asymptotically stable for $\mu \in [0, 1]$



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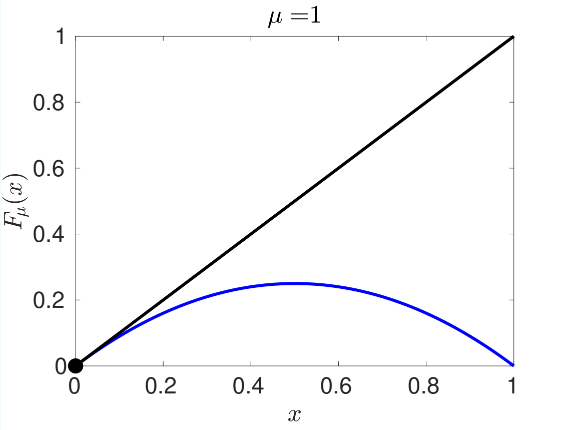

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$\alpha_1 = 0$ is unstable for $\mu > 1$

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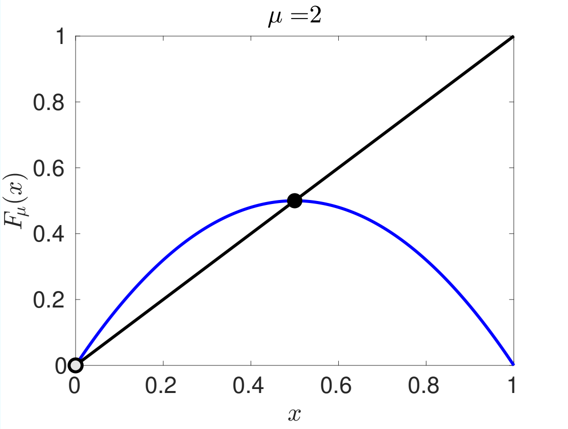

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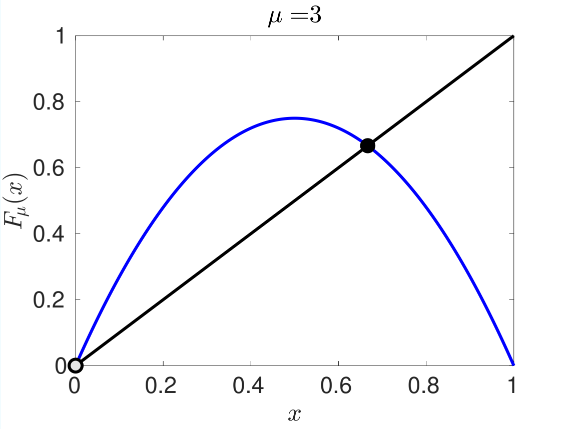

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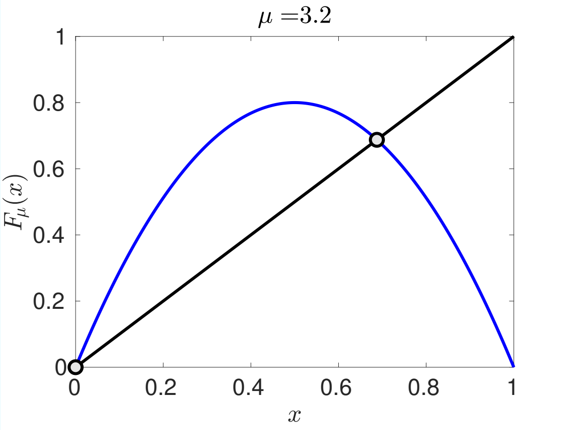

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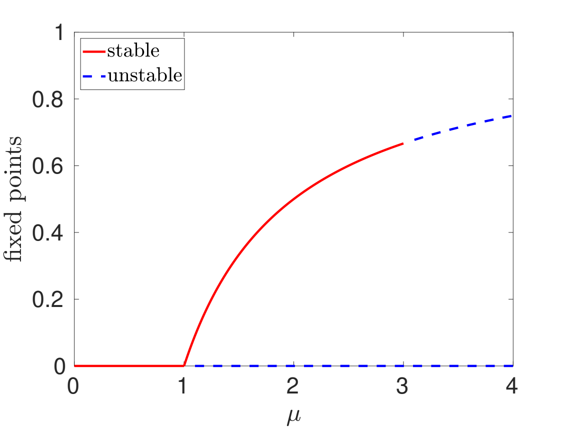

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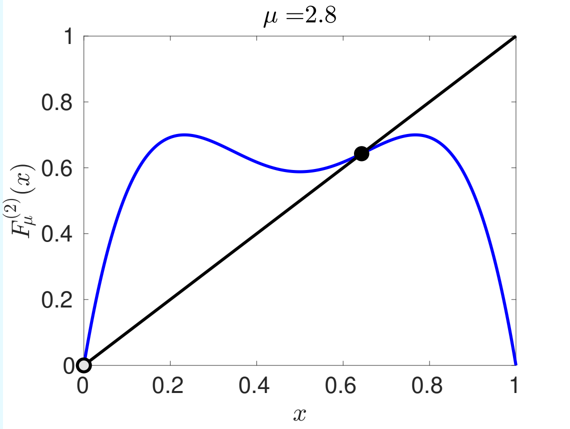
2-cycles: solve $x = F_\mu^{(2)}(x)$

$$x = \mu^2\, x\,(1-x)(1 - \mu\, x\,(1-x))$$



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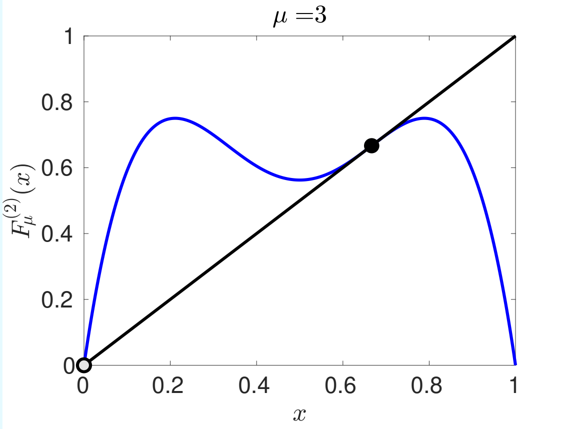

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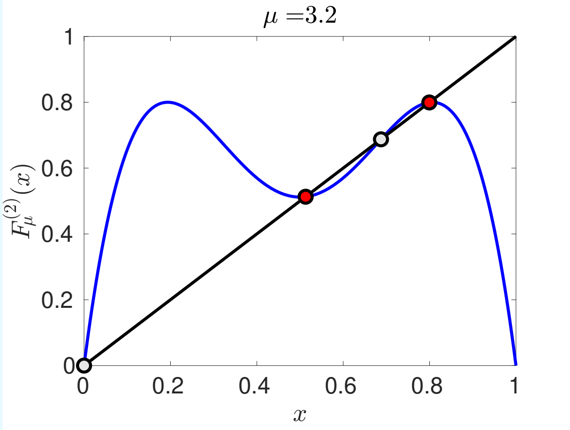

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2-cycle is asymptotically stable for $\mu \in \left(3, 1+\sqrt{6}\right]$

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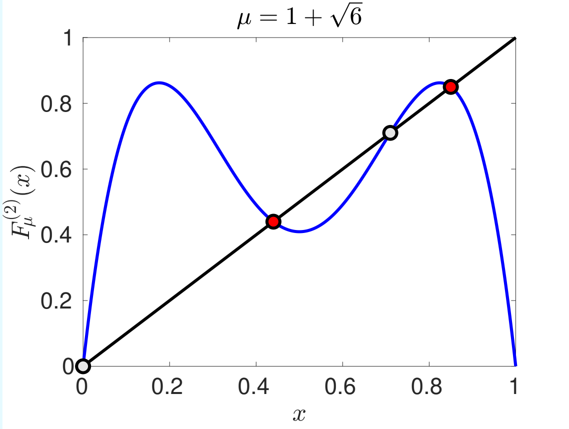

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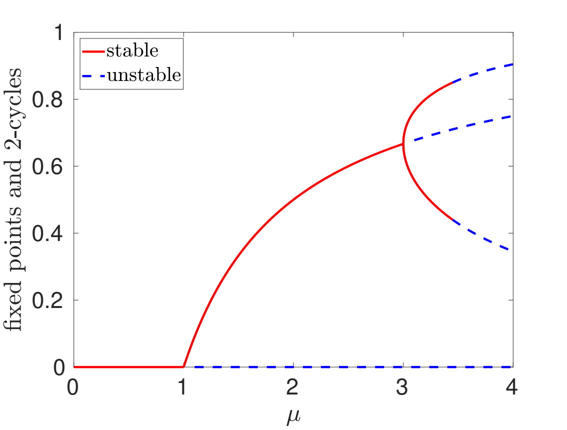

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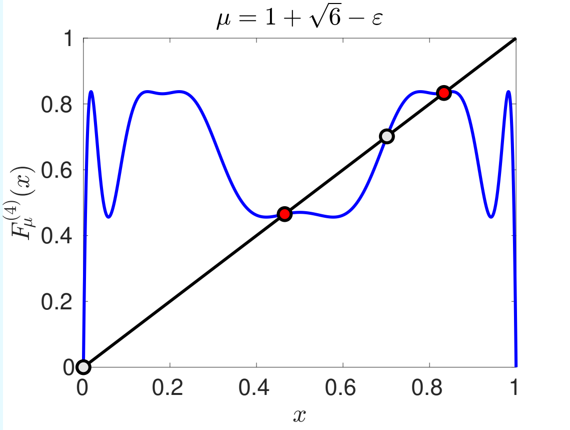

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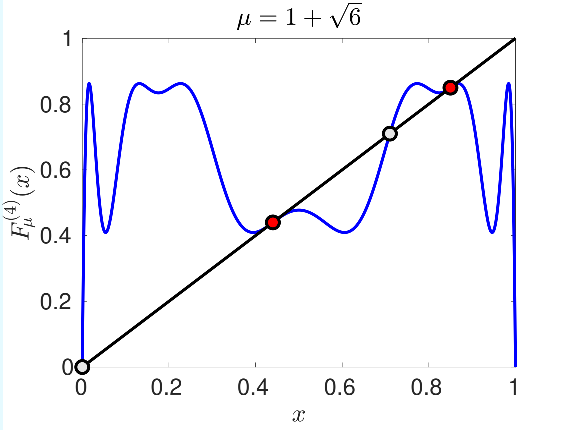

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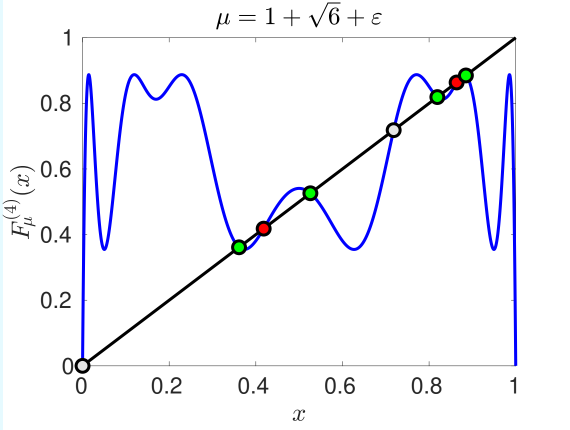
4-cycle exists and is asymptotically stable for $\mu \in \left(1+\sqrt{6},\, 3.544090\ldots\right]$



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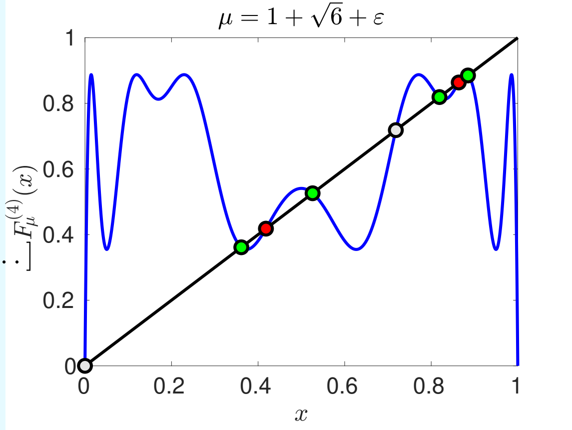
8-cycle exists and is asymptotically stable for $\mu \in (3.544090\ldots,\, 3.564407\ldots]$

this is called the period doubling route to chaos

additional example:
Question 3 on Problem Sheet 2



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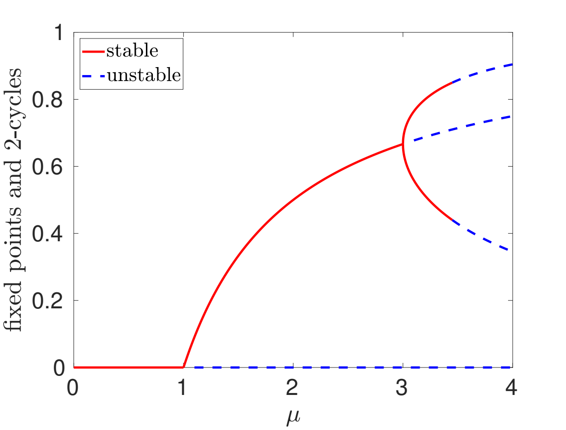
bifurcation diagram



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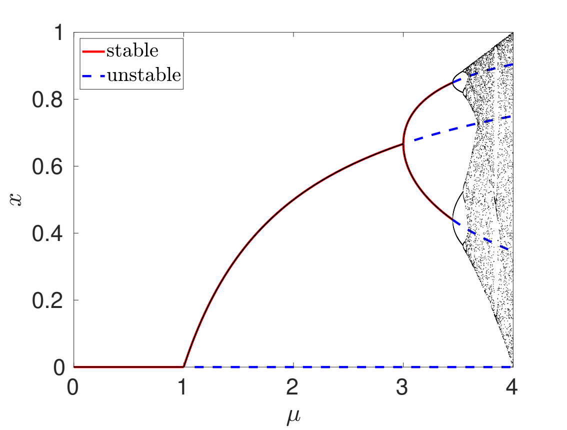
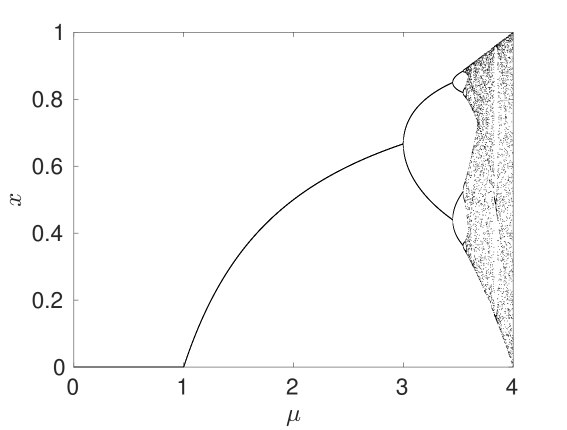


Example: logistic map $x_{k+1} = \mu\, x_k\,(1 - x_k)$

$F : \Omega \times \Theta \to \Omega$, where $\Omega = [0, 1]$, $\Theta = [0, 4]$ and $F(x; \mu) = F_\mu(x) = \mu\, x\,(1 - x)$

bifurcation diagram



α_2 is asymptotically stable for $\mu \in (1, 3]$

asymptotically stable 2-cycle exists for $\mu \in \left(3, 1 + \sqrt{6}\right]$

asymptotically stable 4-cycle exists for $\mu \in \left(1 + \sqrt{6},\, 3.544090\ldots\right]$

asymptotically stable 8-cycle exists for $\mu \in (3.544090\ldots,\, 3.564407\ldots]$

16-cycle, 32-cycle, 64-cycle, ...

period doubling route to chaos

Example: logistic map $x_{k+1} = \mu\, x_k\, (1 - x_k)$

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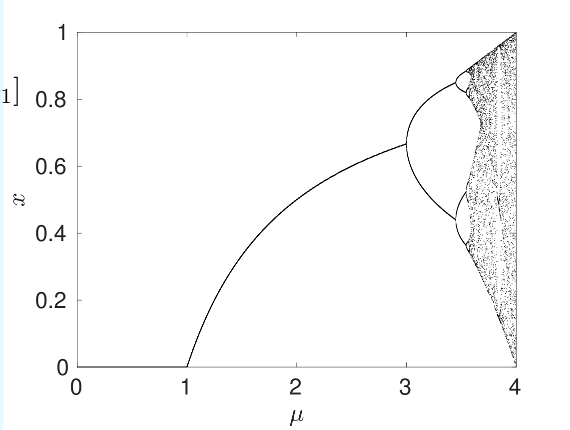


α_2 is asymptotically stable for $\mu \in (1, b_1]$

asymptotically stable 2-cycle exists for $\mu \in (b_1, b_2]$

asymptotically stable 2^k-cycle exists for $\mu \in (b_k, b_{k+1}]$

Feigenbaum's constant:

$$\lim_{k \to \infty} \frac{b_k - b_{k-1}}{b_{k+1} - b_k} = 4.6692016\ldots$$

additional example: Question 3 on Problem Sheet 2

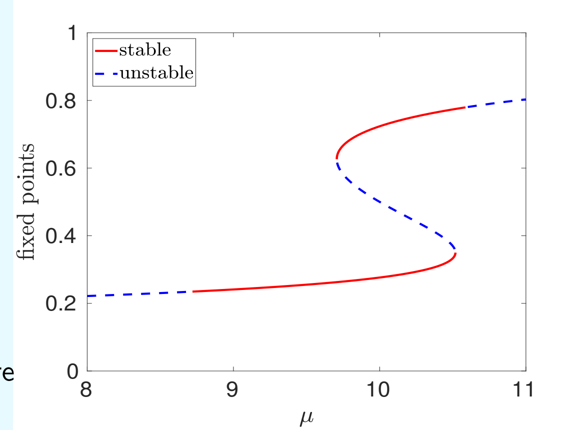
Example (from Lecture 6)

$$x_{k+1} = (1 - x_k)(1 - 5\,x_k + \mu\,x_k^2)$$

$$F(x;\mu) = (1 - x)(1 - 5\,x + \mu\,x^2)$$

If $\mu \in \Theta = [6.3, 11.8]$,
then $F(x;\mu) \in [0,1]$ for all $x \in [0,1]$.

We have studied dynamics
of $F : \Omega \times \Theta \to \Omega$, where $\Omega = [0,1]$.



three fixed points for $\mu \in (\mu_1, \mu_2)$ where
$\mu_1 = 9.7066\ldots$ and $\mu_2 = 10.518\ldots$

one fixed point for $\mu < \mu_1$ and $\mu > \mu_2$

we have saddle-node bifurcations at $\mu = \mu_1$ and $\mu = \mu_2$

we also have period doubling bifurcations at $\mu \approx 8.71988\ldots$ and $\mu \approx 10.5877\ldots$

Example (from Lecture 6) – bifurcation diagram

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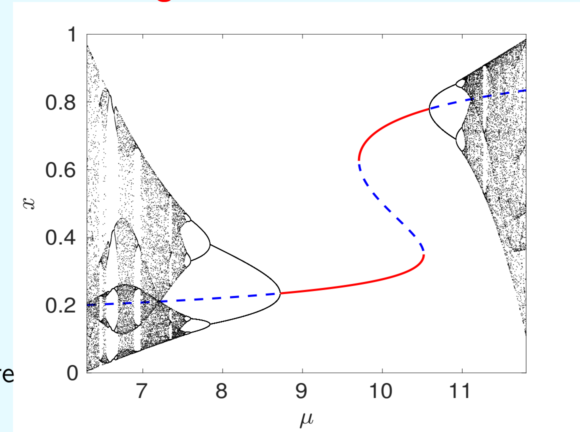

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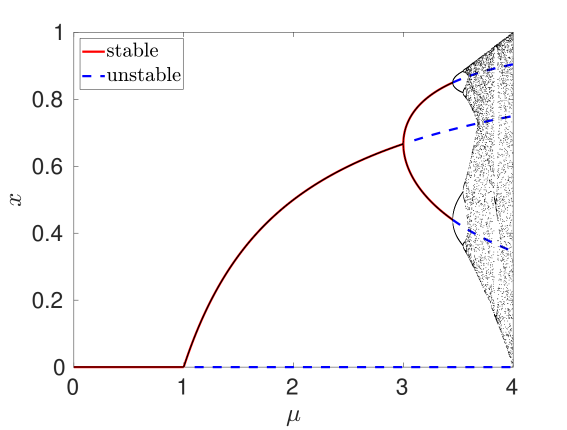
3-cycles: logistic map $\quad x_{k+1} = \mu\, x_k\,(1 - x_k)$

asymptotically stable 2^k-cycle exists
for $\mu \in (b_k, b_{k+1}]$ where

$$b_1 = 3$$
$$b_2 = 1 + \sqrt{6}$$
$$b_3 = 3.544090\ldots$$
$$b_4 = 3.564407\ldots$$

$$\lim_{k\to\infty} b_k = 3.56994567\ldots$$



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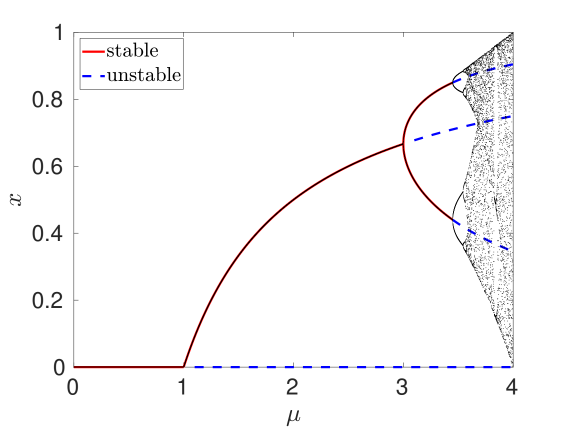

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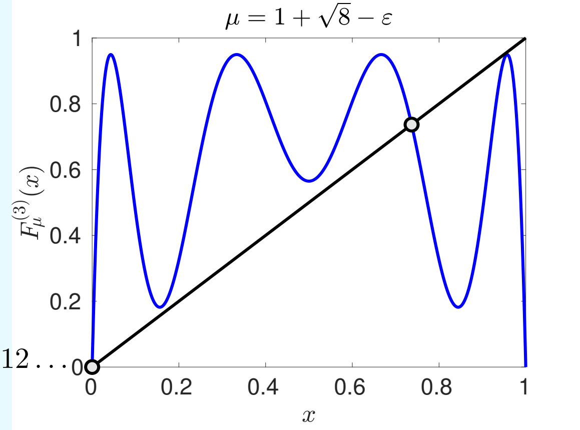

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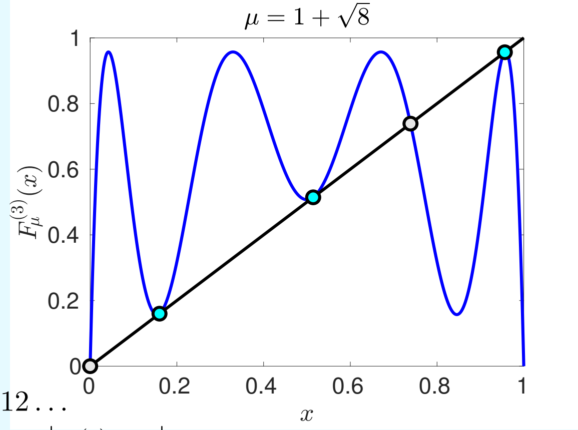

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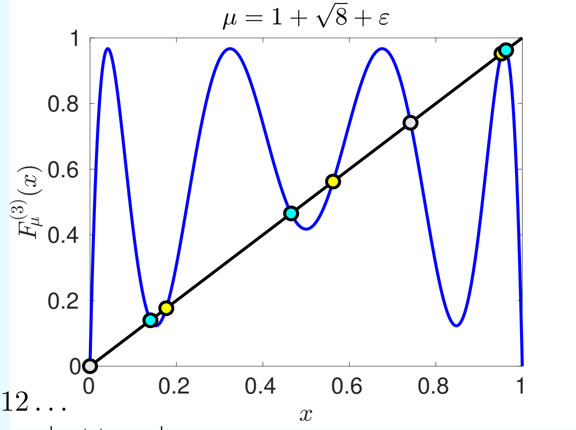

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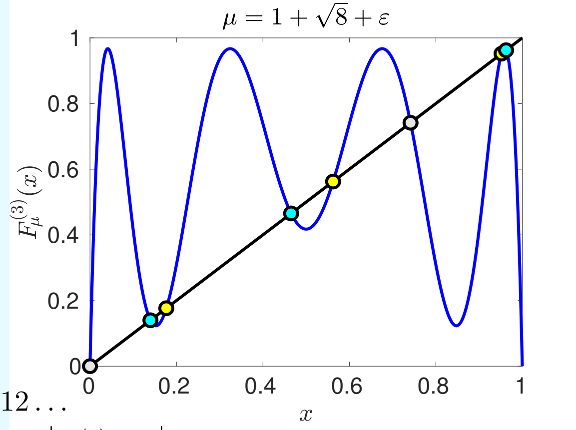

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'cyan 3-cycle' is stable for $\mu = 1+\sqrt{8}+\varepsilon$ for sufficiently small ε

'cyan 3-cycle' is super-attracting for $\varepsilon = 0.00344693\ldots$, i.e. for $\mu = 3.831874\ldots$

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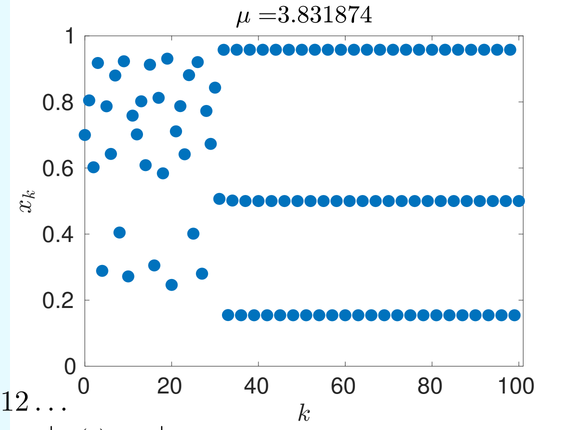

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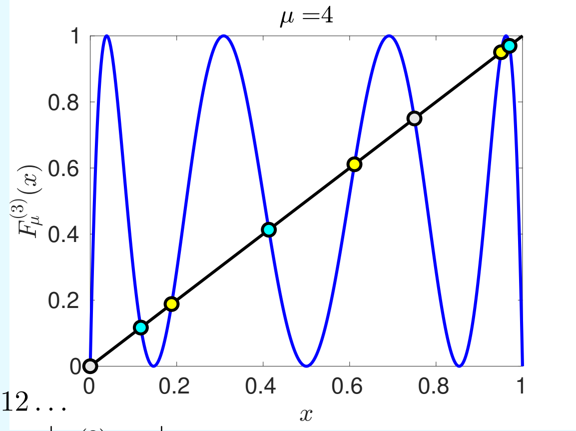

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Question 4 on Problem Sheet 2: closed formulas for both 3-cycles derived for $\mu = 4$

both 3-cycles are unstable because $F_\mu^{(3)\prime}(c_1) = F'_\mu(c_1)\, F'_\mu(c_2) F'_\mu(c_3) = \pm 2^3 = \pm 8$

Sharkovsky's Theorem

Sharkovsky's ordering:

$$3 \triangleright 5 \triangleright 7 \triangleright \ldots \triangleright 2 \times 3 \triangleright 2 \times 5 \triangleright \ldots \triangleright 2^2 \times 3 \triangleright 2^2 \times 5 \triangleright \ldots \triangleright 2^3 \times 3 \triangleright 2^3 \times 5 \triangleright \ldots$$

$$\ldots \triangleright 2^n \times 3 \triangleright 2^n \times 5 \triangleright \ldots \triangleright 2^n \triangleright 2^{n-1} \triangleright 2^3 \triangleright 2^2 \triangleright 2 \triangleright 1$$

Sharkovsky's Theorem (1964):
Let $\Omega = [a, b] \subset \mathbb{R}$ be an interval and $F : \Omega \to \Omega$ be continuous.
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B5.6: Nonlinear Dynamics, Bifurcations and Chaos (Lecture 8)

- summary of Lecture 7: we discussed
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  (Questions 3 and 4 on Problem Sheet 2)
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- course synopsis of **Lectures 1-8** (taken by both Part B and MSc students):
  Discrete-time (maps) and continuous-time (differential equations) dynamical systems. Notion of flows, stability of fixed points, Lyapunov function, invariant manifolds, stable manifold theorem, notion of hyperbolicity, center manifold. Chemical reaction networks. Stable, unstable and center subspaces. Poincaré-Bendixson theorem. Periodic solutions, stable and unstable limit cycles. Introduction to bifurcation theory, covering saddle-node, transcritical, supercritical pitchfork and subcritical pitchfork bifurcations. Extended center manifold. Logistic map. Periodic points of maps. Stability of N-cycles. Period-doubling bifurcation. Sharkovsky's theorem. Invariant distribution.

Sharkovsky's Theorem (last slide of Lecture 7)

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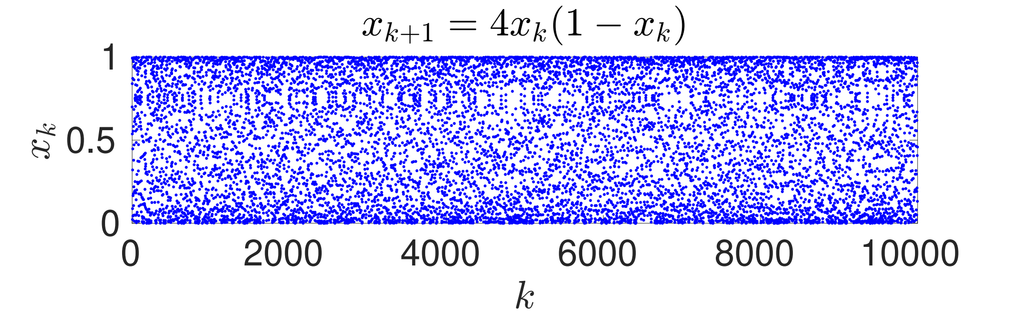
Invariant distribution (Question 7 on Problem Sheet 2)

Questions 3 and 4 on Problem Sheet 0: Starting with $x_0 = 0.7$, we obtain x_k as:



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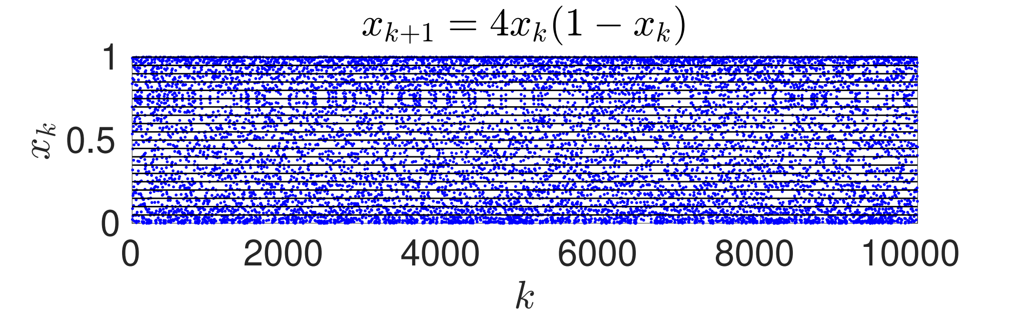
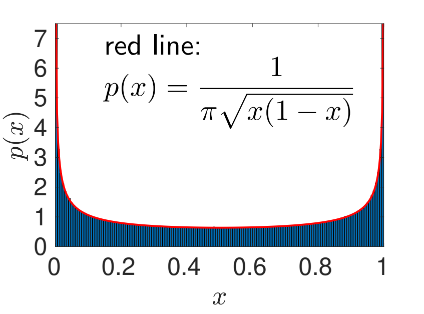


Invariant distribution (Question 7 on Problem Sheet 2)

Histogram of values x_k, for $k = 0, 1, 2, \ldots, 10^6$ (blue bars): $x_{k+1} = 4\, x_k\, (1 - x_k)$



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[Prelims Probability and Calculus]

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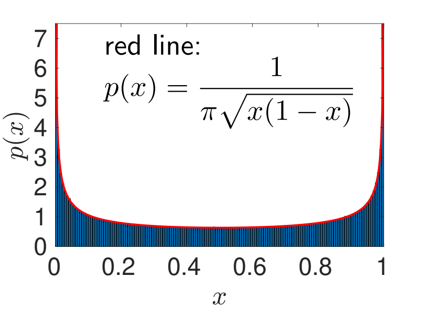

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[Prelims Probability and Calculus]

invariant distribution $p(x)$: if the random variable X is distributed according to $p(x)$, then the random variable $F(X)$ is also distributed according to $p(x)$

Question 7 on Problem Sheet 2: calculate invariant distributions for some other chaotic maps and compare them with the histograms of orbits

theoretical justification is given by ergodic theory (Birkhoff ergodic theorem)

Problem Sheet 2: bifurcations of continuous-time dynamical systems

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu}) \quad \text{with the initial condition} \quad \mathbf{x}(0) = \mathbf{x}_0 \in \Omega$$

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Questions 1, 2, 5 and 6 on Problem Sheet 2 cover bifurcations of fixed points, which can occur for $n \geq 1$ and $m \geq 1$:

- saddle-node bifurcation
- transcritical bifurcation
- supercritical pitchfork bifurcation
- subcritical pitchfork bifurcation

We have explained them in our lectures on examples with $n = 1, 2$ and $m = 1$.

Next, we will discuss some additional examples to help you solve Problem Sheet 2, including examples with $m \geq 2$ and $n = 3$.

Example: $n = 1$, $m = 2$

$$\frac{\mathsf{d}x_1}{\mathsf{d}t} = \mu_2 \; + \; \mu_1 \, x_1 \; - \; x_1^3$$

$$f(x_1; \boldsymbol{\mu}) = \mu_2 + \mu_1 x_1 - x_1^3$$

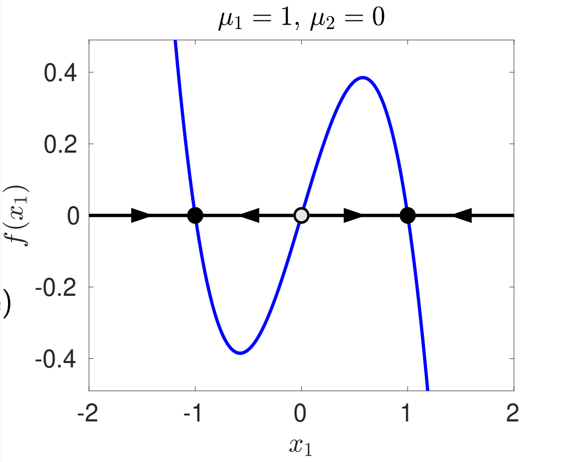
$\mu_2 = 0$: supercritical pitchfork bifurcation

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2 + \mu_1 x_1 - x_1^3$$

$$f(x_1;\boldsymbol{\mu}) = \mu_2 + \mu_1 x_1 - x_1^3$$

$\mu_1 > 0,\ \mu_2 = 0$

three fixed points at $x_1 = \pm\sqrt{\mu_1}$ (stable)
and $x_1 = 0$ (unstable)



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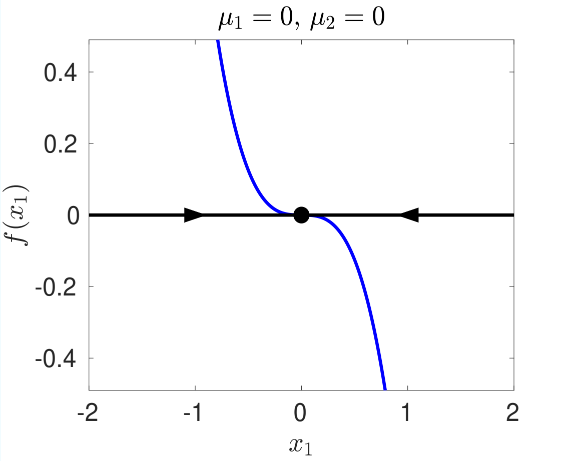
$$f(x_1; \boldsymbol{\mu}) = \mu_2 + \mu_1 x_1 - x_1^3$$

as μ_1 approaches zero from above, two fixed points $\sqrt{\mu_1}$ and $-\sqrt{\mu_1}$ move toward the third one

$\mu_1 = 0$: the fixed points coalesce into a stable fixed point at $x_1 = 0$



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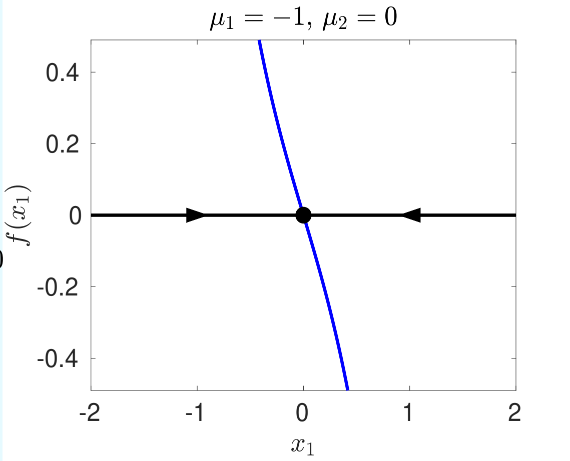
$\mu_1 < 0$: one stable fixed point at $x_1 = 0$



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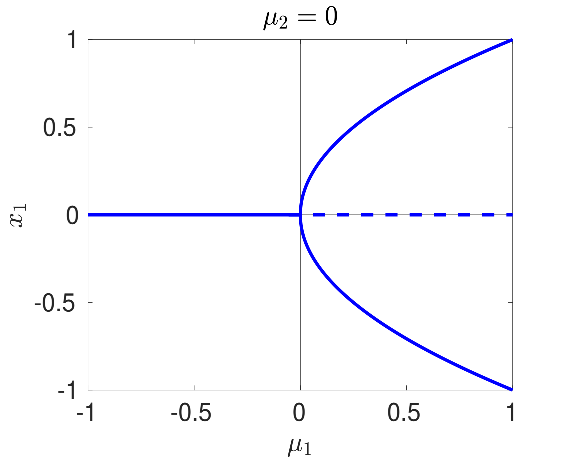
bifurcation diagram



$\mu_2 = 0$: supercritical pitchfork bifurcation ($n = 2$)

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2 + \mu_1 x_1 - x_1^3$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -x_2$$

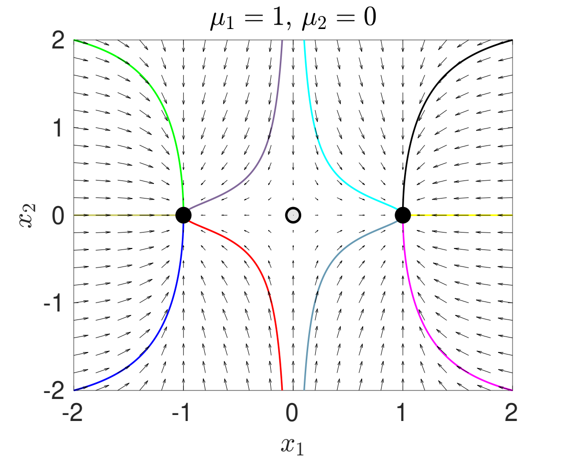
$\mu_1 > 0$, $\mu_2 = 0$

three fixed points at

$\mathbf{x} = [-\sqrt{\mu_1}, 0]$ (stable node)

$\mathbf{x} = [0, 0]$ (saddle)

$\mathbf{x} = [\sqrt{\mu_1}, 0]$ (stable node)



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as μ_1 approaches zero from above, two fixed points $[-\sqrt{\mu_1}, 0]$ and $\sqrt{\mu_1}, 0]$ move toward the third one

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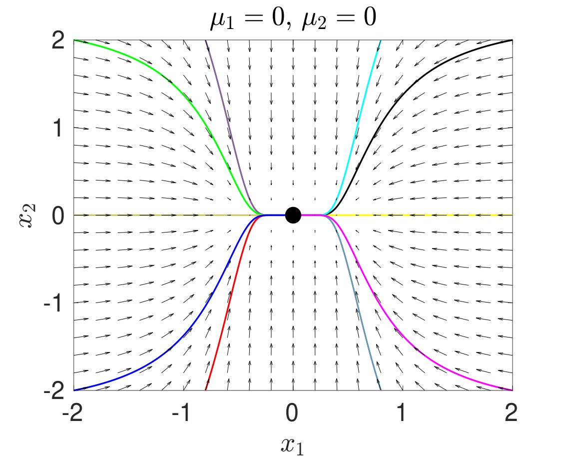

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$\mu_1 < 0$:
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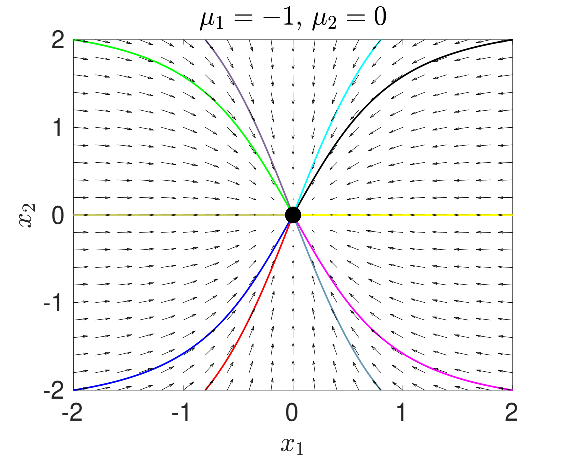

$\mu_2 = 0$: supercritical pitchfork bifurcation ($n = 2$)

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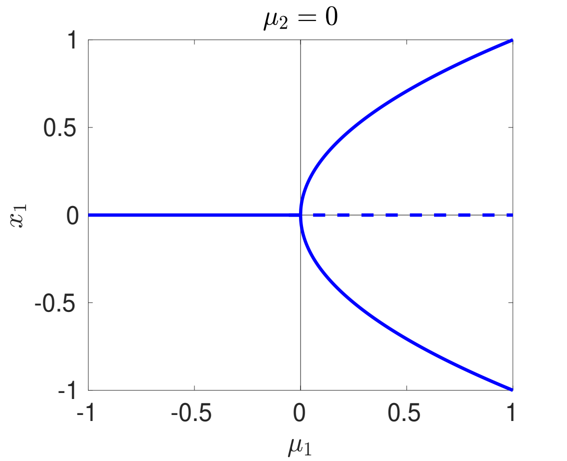


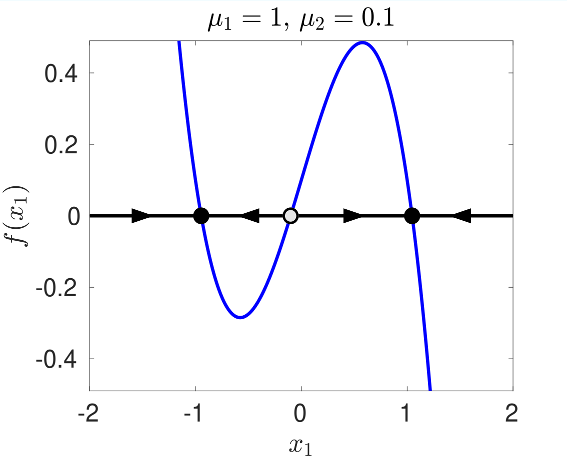
Cusp catastrophe ($\mu_2 = 0.1$)

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2 + \mu_1 x_1 - x_1^3$$

$$f(x_1; \boldsymbol{\mu}) = \mu_2 + \mu_1 x_1 - x_1^3$$

$\mu_1 = 1,\ \mu_2 = 0.1$

three fixed points given as solutions of $\mu_2 + \mu_1 x_1 - x_1^3 = 0$



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$\mu_1 = 0.5, \ \mu_2 = 0.1$

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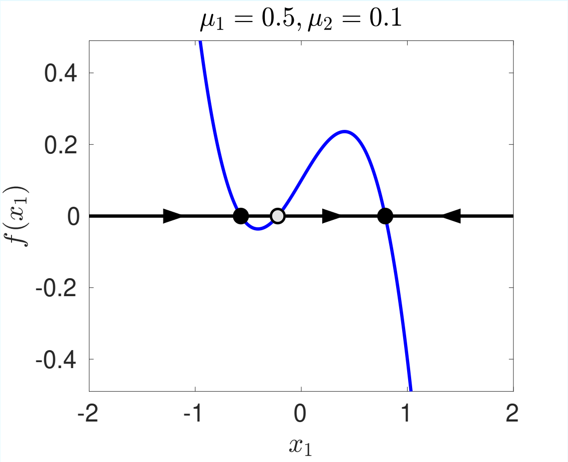

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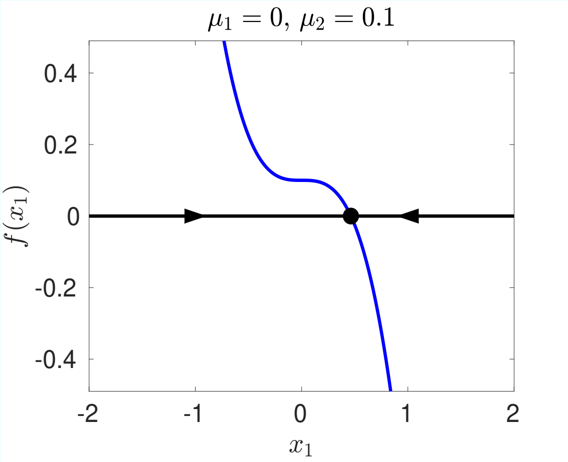
as μ_1 approaches the bifurcation value

$$\mu_c = \left(\frac{27\mu_2^2}{4}\right)^{1/3}$$

from above, two (smaller) fixed points move toward each other (saddle-node bifurcation)

$\mu_1 < \mu_c$: one stable fixed point



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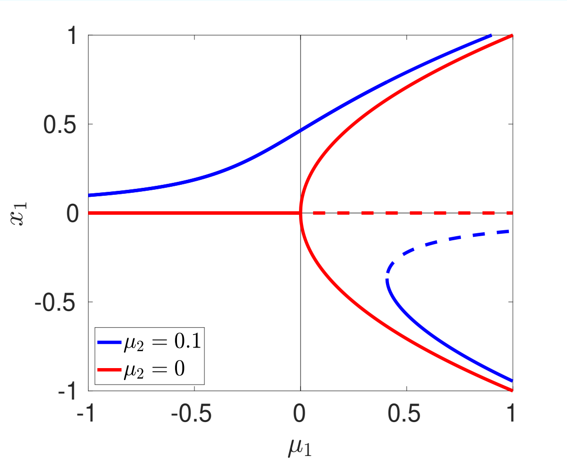

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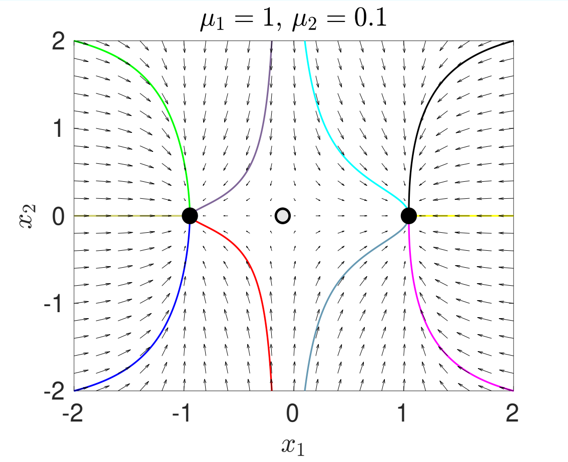

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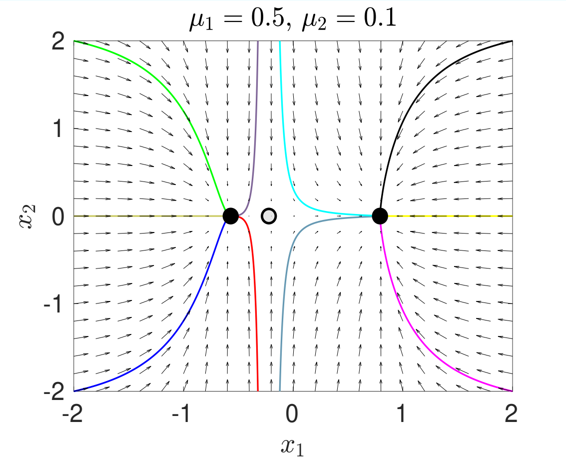

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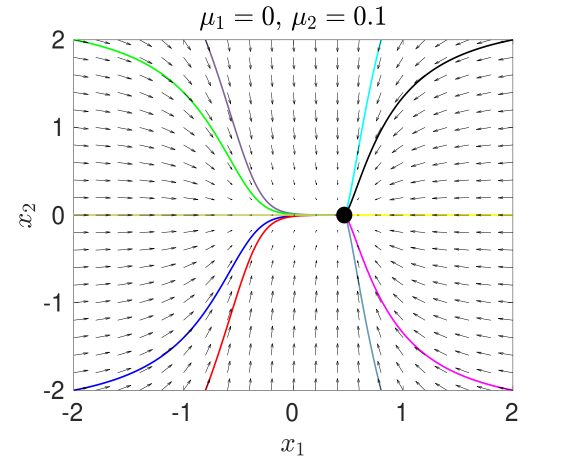

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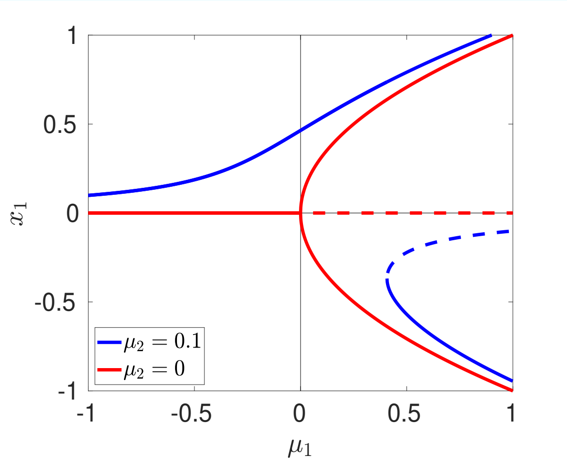

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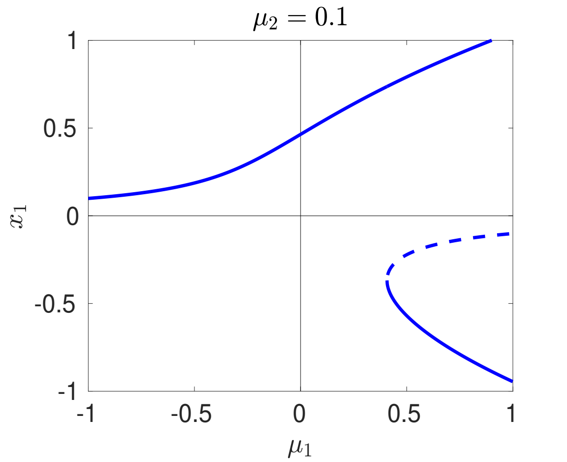

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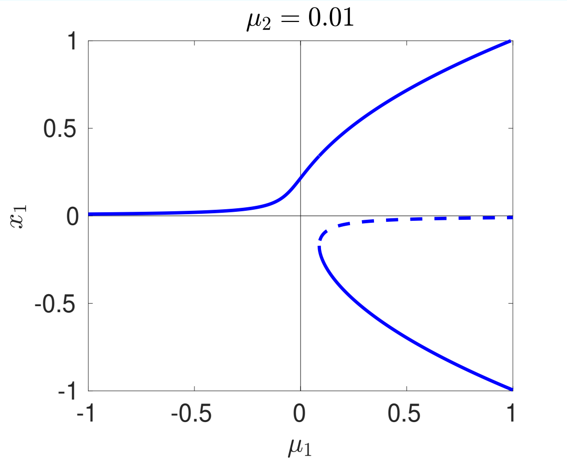

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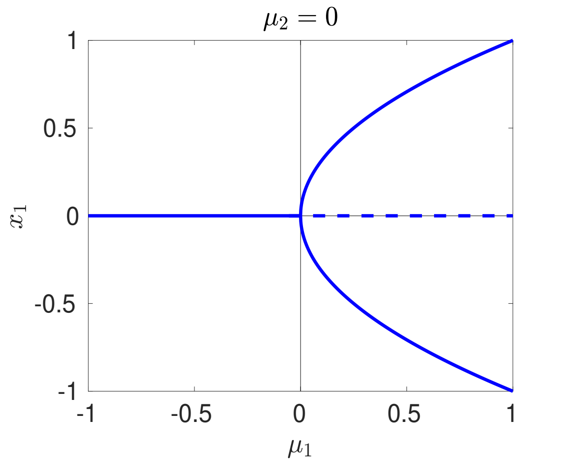

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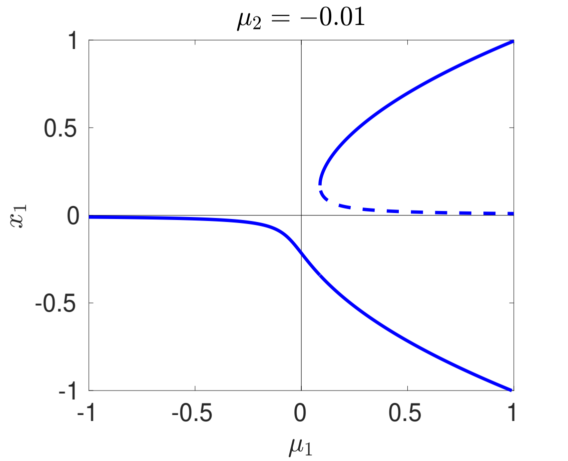

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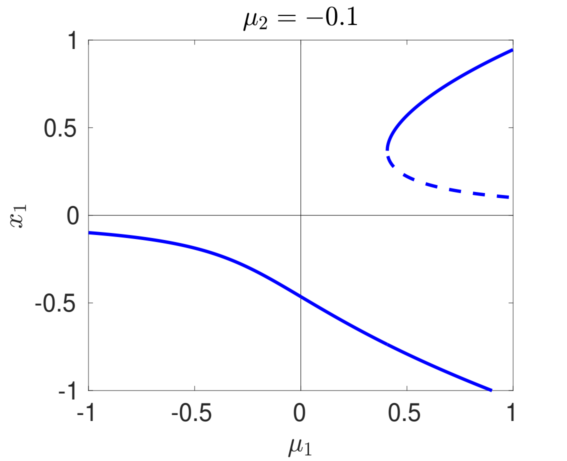

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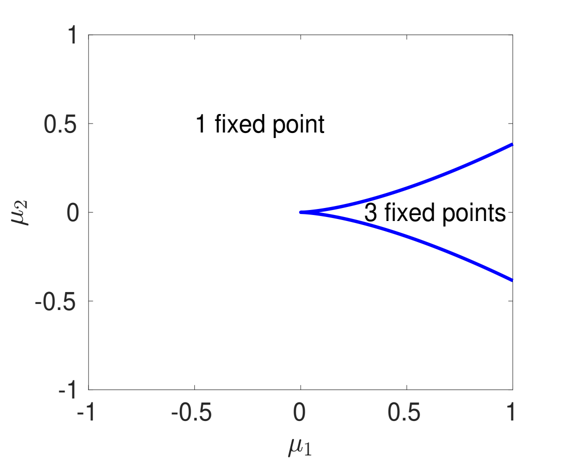

Example: $n = 2$, $m = 2$

$$\frac{\mathsf{d}x_1}{\mathsf{d}t} = \mu_2 + \mu_1 x_1 - x_1^2$$

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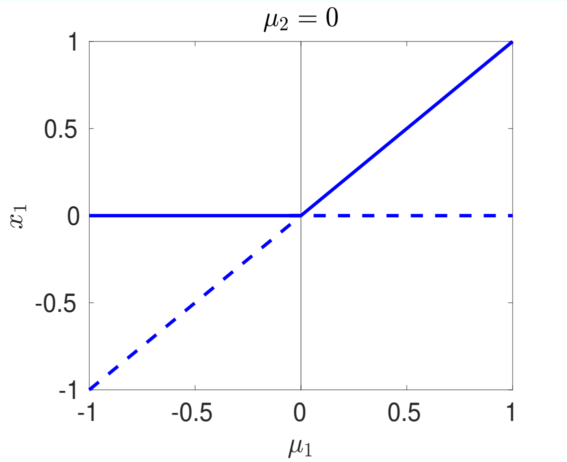

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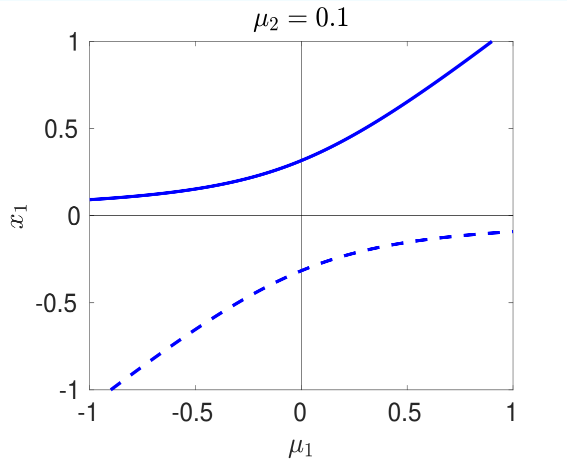
The saddle-node bifurcation is robust under small changes of parameters, but transcritical and pitchfork bifurcations change under small perturbations.



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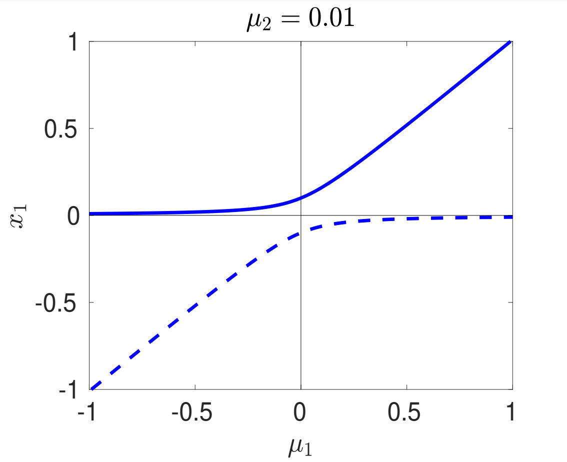

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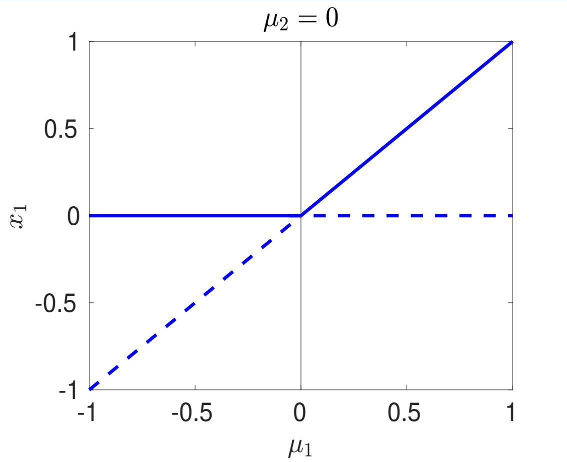

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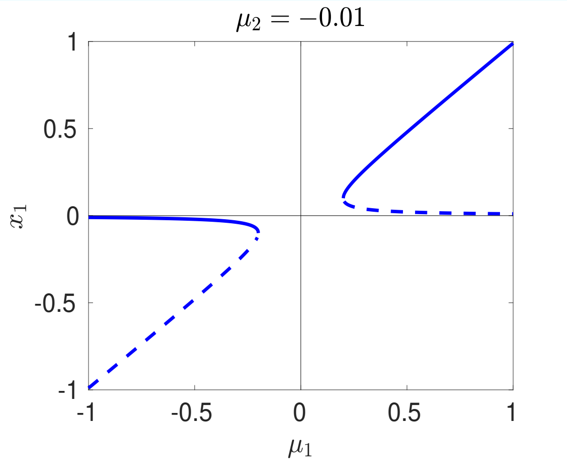

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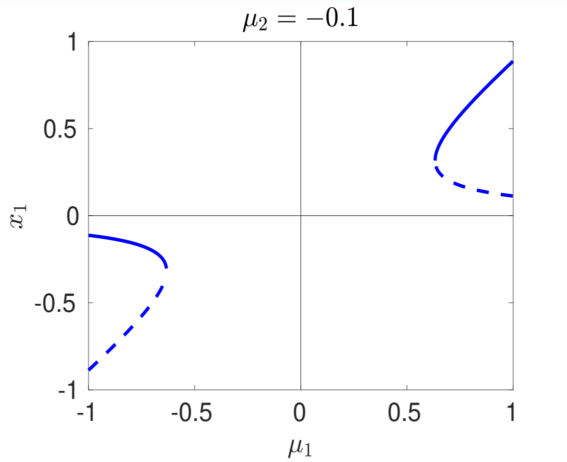

Example with $n = 3$, $m = 3$: Lorenz equations

$$
\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= \mu_2 \, (x_2 - x_1) \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= \mu_1 \, x_1 - x_2 - x_1 \, x_3 \\
\frac{\mathrm{d}x_3}{\mathrm{d}t} &= x_1 \, x_2 - \mu_3 \, x_3
\end{aligned}
$$

Lorenz equations: summary of our 3D visualization of dynamics

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2\,(x_2 - x_1)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu_1\,x_1 - x_2 - x_1\,x_3$$
$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = x_1\,x_2 - \mu_3\,x_3$$

We will study the Lorenz system again
in the second part of our course,
when we will discuss chaos in ODEs.

TODAY: we illustrated the convergence
to fixed points, limit cycles, chaos,
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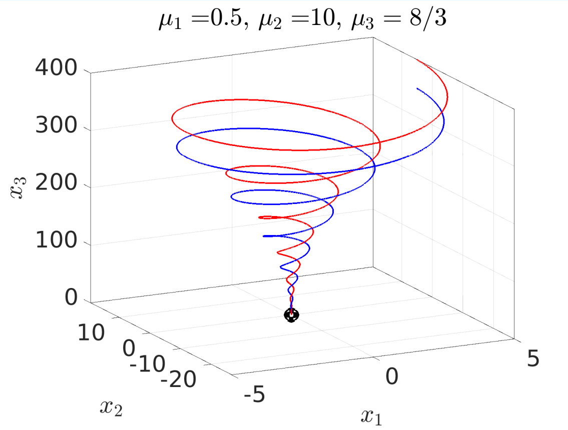
Lorenz equations: summary of our 3D visualization of dynamics

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 10\,(x_2 - x_1)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu_1\,x_1 - x_2 - x_1\,x_3$$
$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = x_1\,x_2 - \frac{8\,x_3}{3}$$



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we varied μ_1, while we fixed the values of parameters μ_2 and μ_3:

$$\mu_2 = 10 \text{ and } \mu_3 = \frac{8}{3} \qquad \text{(Lorenz used } \mu_1 = 28 \text{ to get chaos)}$$

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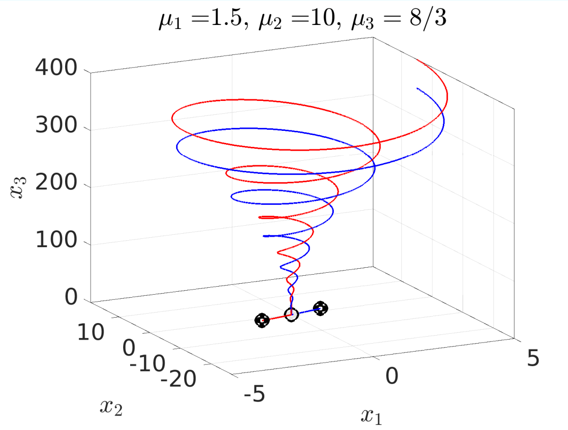

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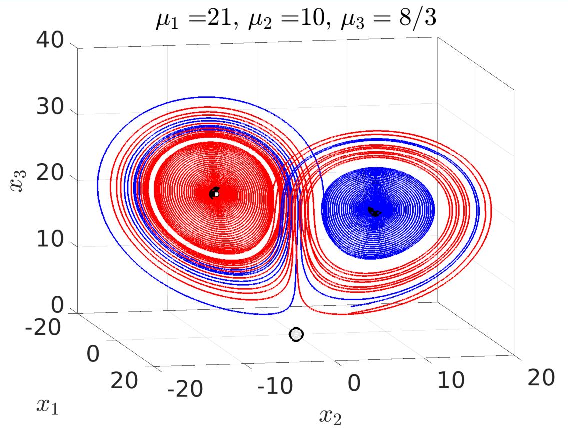

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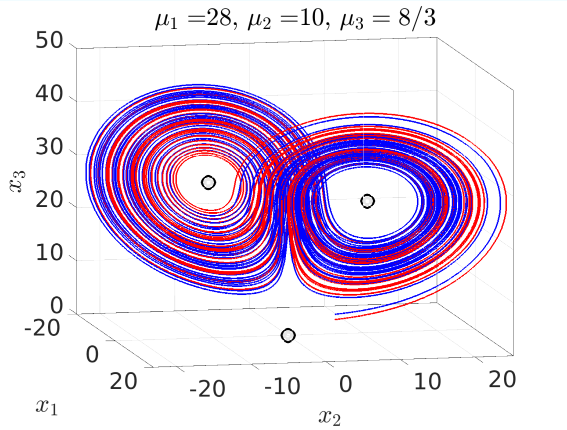
$$\mu_2 = 10 \text{ and } \mu_3 = \frac{8}{3} \qquad \text{(Lorenz used } \mu_1 = 28 \text{ to get chaos)}$$

Lorenz equations: summary of our 3D visualization of dynamics

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 10\,(x_2 - x_1)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu_1\,x_1 - x_2 - x_1\,x_3$$
$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = x_1\,x_2 - \frac{8\,x_3}{3}$$

We will study the Lorenz system again in the second part of our course, when we will discuss chaos in ODEs.



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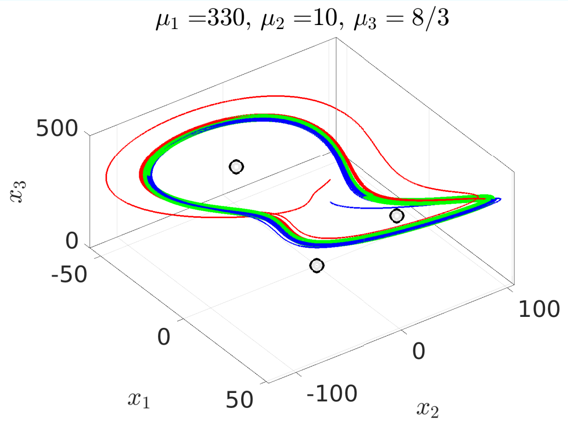

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Lorenz equations: Question 6 on Problem Sheet 2

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2\,(x_2 - x_1)$$
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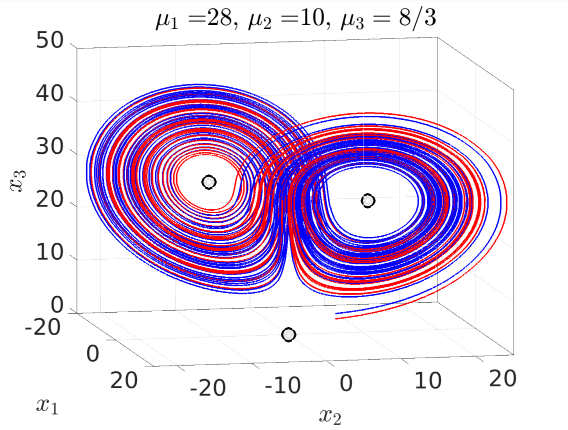

Question 6 on Problem Sheet 2:
we use the Lorenz system to further practice some techniques studied in Lectures 1-8

Lorenz equations: Question 6 on Problem Sheet 2

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2 \, (x_2 - x_1)$$
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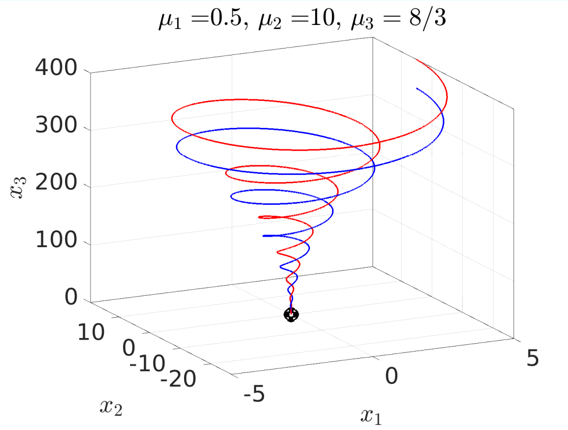

Question 6 on Problem Sheet 2:
we use the Lorenz system to further practice some techniques studied in Lectures 1-8 including:

- finding the Lyapunov function to prove the global stability of the fixed point at the origin $\mathbf{0} = [0, 0, 0]$ for $\mu_1 < 1$

[Question 6(d)]

Lorenz equations: Question 6 on Problem Sheet 2

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2\,(x_2 - x_1)$$
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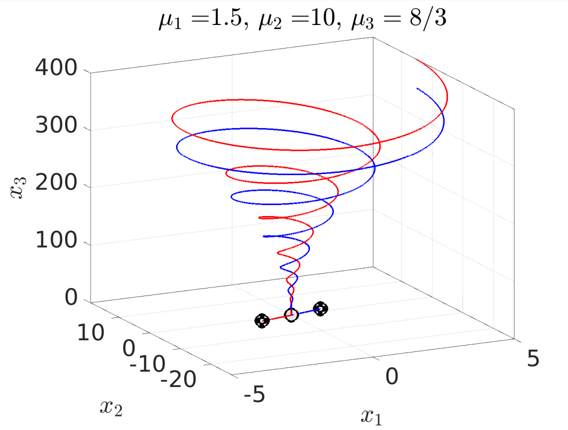

Question 6 on Problem Sheet 2:
we use the Lorenz system to further practice some techniques studied in Lectures 1-8 including:

- finding the Lyapunov function to prove the global stability of the fixed point at the origin $\mathbf{0} = [0, 0, 0]$ for $\mu_1 < 1$ [Question 6(d)]
- using the extended center manifold theory to analyze the bifurcation at $\mu_1 = 1$, calculating the center manifold and the dynamics on it [Question 6(e)]

Lorenz equations: Question 6 on Problem Sheet 2

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 3\,(x_2 - x_1)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu_1\,x_1 - x_2 - x_1\,x_3$$
$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = x_1\,x_2 - x_3$$

- fixed points $\mathbf{x}_{c1} = \mathbf{0} = [0, 0, 0]$

  $\mathbf{x}_{c2} = \left[\sqrt{\mu_1 - 1}, \sqrt{\mu_1 - 1}, \mu_1 - 1\right]$

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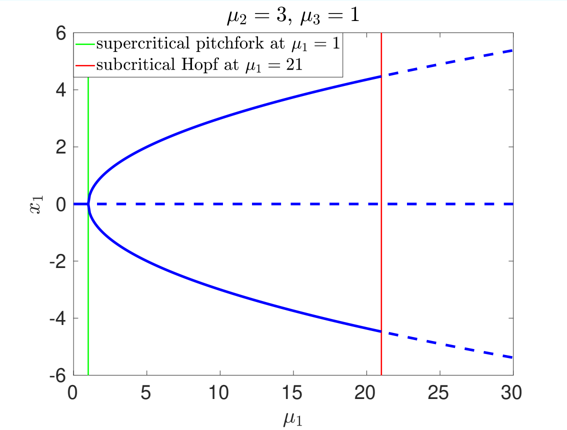
  $\mathbf{x}_{c2}$ and $\mathbf{x}_{c2}$ only exist for $\mu_1 > 1$



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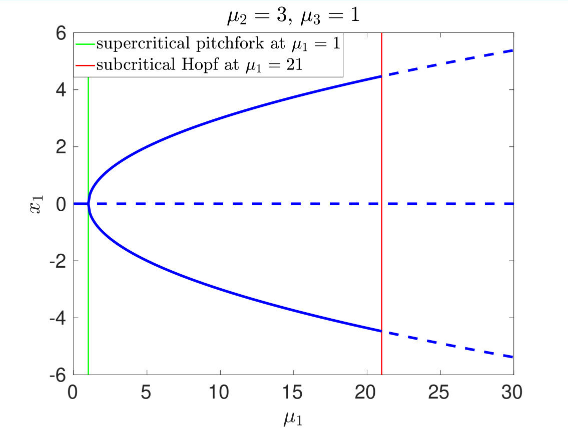

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  $\mathbf{x}_{c2}$ and $\mathbf{x}_{c2}$ only exist for $\mu_1 > 1$
- supercritical pitchfork bifurcation
  at $\mu_1 = 1$ ($\mathbf{x}_{c1} = \mathbf{0}$ is stable for $\mu_1 < 1$ and unstable for $\mu_1 > 1$)

Lorenz equations: Question 6 on Problem Sheet 2

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 3\,(x_2 - x_1)$$
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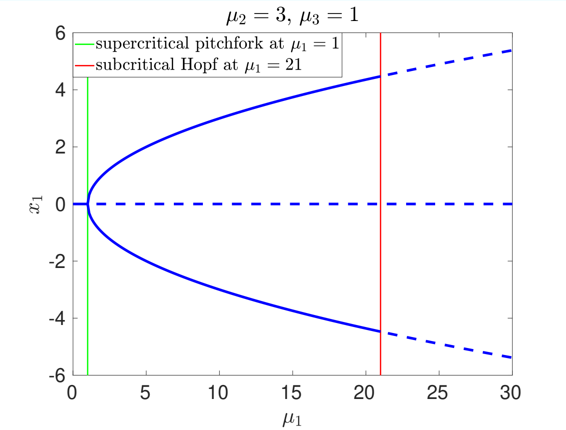

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- $\mathbf{x}_{c2}$ and $\mathbf{x}_{c2}$ are stable for $\mu_1 < 21$ and unstable for $\mu_1 > 21$
  [Hopf bifurcations will be discussed in the second part of the course]

Lorenz equations: Question 6 on Problem Sheet 2

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 3\,(x_2 - x_1)$$
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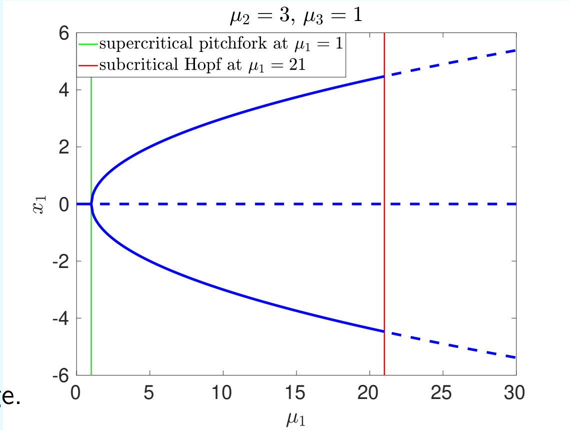
Question 6(c) on Problem Sheet 2:
All trajectories eventually enter and remain inside a large sphere of the form
$x_1^2 + x_2^2 + (x_3 - \mu_1 - 3)^2 = C(\mu_1)$
where constant $C(\mu_1)$ is sufficiently large.



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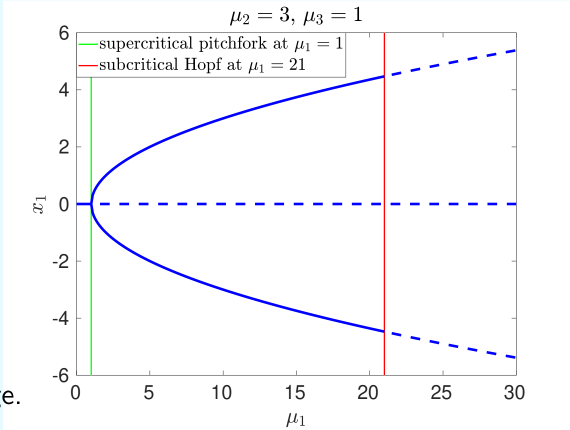

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All trajectories eventually enter and remain inside a large sphere of the form
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Question 6(a): Let $U \equiv U(0) \subset \mathbb{R}^3$ be a compact connected subset of initial conditions. Let $U(t) = \phi_t(U)$ and $v(t) = |U(t)| = |\phi_t(U)|$ be the volume of $U(t)$. Then
$$\lim_{t\to\infty} v(t) = 0$$

Lorenz equations: Question 6 on Problem Sheet 2

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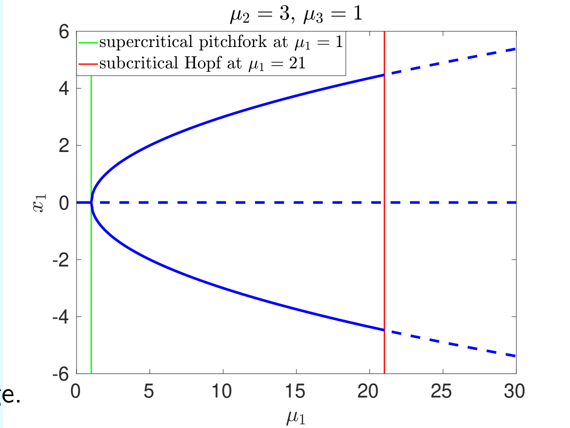

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All trajectories eventually enter and remain inside a large sphere of the form
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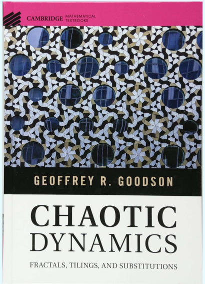
where constant $C(\mu_1)$ is sufficiently large.

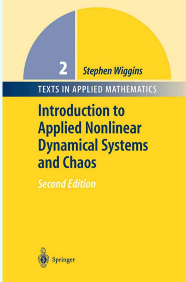
We have a trapping region and volume contraction, but there are no stable critical points or stable limit cycles for some parameter values!
We will continue with the analysis of the Lorenz system in the second part of the course after we introduce Hopf bifurcations.

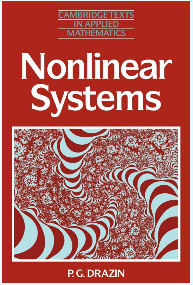
End of Lecture 8: Goodbye, MSc students!

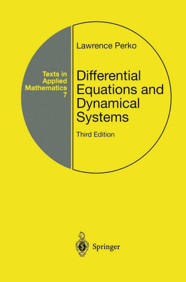
Course synopsis of **Lectures 1-8** (taken by both Part B and MSc students)

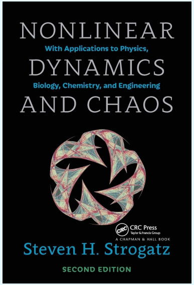
Discrete-time (maps) and continuous-time (differential equations) dynamical systems. Notion of flows, stability of fixed points, Lyapunov function, invariant manifolds, stable manifold theorem, notion of hyperbolicity, center manifold. Chemical reaction networks. Stable, unstable and center subspaces. Poincaré-Bendixson theorem. Periodic solutions, stable and unstable limit cycles. Introduction to bifurcation theory, covering saddle-node, transcritical, supercritical pitchfork and subcritical pitchfork bifurcations. Extended center manifold. Logistic map. Periodic points of maps. Stability of N-cycles. Period-doubling bifurcation. Sharkovsky's theorem. Invariant distribution.

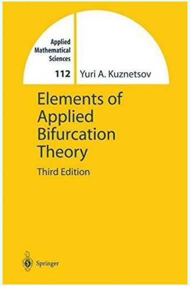








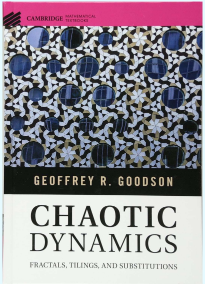


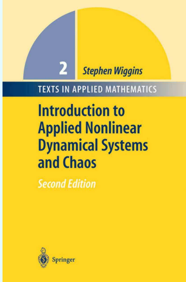


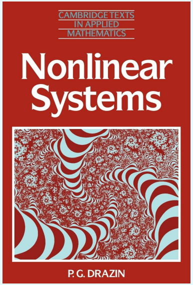
Next week: Part B students only

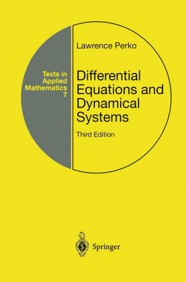
Course synopsis of **Lectures 9-16** (Problem Sheets 3 and 4)

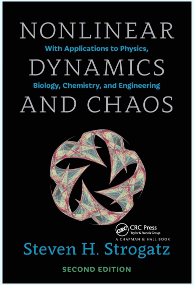
Bifurcations of limit cycles, covering supercritical and subcritical Hopf bifurcations, saddle-node bifurcation of cycles, infinite-period (SNIC) bifurcation and homoclinic (saddle-loop) bifurcation. Oscillations in chemical reaction networks. Weakly nonlinear oscillators. Poincaré-Lindstedt method. Conservative and non-conservative systems. Liénard systems, van der Pol oscillator. Hilbert's 16th problem. Lorenz equations. Lorenz map. Poincaré section. Poincaré map. Converse of Sharkovsky's theorem. Bernoulli shift map, symbolic dynamics. Tent map. Dynamics on metric spaces, sensitive dependence on initial conditions, transitivity, conjugate maps, chaotic dynamics.

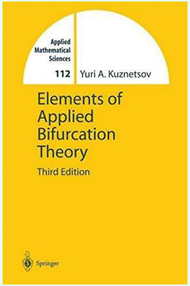












B5.6: Nonlinear Dynamics, Bifurcations and Chaos (Lecture 9)

- summary of Lecture 8: we discussed
  Sharkovsky's theorem. Invariant distribution. Extended center manifold. Bifurcations of fixed points. Lorenz equations: analysis using techniques in Lectures 1-8 (taken by both Part B and MSc students)
- today: we will start our discussion of Problem Sheet 3 (Part B students only)

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- course synopsis of **Lectures 9-16**:
  Bifurcations of limit cycles, covering supercritical and subcritical Hopf bifurcations, saddle-node bifurcation of cycles, infinite-period (SNIC) bifurcation and homoclinic (saddle-loop) bifurcation. Oscillations in chemical reaction networks. Weakly nonlinear oscillators. Poincaré-Lindstedt method. Conservative and non-conservative systems. Liénard systems, van der Pol oscillator. Hilbert's 16th problem. Lorenz equations. Lorenz map. Poincaré section. Poincaré map. Converse of Sharkovsky's theorem. Bernoulli shift map, symbolic dynamics. Tent map. Dynamics on metric spaces, sensitive dependence on initial conditions, transitivity, conjugate maps, chaotic dynamics.

Bifurcations

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu}) \quad \text{with the initial condition} \quad \mathbf{x}(0) = \mathbf{x}_0 \in \Omega$$

Bifurcations: The qualitative structure of the flow can change as parameters $\boldsymbol{\mu}$ are varied. For example, critical points (fixed points) can be created or destroyed, or limit cycles can be created or destroyed. The parameter values at which these qualitative changes in the dynamics occur are called bifurcation points.

Bifurcations of fixed points

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

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Problem Sheet 2: bifurcations of fixed points

they can occur for $n \geq 1$, we studied examples with $n = 1$, $n = 2$ and $n = 3$

- saddle-node bifurcation
- transcritical bifurcation
- supercritical pitchfork bifurcation
- subcritical pitchfork bifurcation

Bifurcations of limit cycles

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

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Problem Sheet 3: bifurcations of limit cycles

they can occur for $n \geq 2$, we will first explain them on the case $n = 2$

- supercritical Hopf bifurcation
- subcritical Hopf bifurcation
- saddle-node bifurcation of cycles
- infinite-period bifurcation (SNIC, SNIPER)
- homoclinic bifurcation (saddle-loop bifurcation)

Supercritical Hopf bifurcation

example:

$$\frac{\mathsf{d}x_1}{\mathsf{d}t} = \mu\, x_1 \,-\, x_2 \,-\, x_1(x_1^2 + x_2^2)$$

$$\frac{\mathsf{d}x_2}{\mathsf{d}t} = x_1 \,+\, \mu\, x_2 \,-\, x_2(x_1^2 + x_2^2)$$

Supercritical Hopf bifurcation

example:

$$\frac{\mathsf{d}x_1}{\mathsf{d}t} = \mu\, x_1 \, - \, x_2 \, - \, x_1(x_1^2 + x_2^2)$$

$$\frac{\mathsf{d}x_2}{\mathsf{d}t} = x_1 \, + \, \mu\, x_2 \, - \, x_2(x_1^2 + x_2^2)$$

fixed point at $\mathbf{0} = [0, 0]$

linearization $D\mathbf{f}(\mathbf{0}) = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}$

eigenvalues $\lambda_\pm = \mu \pm i$

Supercritical Hopf bifurcation

example:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu\, x_1 \,-\, x_2 \,-\, x_1(x_1^2 + x_2^2)$$

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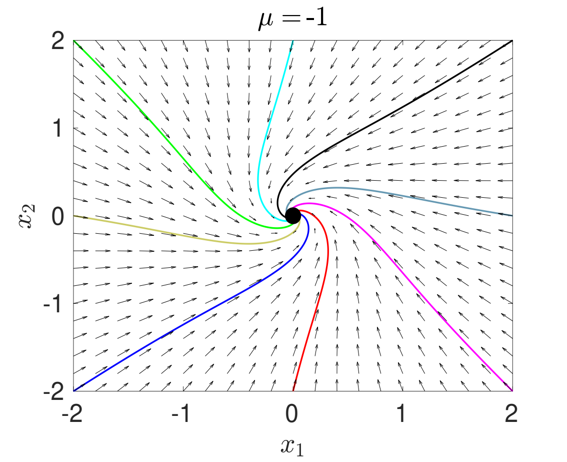

$\mu < 0$: fixed point $\mathbf{0} = [0, 0]$ is a stable spiral

Supercritical Hopf bifurcation

example:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu\, x_1 \;-\; x_2 \;-\; x_1(x_1^2 + x_2^2)$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1 \;+\; \mu\, x_2 \;-\; x_2(x_1^2 + x_2^2)$$

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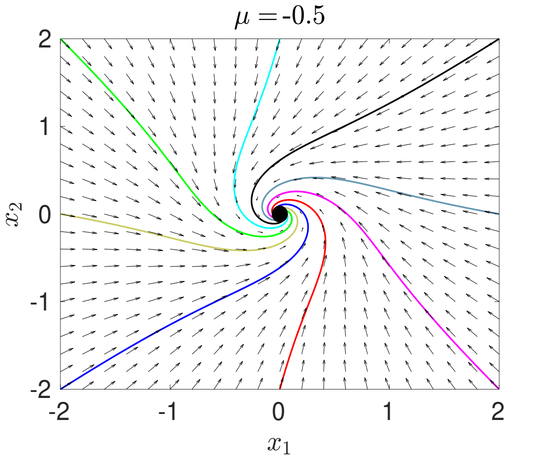

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Supercritical Hopf bifurcation

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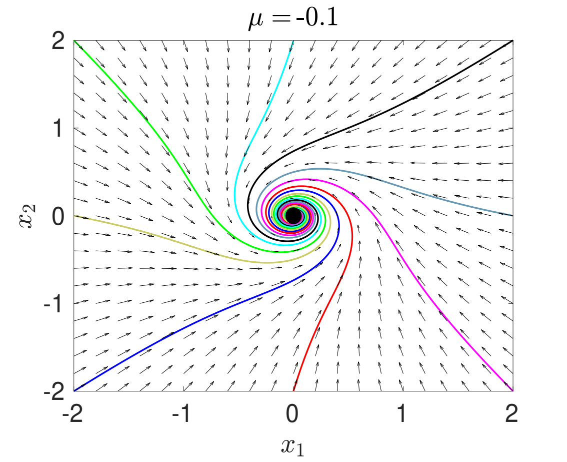


$\mu < 0$: fixed point $\mathbf{0} = [0, 0]$ is a stable spiral

Supercritical Hopf bifurcation

example:

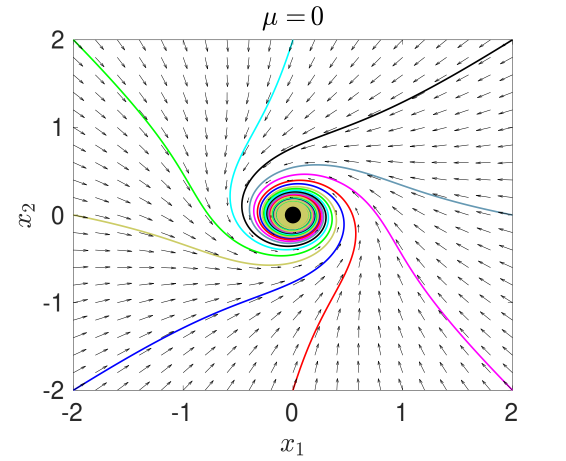
$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu\, x_1 \;-\; x_2 \;-\; x_1(x_1^2 + x_2^2)$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1 \;+\; \mu\, x_2 \;-\; x_2(x_1^2 + x_2^2)$$

fixed point at $\mathbf{0} = [0, 0]$

linearization $D\mathbf{f}(\mathbf{0}) = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}$

eigenvalues $\lambda_\pm = \mu \pm i$



as μ increases from negative to positive values, eigenvalues cross the imaginary axis from left to right

$\mu = 0$: fixed point $\mathbf{0} = [0, 0]$ is a still stable spiral, though a very weak one

Supercritical Hopf bifurcation

example:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu\, x_1 \;-\; x_2 \;-\; x_1(x_1^2 + x_2^2)$$

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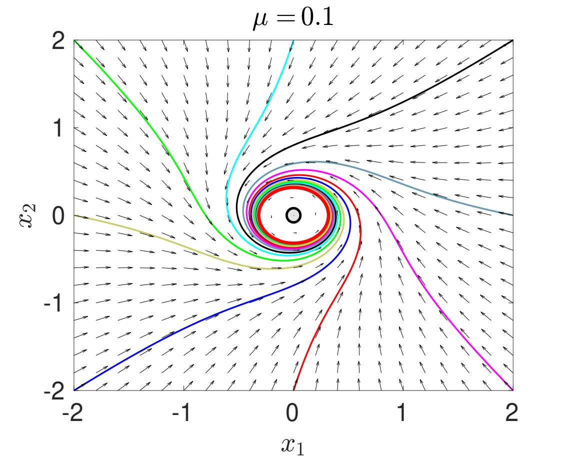

$\mu > 0$: fixed point $\mathbf{0} = [0, 0]$ is an unstable spiral

stable circular limit cycle of radius $r = \sqrt{\mu}$

Supercritical Hopf bifurcation

example:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu\, x_1 \; - \; x_2 \; - \; x_1(x_1^2 + x_2^2)$$

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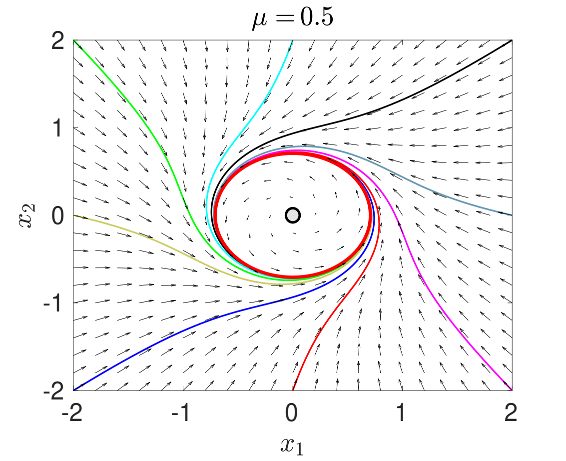

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Supercritical Hopf bifurcation

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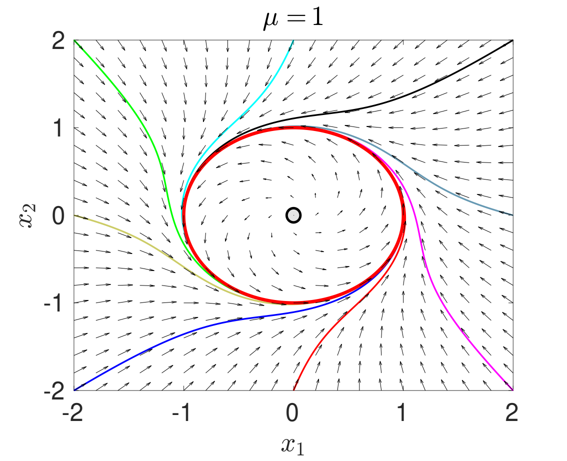

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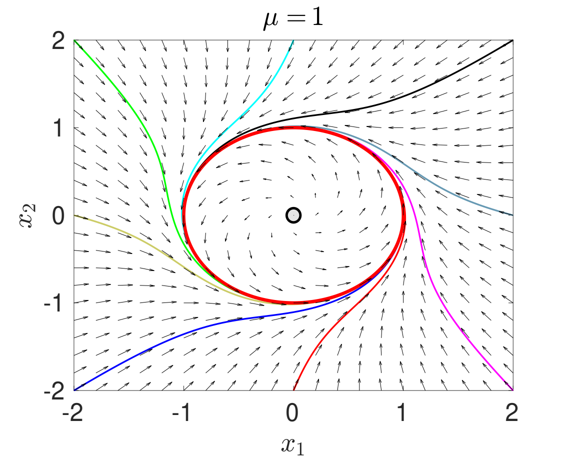

We transform the ODEs to polar coordinates by using variables $r(t)$ and $\theta(t)$, where $x_1(t) = r(t)\cos\theta(t)$ and $x_2(t) = r(t)\sin\theta(t)$. We obtain

$$\frac{\mathrm{d}r}{\mathrm{d}t} = r(\mu - r^2) \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = 1$$

Supercritical Hopf bifurcation

example:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu\, x_1 - x_2 - x_1(x_1^2 + x_2^2)$$

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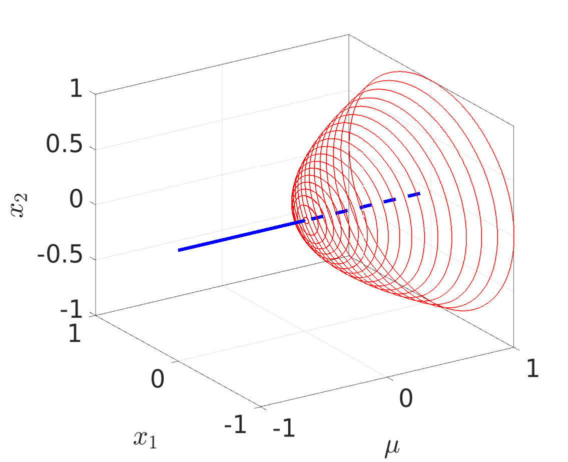

bifurcation diagram

[show 3D animation]

Hopf bifurcation - general case

general case: eigenvalues $\lambda(\mu) = \alpha(\mu) \pm i\,\omega(\mu)$ with $\alpha(0) = 0$ and $\omega(0) \neq 0$

behaviour close to the fixed point: normal form (in polar coordinates)

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha(\mu)\,r + a(\mu)\,r^3 + \mathcal{O}(r^5)$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega(\mu) + b(\mu)\,r^2 + \mathcal{O}(r^4)$$

Taylor expanding:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha'(0)\,\mu\,r + a(0)\,r^3 + \mathcal{O}(\mu^2 r, \mu r^3, r^5)$$

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our previous example: $\alpha'(0) = 1$, $a(0) = -1$, $\omega(0) = 1$, $\omega'(0) = b(0) = 0$

supercritical Hopf bifurcation: $a(0) < 0$ (periodic orbit is asymptotically stable)

subcritical Hopf bifurcation: $a(0) > 0$ (periodic orbit is unstable)

Supercritical Hopf bifurcation

general case: $a(0) < 0$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha'(0)\,\mu\,r \;+\; a(0)\,r^3$$

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eigenvalues $\lambda_\pm = \alpha'(0)\,\mu \pm i\,\omega(0)$

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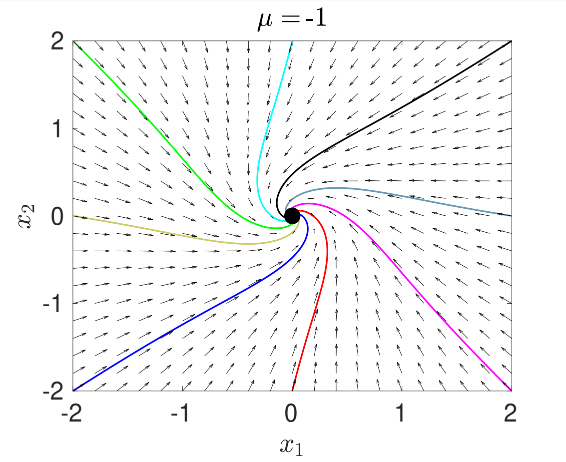
$\mu < 0$: $\mathbf{0} = [0, 0]$ is a stable spiral

$\mu > 0$: $\mathbf{0} = [0, 0]$ is an unstable spiral

stable circular limit cycle of radius $r = \sqrt{\mu}$



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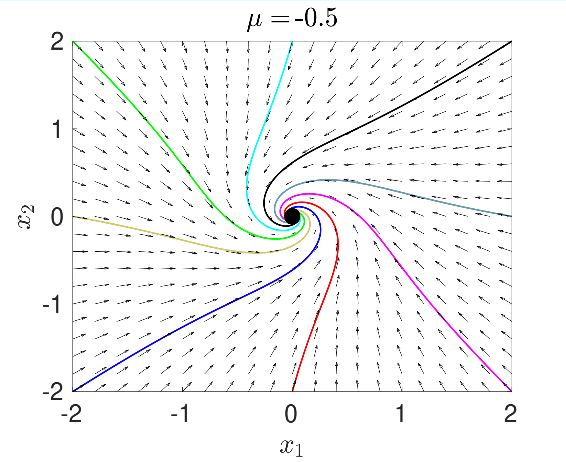

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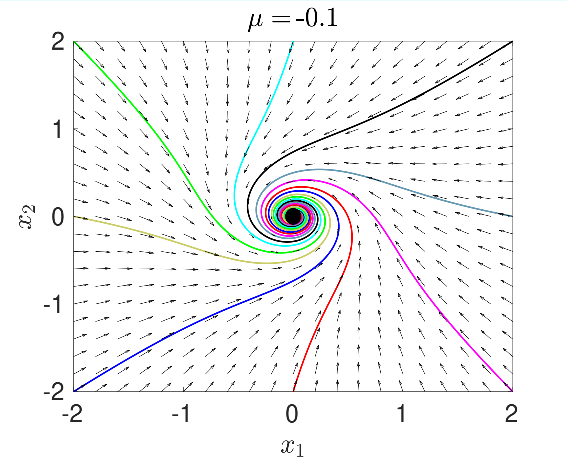

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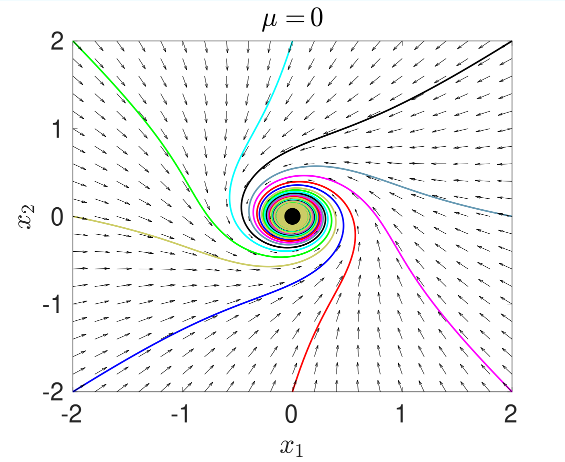

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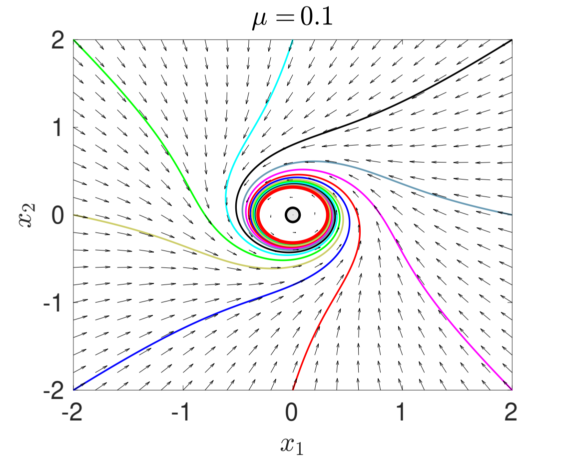

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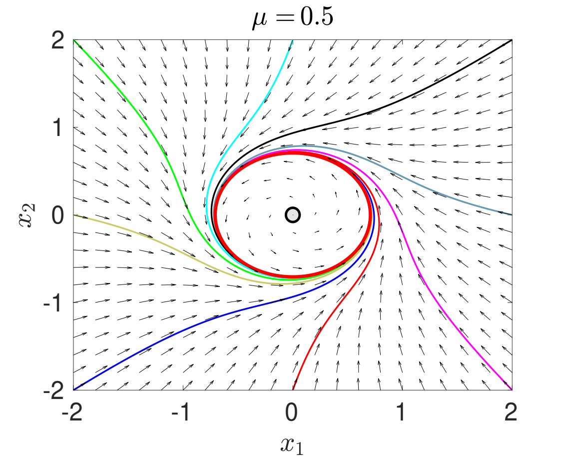

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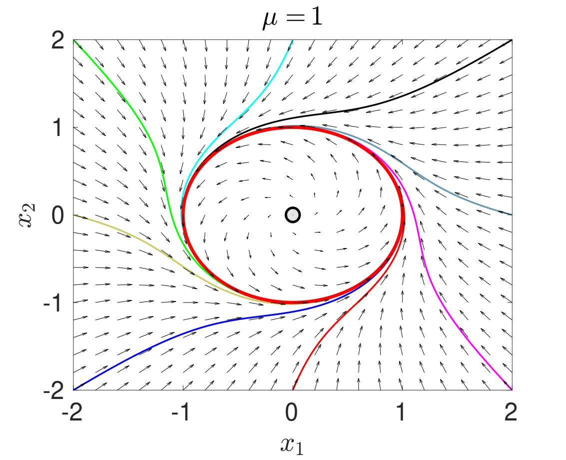

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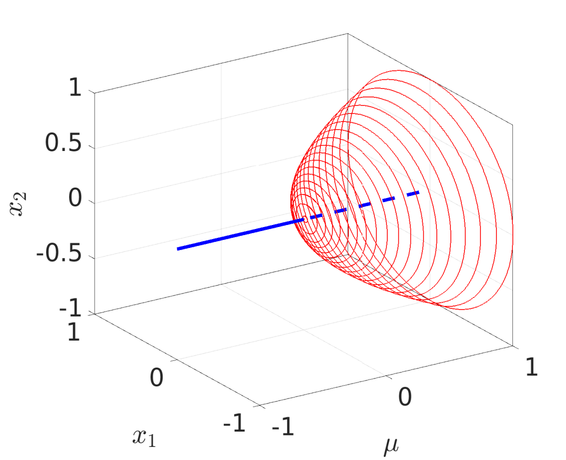
stable circular limit cycle of radius $r = \sqrt{\mu}$



Subcritical Hopf bifurcation

general case: $a(0) > 0$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha'(0)\,\mu\,r \; + \; a(0)\,r^3$$

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Subcritical Hopf bifurcation

general case: $a(0) > 0$

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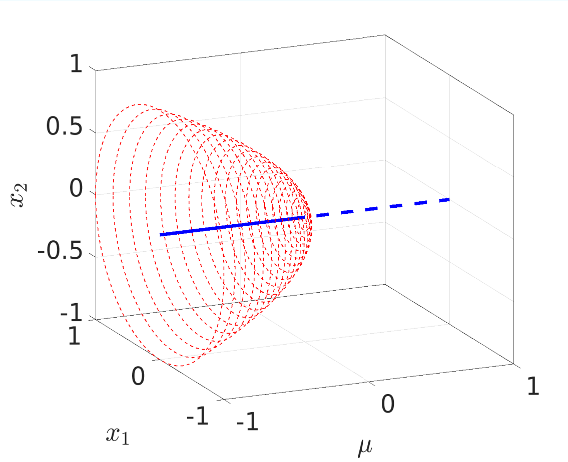

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unstable circular limit cycle of radius $r = \sqrt{\mu}$

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Subcritical Hopf bifurcation

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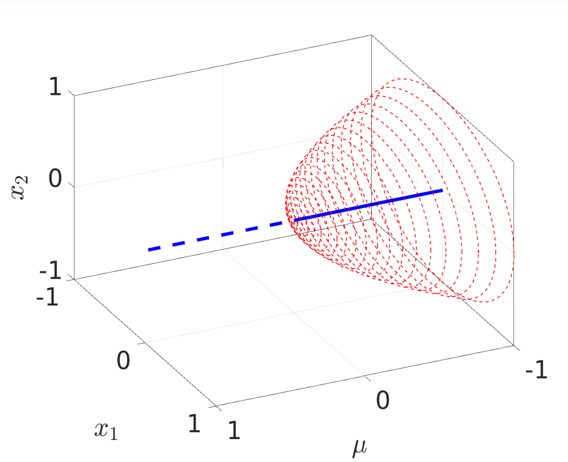

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subcritical Hopf bifurcation at $\mu = 0$

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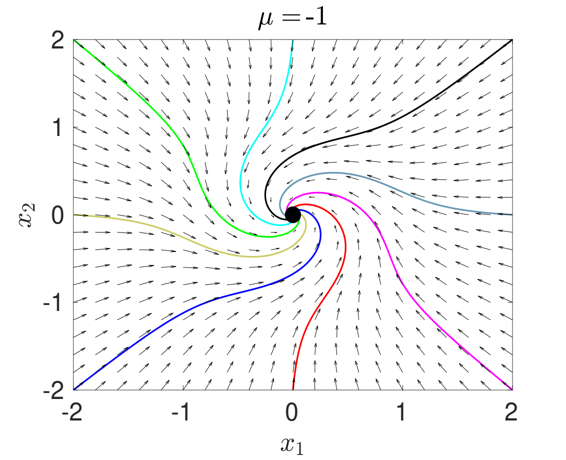

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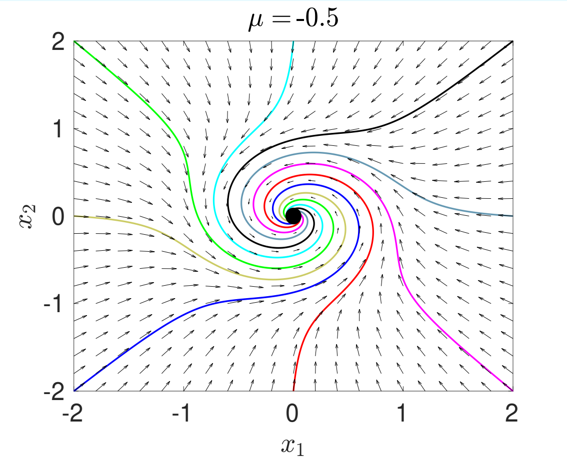

Saddle-node bifurcation of cycles

general case: $a(0) > 0$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha'(0)\,\mu\,r \;+\; a(0)\,r^3 \;-\; r^5$$

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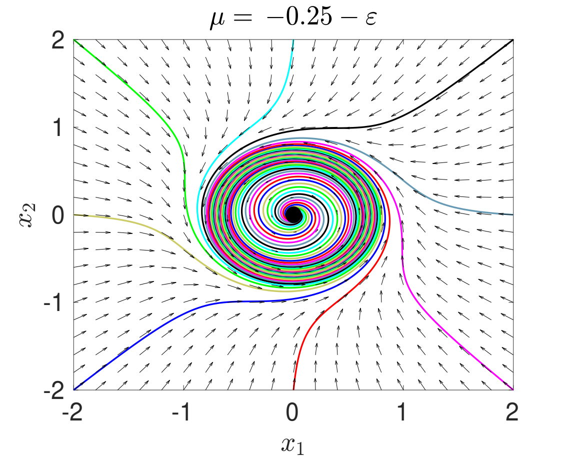
subcritical Hopf bifurcation at $\mu = 0$



saddle-node bifurcation of cycles at $\mu = -1/4$: a half-stable cycle appears, it splits into a pair of limit cycles for $\mu > -1/4$, one stable, one unstable

Saddle-node bifurcation of cycles

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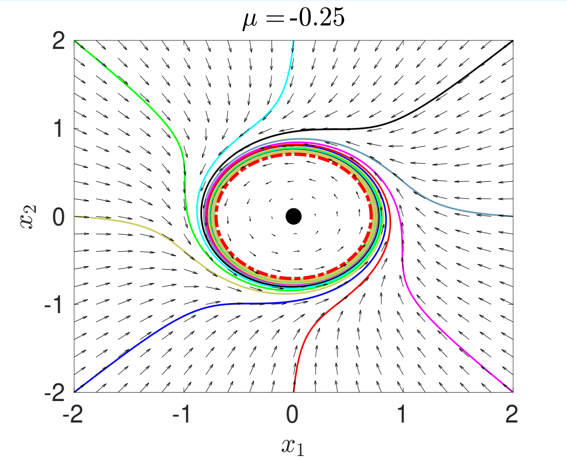
subcritical Hopf bifurcation at $\mu = 0$



saddle-node bifurcation of cycles at $\mu = -1/4$: a half-stable cycle appears, it splits into a pair of limit cycles for $\mu > -1/4$, one stable, one unstable

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example: $\alpha'(0) = 1,\ a(0) = 1,$
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$\mu < 0$: $\mathbf{0} = [0, 0]$ is a stable spiral

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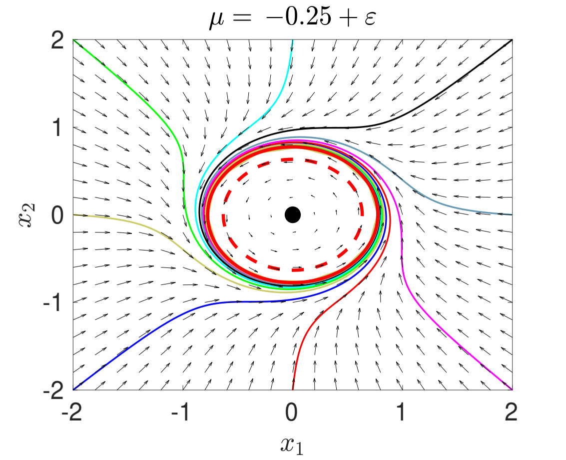
subcritical Hopf bifurcation at $\mu = 0$



saddle-node bifurcation of cycles at $\mu = -1/4$: viewed in the other direction, a stable and unstable cycle collide and disappear as μ decreases through $\mu = -1/4$

Subcritical Hopf bifurcation

general case: $a(0) > 0$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha'(0)\,\mu\, r \;+\; a(0)\, r^3 \;-\; r^5$$

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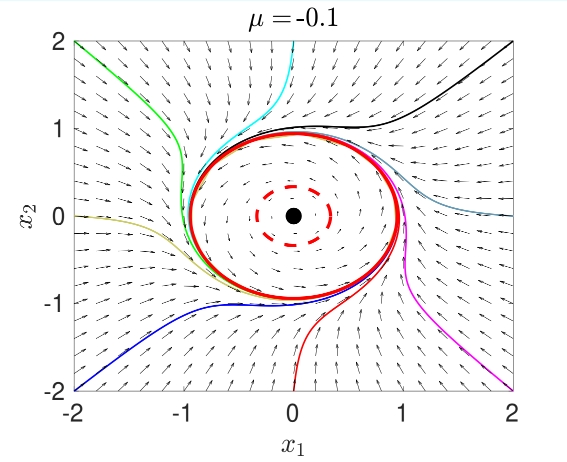

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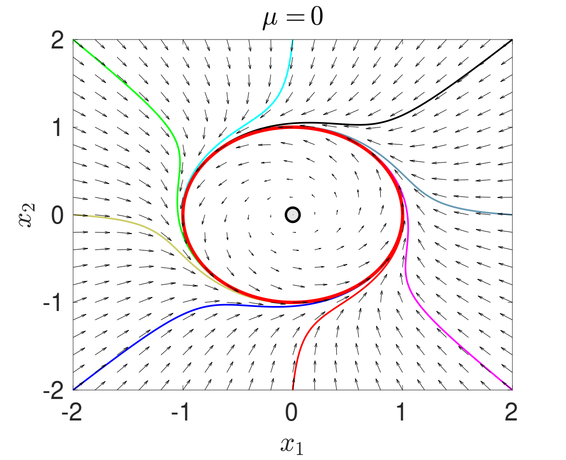

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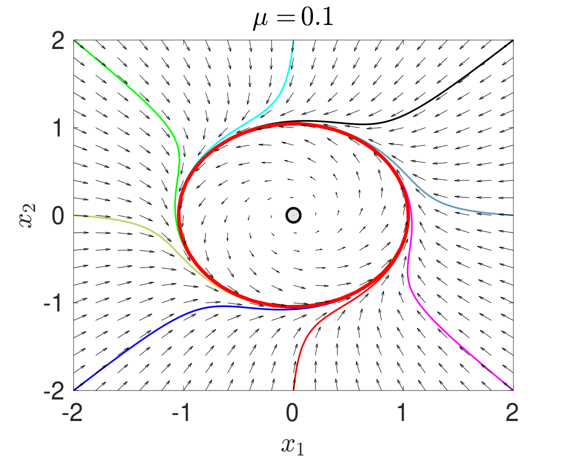

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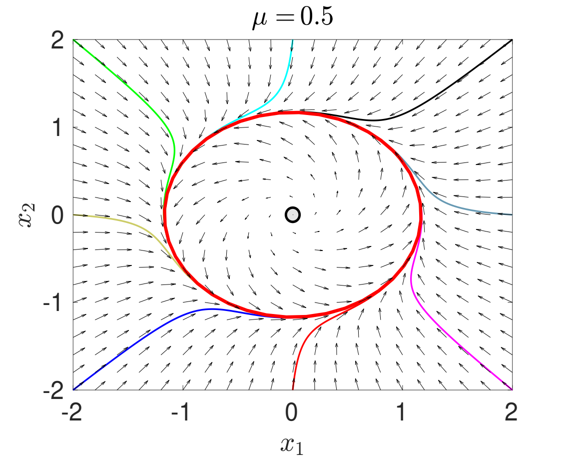

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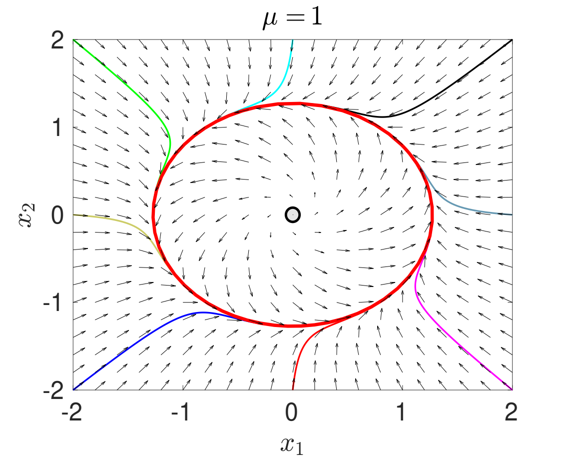

Subcritical Hopf bifurcation and saddle-node bifurcation of cycles

general case: $a(0) > 0$

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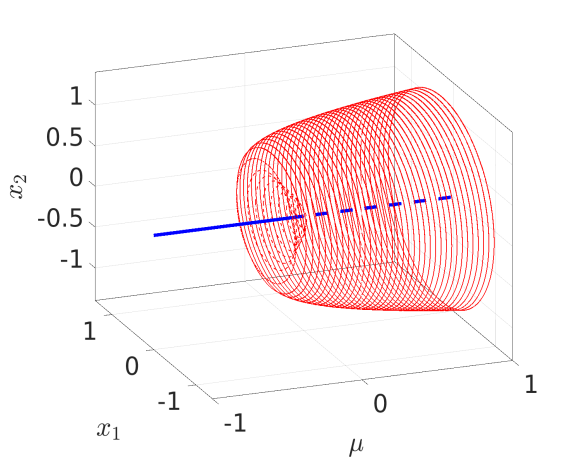
saddle-node bifurcation of cycles at $\mu = -1/4$



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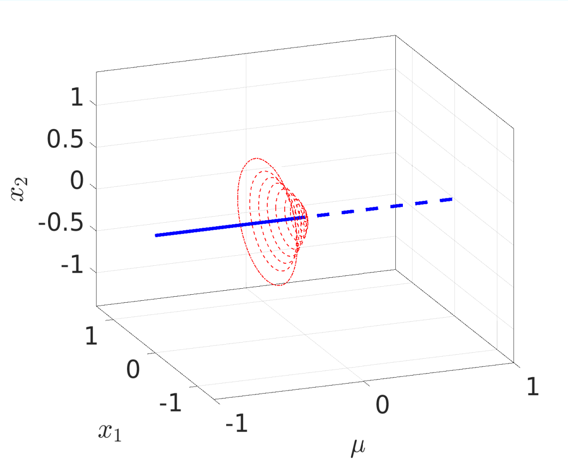

B5.6: Nonlinear Dynamics, Bifurcations and Chaos (Lecture 10)

- summary of Lecture 9: we discussed
  Bifurcations of limit cycles, covering supercritical and subcritical Hopf bifurcations, saddle-node bifurcation of cycles.

- today: we will continue in our discussion of Problem Sheet 3

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- course synopsis of **Lectures 9-16**:
  Bifurcations of limit cycles, covering supercritical and subcritical Hopf bifurcations, saddle-node bifurcation of cycles, infinite-period (SNIC) bifurcation and homoclinic (saddle-loop) bifurcation. Oscillations in chemical reaction networks. Weakly nonlinear oscillators. Poincaré-Lindstedt method. Conservative and non-conservative systems. Liénard systems, van der Pol oscillator. Hilbert's 16th problem. Lorenz equations. Lorenz map. Poincaré section. Poincaré map. Converse of Sharkovsky's theorem. Bernoulli shift map, symbolic dynamics. Tent map. Dynamics on metric spaces, sensitive dependence on initial conditions, transitivity, conjugate maps, chaotic dynamics.

Question 6 on Problem Sheet 1

System of $n = 2$ chemical species X_1 and X_2 which are subject to $\ell = 4$ reactions:

$$2X_1 + X_2 \xrightarrow{k_1} 3X_1 \qquad \emptyset \xrightarrow{k_2} X_1 \qquad X_1 \xrightarrow{k_3} \emptyset \qquad \emptyset \xrightarrow{k_4} X_2$$

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Assuming mass action kinetics, concentrations $x_1(t)$ and $x_2(t)$ evolve by

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= k_1\, x_1^2\, x_2 + k_2 - k_3\, x_1 \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -\, k_1\, x_1^2\, x_2 + k_4 \end{aligned}$$

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Using $k_1 = k_2 = 1$, $k_3 = \mu$ and $k_4 = 2$, we get:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2\, x_2 + 1 - \mu\, x_1$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -\,x_1^2\, x_2 + 2$$

Question 6 on Problem Sheet 1: We considered $\mu = 9$. We showed that the fixed point $[1/3, 18]$ is unstable and we found a trapping region (closed bounded connected set such that the vector field points inward everywhere on its boundary). We applied the Poincaré-Bendixson theorem to prove that there exists a periodic solution.

Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2\,x_2 + 1 - \mu\,x_1$$

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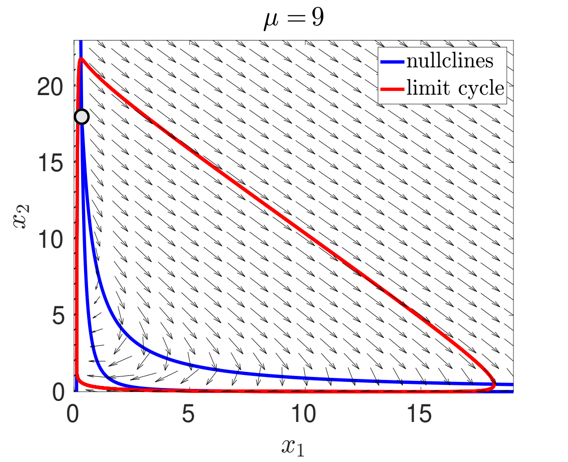
$\mu = 9$



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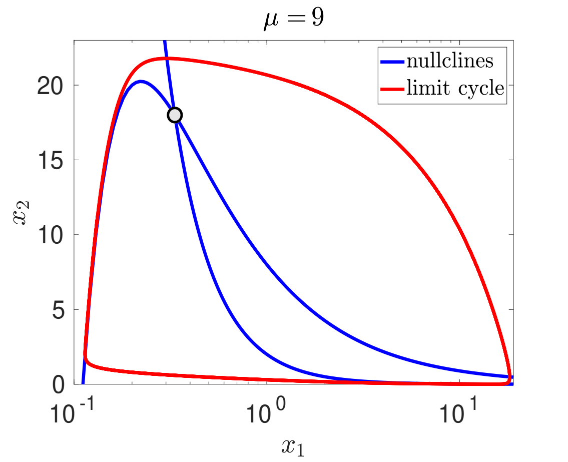

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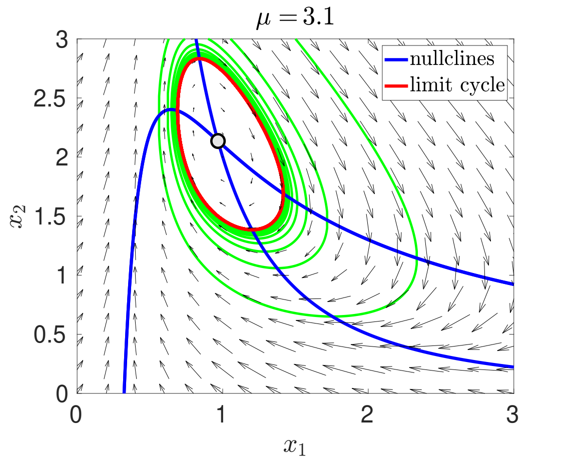
We decrease the value of parameter μ and the limit cycle shrinks.



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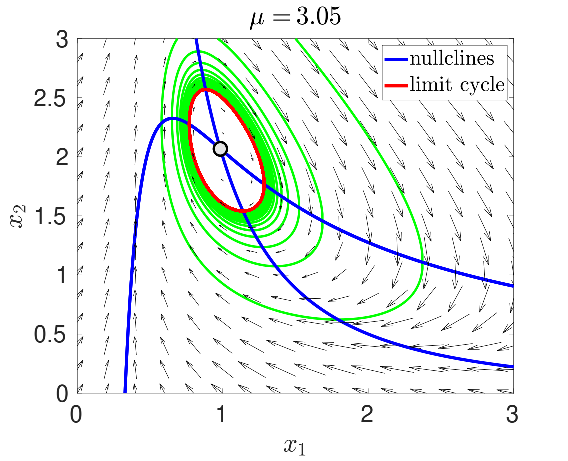

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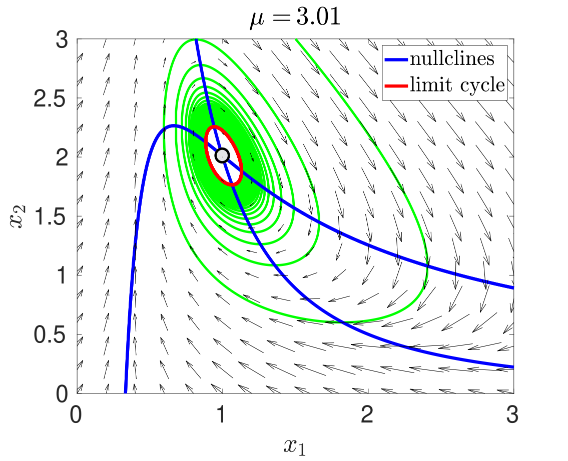

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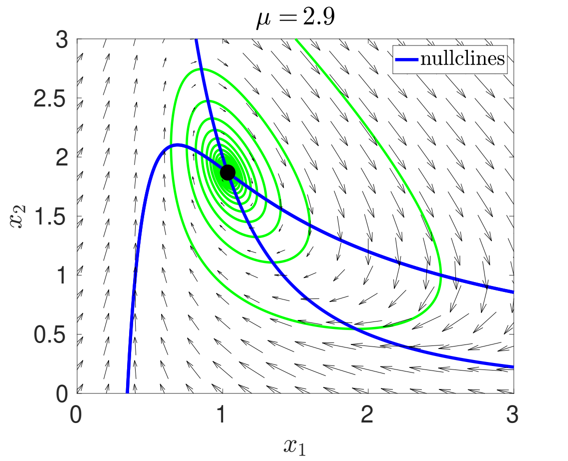
There is no limit cycle for $\mu < 3$.



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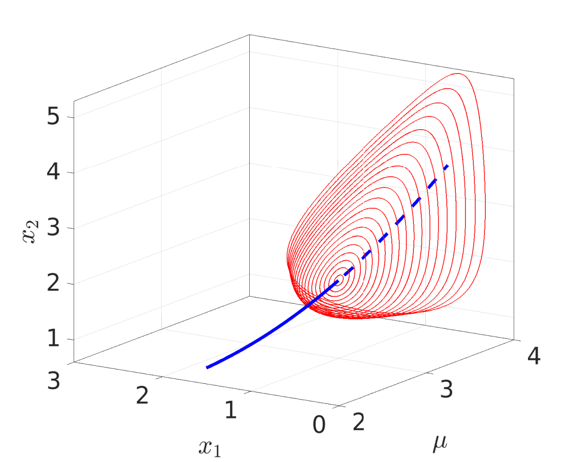
bifurcation diagram

[show 3D animation]



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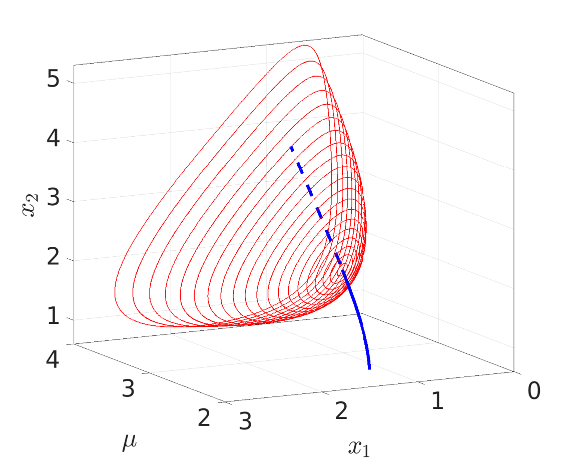

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2\,x_2 + 1 - \mu\,x_1$$

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fixed point at $\mathbf{x}_c = \left[\dfrac{3}{\mu},\ \dfrac{2\mu^2}{9}\right]$

Jacobian $D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} 2x_1x_2 - \mu & x_1^2 \\ -2x_1x_2 & -x_1^2 \end{pmatrix}$

$$D\mathbf{f}(\mathbf{x}_c) = \begin{pmatrix} \mu/3 & 9/\mu^2 \\ -4\mu/3 & -9/\mu^2 \end{pmatrix}$$

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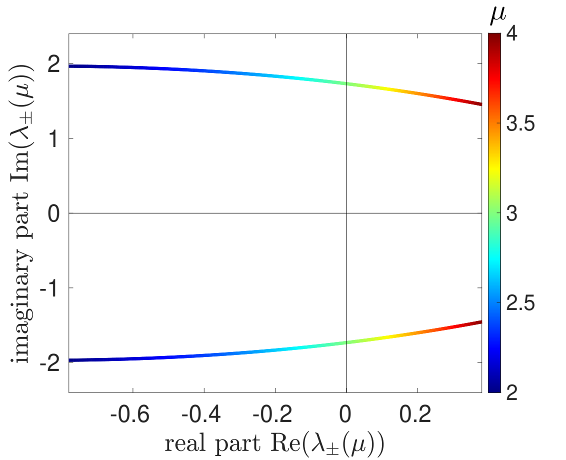

solving $\lambda^2 + \left(\dfrac{9}{\mu^2} - \dfrac{\mu}{3}\right)\lambda + \dfrac{9}{\mu} = 0$, we get $\lambda_\pm = \dfrac{1}{2}\left(\dfrac{\mu}{3} - \dfrac{9}{\mu^2} \pm \sqrt{\dfrac{\mu^2}{9} + \dfrac{81}{\mu^4} - \dfrac{42}{\mu}}\right)$

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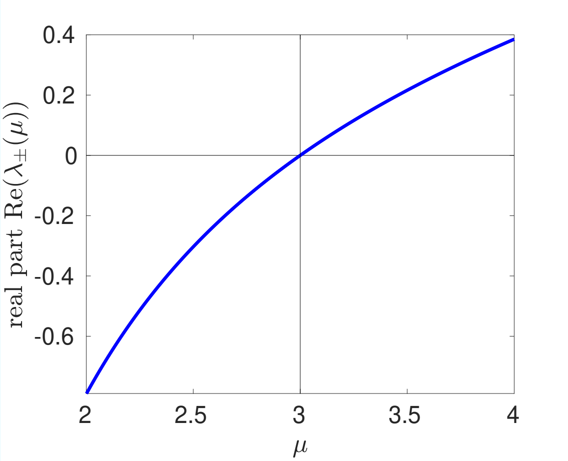

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bifurcation at $\mu = 3$, when $\lambda_\pm = \pm\, i\,\sqrt{3}$

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bifurcation at $\mu = 3$, when $\lambda_\pm = \pm\, i\,\sqrt{3}$

using new variables $\overline{x}_1 = x_1 - \frac{3}{\mu}$, $\overline{x}_2 = x_2 - \frac{2\mu^2}{9}$, $\overline{\mu} = \frac{\mu - 3}{3}$, we obtain

$$\frac{\mathrm{d}\overline{x}_1}{\mathrm{d}t} = (1+\overline{\mu})\,\overline{x}_1 + \frac{1}{(1+\overline{\mu})^2}\,\overline{x}_2 + 2(1+\overline{\mu})^2\,\overline{x}_1^2 + \frac{2}{1+\overline{\mu}}\,\overline{x}_1\,\overline{x}_2 + \overline{x}_1^2\,\overline{x}_2$$

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Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}\overline{x}_1}{\mathrm{d}t} = (1+\overline{\mu})\,\overline{x}_1 + \frac{1}{(1+\overline{\mu})^2}\,\overline{x}_2 + 2(1+\overline{\mu})^2\,\overline{x}_1^2 + \frac{2}{1+\overline{\mu}}\,\overline{x}_1\,\overline{x}_2 + \overline{x}_1^2\,\overline{x}_2$$

$$\frac{\mathrm{d}\overline{x}_2}{\mathrm{d}t} = -4\,(1+\overline{\mu})\,\overline{x}_1 - \frac{1}{(1+\overline{\mu})^2}\,\overline{x}_2 - 2(1+\overline{\mu})^2\,\overline{x}_1^2 - \frac{2}{1+\overline{\mu}}\,\overline{x}_1\,\overline{x}_2 - \overline{x}_1^2\,\overline{x}_2$$

bifurcation at $\overline{\mu} = 0$, fixed point $\mathbf{0}$ with $D\mathbf{f}(\mathbf{0}) = M(\overline{\mu}) = \begin{pmatrix} 1+\overline{\mu} & (1+\overline{\mu})^{-2} \\ -4\,(1+\overline{\mu}) & -(1+\overline{\mu})^{-2} \end{pmatrix}$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}\overline{x}_1}{\mathrm{d}t} = (1+\overline{\mu})\,\overline{x}_1 + \frac{1}{(1+\overline{\mu})^2}\,\overline{x}_2 + 2(1+\overline{\mu})^2\,\overline{x}_1^2 + \frac{2}{1+\overline{\mu}}\,\overline{x}_1\,\overline{x}_2 + \overline{x}_1^2\,\overline{x}_2$$

$$\frac{\mathrm{d}\overline{x}_2}{\mathrm{d}t} = -4\,(1+\overline{\mu})\,\overline{x}_1 - \frac{1}{(1+\overline{\mu})^2}\,\overline{x}_2 - 2(1+\overline{\mu})^2\,\overline{x}_1^2 - \frac{2}{1+\overline{\mu}}\,\overline{x}_1\,\overline{x}_2 - \overline{x}_1^2\,\overline{x}_2$$

bifurcation at $\overline{\mu} = 0$, fixed point $\mathbf{0}$ with $D\mathbf{f}(\mathbf{0}) = M(\overline{\mu}) = \begin{pmatrix} 1+\overline{\mu} & (1+\overline{\mu})^{-2} \\ -4\,(1+\overline{\mu}) & -(1+\overline{\mu})^{-2} \end{pmatrix}$

denote $g(\overline{x}_1, \overline{x}_2; \overline{\mu}) = 2(1+\overline{\mu})^2\,\overline{x}_1^2 + \frac{2}{1+\overline{\mu}}\,\overline{x}_1\,\overline{x}_2 + \overline{x}_1^2\,\overline{x}_2$ and rewrite the system as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\overline{x}_1 \\ \overline{x}_2\end{pmatrix} = M(\overline{\mu})\begin{pmatrix}\overline{x}_1 \\ \overline{x}_2\end{pmatrix} + g(\overline{x}_1, \overline{x}_2; \overline{\mu})\begin{pmatrix}1 \\ -1\end{pmatrix}$$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}\overline{x}_1}{\mathrm{d}t} = (1+\overline{\mu})\,\overline{x}_1 + \frac{1}{(1+\overline{\mu})^2}\,\overline{x}_2 + 2(1+\overline{\mu})^2\,\overline{x}_1^2 + \frac{2}{1+\overline{\mu}}\,\overline{x}_1\,\overline{x}_2 + \overline{x}_1^2\,\overline{x}_2$$

$$\frac{\mathrm{d}\overline{x}_2}{\mathrm{d}t} = -4\,(1+\overline{\mu})\,\overline{x}_1 - \frac{1}{(1+\overline{\mu})^2}\,\overline{x}_2 - 2(1+\overline{\mu})^2\,\overline{x}_1^2 - \frac{2}{1+\overline{\mu}}\,\overline{x}_1\,\overline{x}_2 - \overline{x}_1^2\,\overline{x}_2$$

bifurcation at $\overline{\mu}=0$, fixed point $\mathbf{0}$ with $D\mathbf{f}(\mathbf{0}) = M(\overline{\mu}) = \begin{pmatrix} 1+\overline{\mu} & (1+\overline{\mu})^{-2} \\ -4\,(1+\overline{\mu}) & -(1+\overline{\mu})^{-2} \end{pmatrix}$

denote $g(\overline{x}_1,\overline{x}_2;\overline{\mu}) = 2(1+\overline{\mu})^2\,\overline{x}_1^2 + \frac{2}{1+\overline{\mu}}\,\overline{x}_1\,\overline{x}_2 + \overline{x}_1^2\,\overline{x}_2$ and rewrite the system as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = M(\overline{\mu})\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + g(\overline{x}_1,\overline{x}_2;\overline{\mu})\begin{pmatrix}1\\ -1\end{pmatrix}$$

at $\overline{\mu}=0$, we have $M(0) = \begin{pmatrix} 1 & 1 \\ -4 & -1 \end{pmatrix}$

eigenvalues $\lambda_\pm = \pm\, i\,\sqrt{3}$, eigenvectors $\mathbf{v}_\pm = \begin{pmatrix}1\\ -4\end{pmatrix} \pm\, i\begin{pmatrix}\sqrt{3}\\ 0\end{pmatrix}$,

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}\overline{x}_1}{\mathrm{d}t} = (1+\overline{\mu})\,\overline{x}_1 + \frac{1}{(1+\overline{\mu})^2}\,\overline{x}_2 + 2(1+\overline{\mu})^2\,\overline{x}_1^2 + \frac{2}{1+\overline{\mu}}\,\overline{x}_1\,\overline{x}_2 + \overline{x}_1^2\,\overline{x}_2$$

$$\frac{\mathrm{d}\overline{x}_2}{\mathrm{d}t} = -4\,(1+\overline{\mu})\,\overline{x}_1 - \frac{1}{(1+\overline{\mu})^2}\,\overline{x}_2 - 2(1+\overline{\mu})^2\,\overline{x}_1^2 - \frac{2}{1+\overline{\mu}}\,\overline{x}_1\,\overline{x}_2 - \overline{x}_1^2\,\overline{x}_2$$

bifurcation at $\overline{\mu}=0$, fixed point $\mathbf{0}$ with $D\mathbf{f}(\mathbf{0}) = M(\overline{\mu}) = \begin{pmatrix} 1+\overline{\mu} & (1+\overline{\mu})^{-2} \\ -4\,(1+\overline{\mu}) & -(1+\overline{\mu})^{-2} \end{pmatrix}$

denote $g(\overline{x}_1,\overline{x}_2;\overline{\mu}) = 2(1+\overline{\mu})^2\,\overline{x}_1^2 + \dfrac{2}{1+\overline{\mu}}\,\overline{x}_1\,\overline{x}_2 + \overline{x}_1^2\,\overline{x}_2$ and rewrite the system as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} = M(\overline{\mu})\begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} + g(\overline{x}_1,\overline{x}_2;\overline{\mu})\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

at $\overline{\mu}=0$, we have $M(0) = \begin{pmatrix} 1 & 1 \\ -4 & -1 \end{pmatrix}$

eigenvalues $\lambda_\pm = \pm\, i\,\sqrt{3}$, eigenvectors $\mathbf{v}_\pm = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \pm i \begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix}$, change of variables

$$\begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{3} \\ -4 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \text{ with inverse } \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix} 0 & -\sqrt{3} \\ 4 & 1 \end{pmatrix}\begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix}$$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = M(0)\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + g(\overline{x}_1,\overline{x}_2;0)\begin{pmatrix}1\\ -1\end{pmatrix}$$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = M(0)\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + g(\overline{x}_1,\overline{x}_2;0)\begin{pmatrix}1\\ -1\end{pmatrix}$$

change of variables: $\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = \begin{pmatrix}1 & \sqrt{3}\\ -4 & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}$ and $\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix}$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathsf{d}}{\mathsf{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = M(0)\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + g(\overline{x}_1,\overline{x}_2;0)\begin{pmatrix}1\\ -1\end{pmatrix}$$

change of variables: $\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = \begin{pmatrix}1 & \sqrt{3}\\ -4 & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}$ and $\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \dfrac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix}$

$$\frac{\mathsf{d}}{\mathsf{d}t}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}\frac{\mathsf{d}}{\mathsf{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix}$$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathsf{d}}{\mathsf{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = M(0)\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + g(\overline{x}_1,\overline{x}_2;0)\begin{pmatrix}1\\ -1\end{pmatrix}$$

change of variables: $\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = \begin{pmatrix}1 & \sqrt{3}\\ -4 & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}$ and $\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \dfrac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix}$

$$\frac{\mathsf{d}}{\mathsf{d}t}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}\frac{\mathsf{d}}{\mathsf{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix}$$

$$\frac{\mathsf{d}}{\mathsf{d}t}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}M(0)\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}g(\overline{x}_1,\overline{x}_2;0)\begin{pmatrix}1\\ -1\end{pmatrix}$$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathsf{d}}{\mathsf{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = M(0)\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + g(\overline{x}_1,\overline{x}_2;0)\begin{pmatrix}1\\ -1\end{pmatrix}$$

change of variables: $\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = \begin{pmatrix}1 & \sqrt{3}\\ -4 & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}$ and $\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \dfrac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix}$

$$\frac{\mathsf{d}}{\mathsf{d}t}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}\frac{\mathsf{d}}{\mathsf{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix}$$

$$\frac{\mathsf{d}}{\mathsf{d}t}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}M(0)\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}g(\overline{x}_1,\overline{x}_2;0)\begin{pmatrix}1\\ -1\end{pmatrix}$$

$$\frac{\mathsf{d}}{\mathsf{d}t}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}M(0)\begin{pmatrix}1 & \sqrt{3}\\ -4 & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} + \frac{1}{4}\begin{pmatrix}1\\ \sqrt{3}\end{pmatrix}g(\overline{x}_1,\overline{x}_2;0)$$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = M(0)\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + g(\overline{x}_1,\overline{x}_2;0)\begin{pmatrix}1\\ -1\end{pmatrix}$$

change of variables: $\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = \begin{pmatrix}1 & \sqrt{3}\\ -4 & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}$ and $\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}M(0)\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}g(\overline{x}_1,\overline{x}_2;0)\begin{pmatrix}1\\ -1\end{pmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}M(0)\begin{pmatrix}1 & \sqrt{3}\\ -4 & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} + \frac{1}{4}\begin{pmatrix}1\\ \sqrt{3}\end{pmatrix}g(\overline{x}_1,\overline{x}_2;0)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}0 & \sqrt{3}\\ -\sqrt{3} & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} + \frac{1}{4}\begin{pmatrix}1\\ \sqrt{3}\end{pmatrix}g(\overline{x}_1,\overline{x}_2;0)$$

where $g(\overline{x}_1,\overline{x}_2;0) = 2\,\overline{x}_1^2 + 2\,\overline{x}_1\,\overline{x}_2 + \overline{x}_1^2\,\overline{x}_2$

$$= -6\,y_1^2 - 4\,y_1\,y_2\sqrt{3} + 6\,y_2^2 - 4\,y_1^3 - 8\,y_1^2\,y_2\sqrt{3} - 12\,y_2^2\,y_1$$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} h(y_1, y_2)$$

where $\quad h(y_1, y_2) = -3\,y_1^2 - 2\,y_1\,y_2\,\sqrt{3} + 3\,y_2^2 - 2\,y_1^3 - 4\,y_1^2\,y_2\,\sqrt{3} - 6\,y_2^2\,y_1$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} h(y_1, y_2)$$

where $h(y_1, y_2) = -3\,y_1^2 - 2\,y_1\,y_2\,\sqrt{3} + 3\,y_2^2 - 2\,y_1^3 - 4\,y_1^2\,y_2\,\sqrt{3} - 6\,y_2^2\,y_1$

$\overline{\mu}$ close to the bifurcation point $\overline{\mu} = 0$: matrix $M(\overline{\mu}) = \begin{pmatrix} 1+\overline{\mu} & (1+\overline{\mu})^{-2} \\ -4\,(1+\overline{\mu}) & -(1+\overline{\mu})^{-2} \end{pmatrix}$

has eigenvalues $\lambda_\pm(\overline{\mu}) = \alpha(\overline{\mu}) \pm i\,\omega(\overline{\mu})$ where

$\alpha(\overline{\mu}) = \dfrac{1}{2}\left(1 + \overline{\mu} - \dfrac{1}{(1+\overline{\mu})^2}\right)$ and $\omega(\overline{\mu}) = -\dfrac{1}{2}\sqrt{\dfrac{14}{1+\overline{\mu}} - 1 - 2\,\overline{\mu} - \overline{\mu}^2 - \dfrac{1}{(1+\overline{\mu})^4}}$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} h(y_1, y_2)$$

where $h(y_1, y_2) = -3\,y_1^2 - 2\,y_1\,y_2\,\sqrt{3} + 3\,y_2^2 - 2\,y_1^3 - 4\,y_1^2\,y_2\,\sqrt{3} - 6\,y_2^2\,y_1$

$\overline{\mu}$ close to the bifurcation point $\overline{\mu} = 0$: matrix $M(\overline{\mu}) = \begin{pmatrix} 1+\overline{\mu} & (1+\overline{\mu})^{-2} \\ -4\,(1+\overline{\mu}) & -(1+\overline{\mu})^{-2} \end{pmatrix}$

has eigenvalues $\lambda_\pm(\overline{\mu}) = \alpha(\overline{\mu}) \pm i\,\omega(\overline{\mu})$ where

$\alpha(\overline{\mu}) = \dfrac{1}{2}\left(1 + \overline{\mu} - \dfrac{1}{(1+\overline{\mu})^2}\right)$ and $\omega(\overline{\mu}) = -\dfrac{1}{2}\sqrt{\dfrac{14}{1+\overline{\mu}} - 1 - 2\,\overline{\mu} - \overline{\mu}^2 - \dfrac{1}{(1+\overline{\mu})^4}}$

which implies $\alpha(0) = 0$, $\omega(0) = -\sqrt{3}$, $\alpha'(0) = \dfrac{3}{2}$ and $\omega'(0) = \dfrac{\sqrt{3}}{2}$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} h(y_1, y_2)$$

where $\quad h(y_1, y_2) = -3\,y_1^2 - 2\,y_1\,y_2\,\sqrt{3} + 3\,y_2^2 - 2\,y_1^3 - 4\,y_1^2\,y_2\,\sqrt{3} - 6\,y_2^2\,y_1$

$\overline{\mu}$ close to the bifurcation point $\overline{\mu} = 0$: matrix $M(\overline{\mu}) = \begin{pmatrix} 1+\overline{\mu} & (1+\overline{\mu})^{-2} \\ -4\,(1+\overline{\mu}) & -(1+\overline{\mu})^{-2} \end{pmatrix}$

has eigenvalues $\lambda_\pm(\overline{\mu}) = \alpha(\overline{\mu}) \pm i\,\omega(\overline{\mu})$ where

$$\alpha(\overline{\mu}) = \frac{1}{2}\left(1 + \overline{\mu} - \frac{1}{(1+\overline{\mu})^2}\right) \text{ and } \omega(\overline{\mu}) = -\frac{1}{2}\sqrt{\frac{14}{1+\overline{\mu}} - 1 - 2\,\overline{\mu} - \overline{\mu}^2 - \frac{1}{(1+\overline{\mu})^4}}$$

which implies $\alpha(0) = 0$, $\omega(0) = -\sqrt{3}$, $\alpha'(0) = \dfrac{3}{2}$ and $\omega'(0) = \dfrac{\sqrt{3}}{2}$

normal form in polar coordinates:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha'(0)\,\overline{\mu}\,r + a(0)\,r^3 + \mathcal{O}(\overline{\mu}^2 r, \overline{\mu} r^3, r^5)$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega(0) + \omega'(0)\,\overline{\mu} + b(0)\,r^2 + \mathcal{O}(\overline{\mu}^2, \overline{\mu} r^2, r^4)$$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} h(y_1, y_2)$$

where $h(y_1, y_2) = -3\,y_1^2 - 2\,y_1\,y_2\,\sqrt{3} + 3\,y_2^2 - 2\,y_1^3 - 4\,y_1^2\,y_2\,\sqrt{3} - 6\,y_2^2\,y_1$

$\overline{\mu}$ close to the bifurcation point $\overline{\mu} = 0$: matrix $M(\overline{\mu}) = \begin{pmatrix} 1+\overline{\mu} & (1+\overline{\mu})^{-2} \\ -4\,(1+\overline{\mu}) & -(1+\overline{\mu})^{-2} \end{pmatrix}$

has eigenvalues $\lambda_\pm(\overline{\mu}) = \alpha(\overline{\mu}) \pm i\,\omega(\overline{\mu})$ where

$$\alpha(\overline{\mu}) = \frac{1}{2}\left(1 + \overline{\mu} - \frac{1}{(1+\overline{\mu})^2}\right) \text{ and } \omega(\overline{\mu}) = -\frac{1}{2}\sqrt{\frac{14}{1+\overline{\mu}} - 1 - 2\,\overline{\mu} - \overline{\mu}^2 - \frac{1}{(1+\overline{\mu})^4}}$$

which implies $\alpha(0) = 0$, $\omega(0) = -\sqrt{3}$, $\alpha'(0) = \dfrac{3}{2}$ and $\omega'(0) = \dfrac{\sqrt{3}}{2}$

normal form in polar coordinates:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{3}{2}\,\overline{\mu}\,r + a(0)\,r^3 + \mathcal{O}(\overline{\mu}^2 r, \overline{\mu} r^3, r^5)$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\sqrt{3} + \frac{\sqrt{3}}{2}\,\overline{\mu} + b(0)\,r^2 + \mathcal{O}(\overline{\mu}^2, \overline{\mu} r^2, r^4)$$

Calculation of $a(0)$

supercritical Hopf bifurcation: $a(0) < 0$ (periodic orbit is asymptotically stable)

subcritical Hopf bifurcation: $a(0) > 0$ (periodic orbit is unstable)

Calculation of $a(0)$

supercritical Hopf bifurcation: $a(0) < 0$ (periodic orbit is asymptotically stable)

subcritical Hopf bifurcation: $a(0) > 0$ (periodic orbit is unstable)

Lemma: Assume that the ODE system with Hopf bifurcation at $\overline{\mu} = 0$ was transformed to

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & -\omega(0) \\ \omega(0) & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} h_1(y_1, y_2) \\ h_2(y_1, y_2) \end{pmatrix}$$

where $h_1(y_1, y_2)$ and $h_2(y_1, y_2)$ contain only higher-order nonlinear terms that vanish at the origin. Then

$$a(0) = \frac{1}{16}\left(\frac{\partial^3 h_1}{\partial y_1^3} + \frac{\partial^3 h_1}{\partial y_1 \partial y_2^2} + \frac{\partial^3 h_2}{\partial y_1^2 \partial y_2} + \frac{\partial^3 h_2}{\partial y_2^3}\right) + \frac{1}{16\,\omega(0)}\left[\frac{\partial^2 h_1}{\partial y_1 \partial y_2}\left(\frac{\partial^2 h_1}{\partial y_1^2} + \frac{\partial^2 h_1}{\partial y_2^2}\right)\right.$$
$$\left. - \frac{\partial^2 h_2}{\partial y_1 \partial y_2}\left(\frac{\partial^2 h_2}{\partial y_1^2} + \frac{\partial^2 h_2}{\partial y_2^2}\right) - \frac{\partial^2 h_1}{\partial y_1^2}\frac{\partial^2 h_2}{\partial y_1^2} + \frac{\partial^2 h_1}{\partial y_2^2}\frac{\partial^2 h_2}{\partial y_2^2}\right]$$

where the partial derivatives are evaluated at the origin $\mathbf{0}$.

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

Our equation $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} h(y_1, y_2)$$

where $h(y_1, y_2) = -3\,y_1^2 - 2\,y_1\,y_2\,\sqrt{3} + 3\,y_2^2 - 2\,y_1^3 - 4\,y_1^2\,y_2\,\sqrt{3} - 6\,y_1\,y_2^2$

is in the form $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & -\omega(0) \\ \omega(0) & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} h_1(y_1, y_2) \\ h_2(y_1, y_2) \end{pmatrix}$$

where $\omega_0 = -\sqrt{3}$, $h_1(y_1, y_2) = h(y_1, y_2)/2$ and $h_2(y_1, y_2) = \sqrt{3}\,h(y_1, y_2)/2$.

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

Our equation $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} h(y_1, y_2)$$

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Substituting (partial derivatives evaluated at the origin $\mathbf{0}$):

$$\frac{\partial^3 h_1}{\partial y_1^3} = -6, \quad \frac{\partial^3 h_1}{\partial y_1 \partial y_2^2} = -6, \quad \frac{\partial^3 h_2}{\partial y_1^2 \partial y_2} = -12, \quad \frac{\partial^3 h_2}{\partial y_2^3} = 0, \quad \frac{\partial^2 h_1}{\partial y_1^2} = -3,$$

$$\frac{\partial^2 h_1}{\partial y_1 \partial y_2} = -\sqrt{3}, \quad \frac{\partial^2 h_1}{\partial y_2^2} = 3, \quad \frac{\partial^2 h_2}{\partial y_1^2} = -3\sqrt{3}, \quad \frac{\partial^2 h_2}{\partial y_1 \partial y_2} = -3, \quad \frac{\partial^2 h_2}{\partial y_2^2} = 3\,\sqrt{3}$$

we get $a(0) = -\dfrac{3}{2}$ $\quad\Longrightarrow\quad$ supercritical Hopf bifurcation

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

normal form:

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Origin $\mathbf{0}$ is stable for $\overline{\mu} < 0 \;\Leftrightarrow\; \mu < 3$

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$$\overline{x}_2^2 \,+\, \frac{1}{3}\,\left(4\,\overline{x}_1 + \overline{x}_2\right)^2 = \frac{16(\mu - 3)}{3}$$

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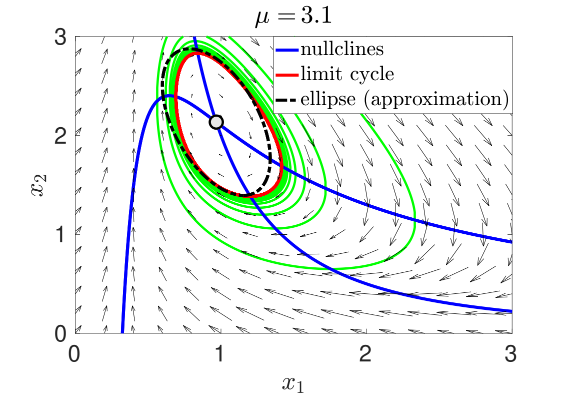

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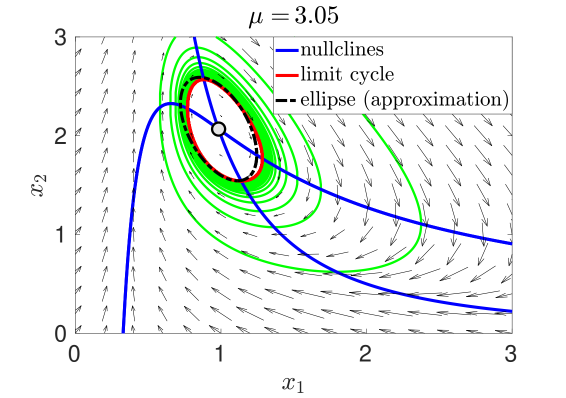

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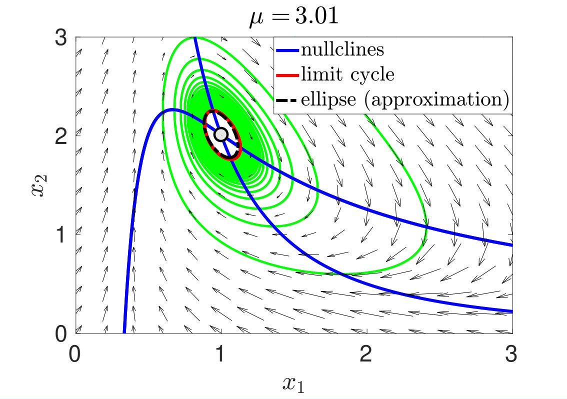

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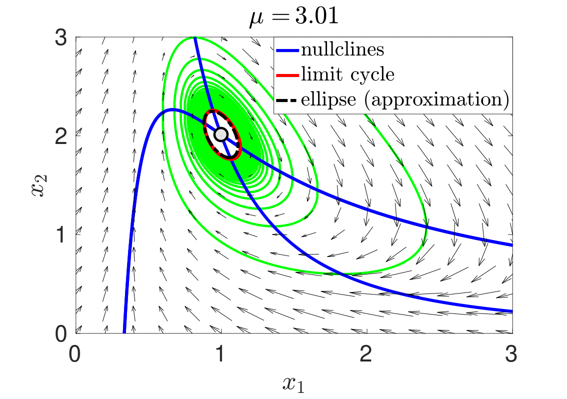

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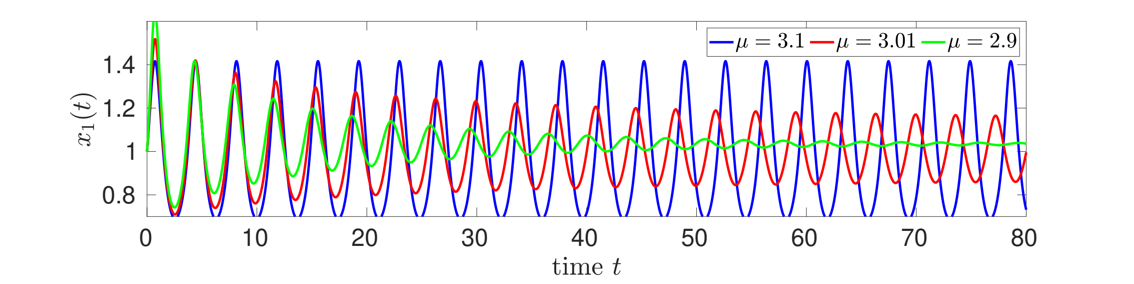
Additional examples: Question 2 on Problem Sheet 3 and
Questions 1, 4, 5 and 6 (formulated in a way that the questions do not specify what bifurcations of limit cycles are there)

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1



A stable limit cycle is born with amplitude $\sqrt{\dfrac{\mu - 3}{3}}$ and period $\dfrac{2\pi}{\sqrt{3}} \approx 3.6$

Close to the bifurcation point $\mu = \mu_c$, the amplitude is $\mathcal{O}\left(\sqrt{|\mu - \mu_c|}\right)$

NEXT: we will consider global bifucations when the amplitude will satisfy $\mathcal{O}\left(1\right)$,
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B5.6: Nonlinear Dynamics, Bifurcations and Chaos (Lecture 11)

- summary of Lecture 10: we discussed
  Bifurcations of limit cycles, covering supercritical and subcritical Hopf bifurcations.
  Oscillations in chemical reaction networks.
- today: we will continue in our discussion of Problem Sheet 3

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- course synopsis of **Lectures 9-16**:
  Bifurcations of limit cycles, covering supercritical and subcritical Hopf bifurcations, saddle-node bifurcation of cycles, infinite-period (SNIC) bifurcation and homoclinic (saddle-loop) bifurcation. Oscillations in chemical reaction networks. Weakly nonlinear oscillators. Poincaré-Lindstedt method. Conservative and non-conservative systems. Liénard systems, van der Pol oscillator. Hilbert's 16th problem. Lorenz equations. Lorenz map. Poincaré section. Poincaré map. Converse of Sharkovsky's theorem. Bernoulli shift map, symbolic dynamics. Tent map. Dynamics on metric spaces, sensitive dependence on initial conditions, transitivity, conjugate maps, chaotic dynamics.

Bifurcations of limit cycles

bifurcation at $\mu = \mu_c$	amplitude	period
supercritical Hopf bifurcation	$\mathcal{O}\left(\sqrt{\lvert\mu - \mu_c\rvert}\right)$	$\mathcal{O}(1)$
subcritical Hopf bifurcation	$\mathcal{O}\left(\sqrt{\lvert\mu - \mu_c\rvert}\right)$	$\mathcal{O}(1)$
saddle-node bifurcation of cycles	$\mathcal{O}(1)$	$\mathcal{O}(1)$
infinite-period (SNIC, SNIPER)	$\mathcal{O}(1)$	$\mathcal{O}\left(\dfrac{1}{\sqrt{\lvert\mu - \mu_c\rvert}}\right)$
homoclinic (saddle-loop) bifurcation	$\mathcal{O}(1)$	$\mathcal{O}\left(\big\lvert \log\lvert\mu - \mu_c\rvert\big\rvert\right)$

supercritical and subcritical Hopf bifurcations: we discussed them last week (including the analysis of the supercritical Hopf bifurcation in the chemical system in Question 6 on Problem Sheet 1)

Lecture 10: chemical system in Question 6 on Problem Sheet 1

normal form (where $\overline{\mu} = (\mu - 3)/3$):

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{3}{2}\,\overline{\mu}\,r - \frac{3}{2}\,r^3 + \dots$$

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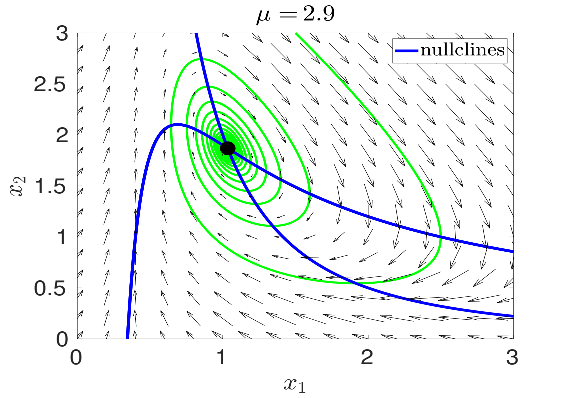

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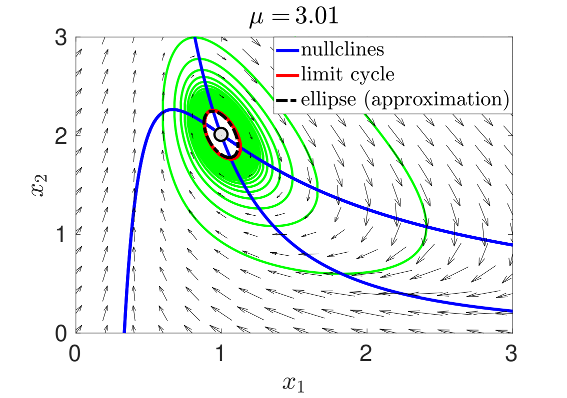

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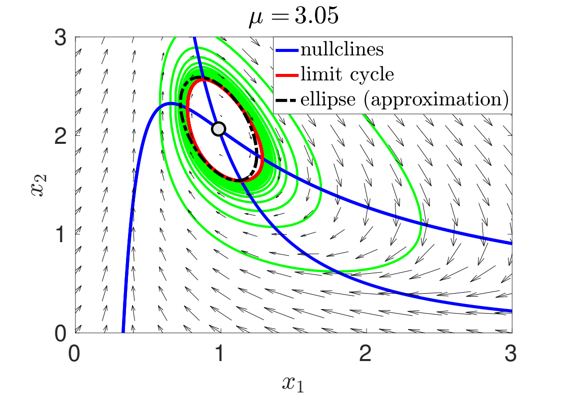

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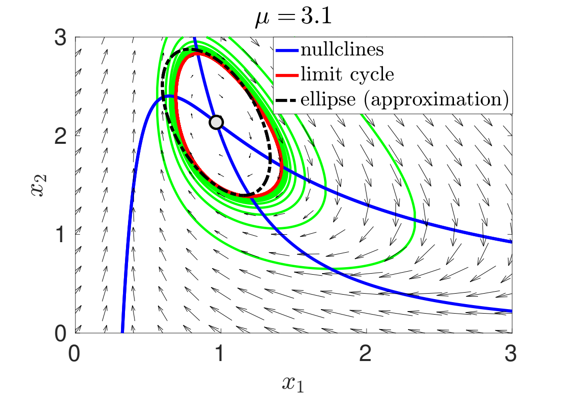

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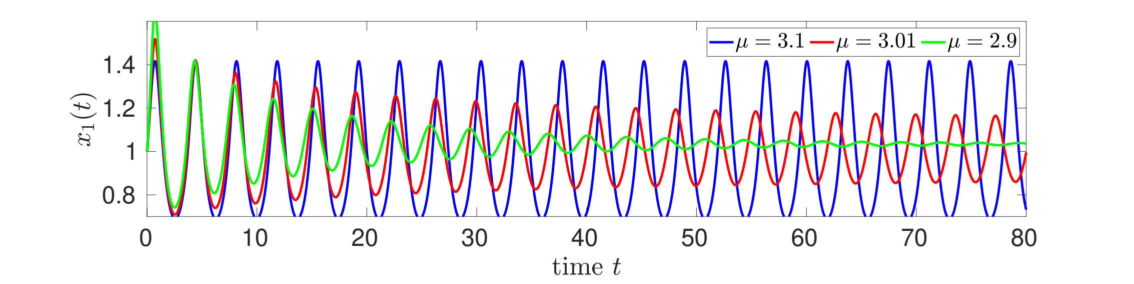
Lecture 10: chemical system in Question 6 on Problem Sheet 1



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saddle-node bifurcation of cycles: we have already presented an example when we discussed the subcritical Hopf bifurcation

Example: saddle-node bifurcation of cycles

saddle-node bifurcation of cycles at $\mu = -1/4$: a half-stable cycle appears, it splits into a pair of limit cycles for $\mu > -1/4$, one stable, one unstable, or, viewed in the other direction, a stable and unstable cycle collide and disappear as μ decreases through $\mu = -1/4$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \mu r + r^3 - r^5$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1$$

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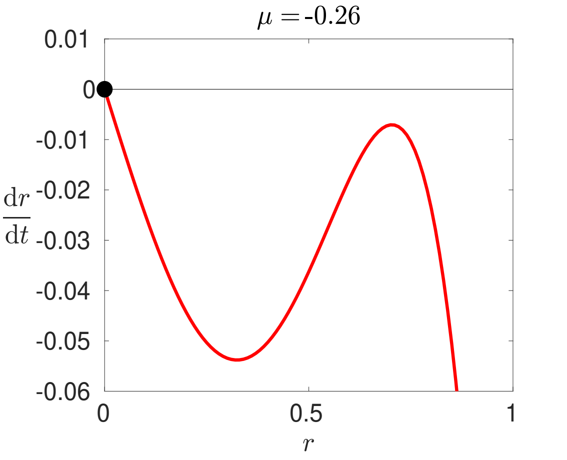

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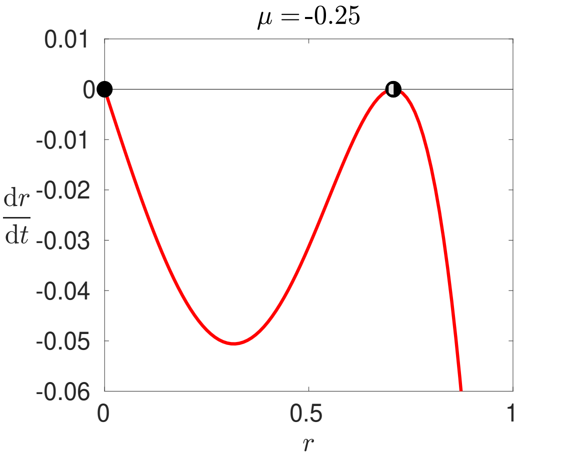

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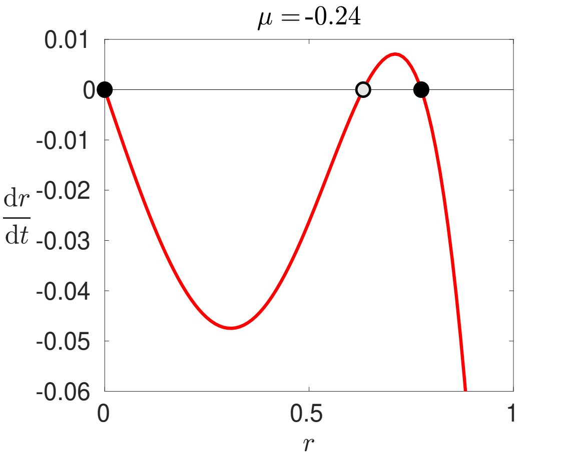

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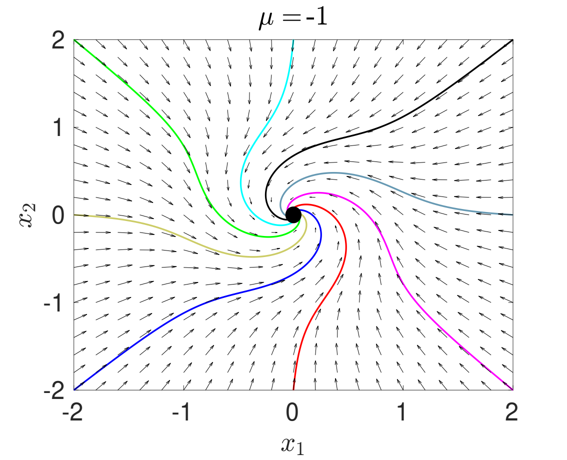

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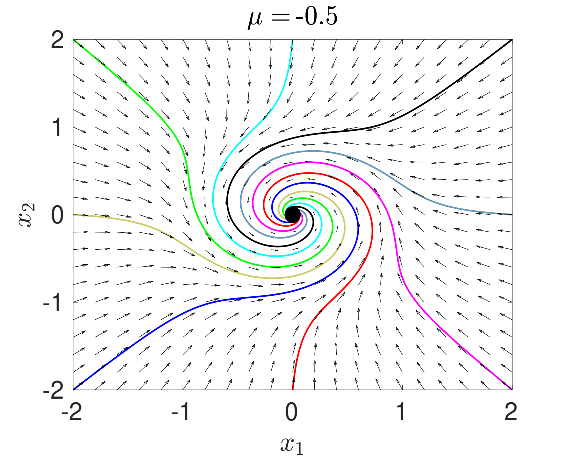

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saddle-node bifurcation of cycles at $\mu = -1/4$: a half-stable cycle appears, it splits into a pair of limit cycles for $\mu > -1/4$, one stable, one unstable, or, viewed in the other direction, a stable and unstable cycle collide and disappear as μ decreases through $\mu = -1/4$

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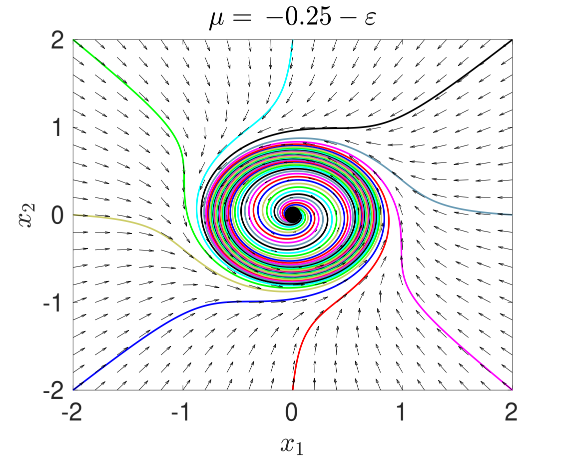

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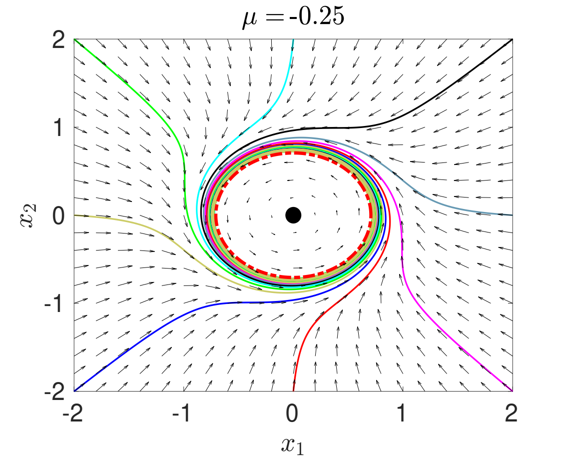

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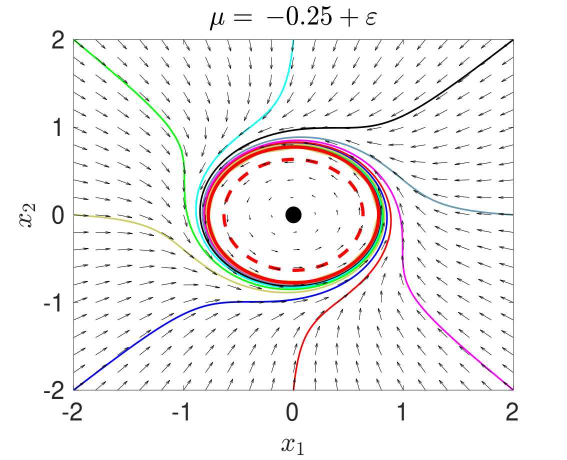

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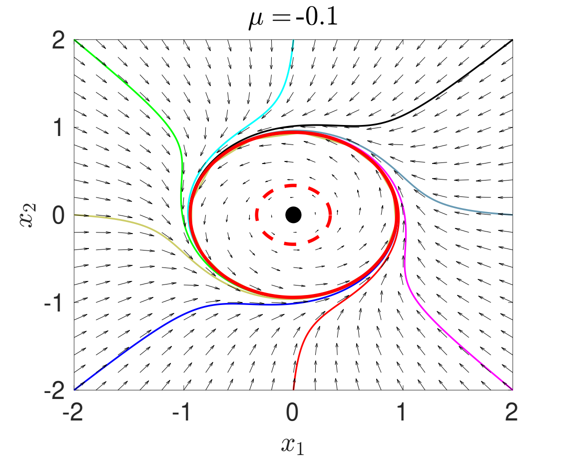

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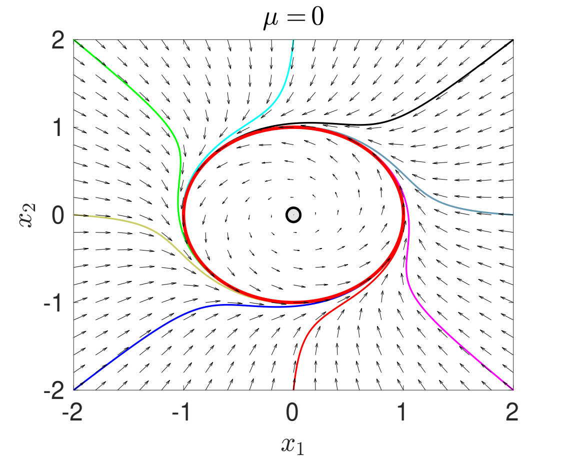

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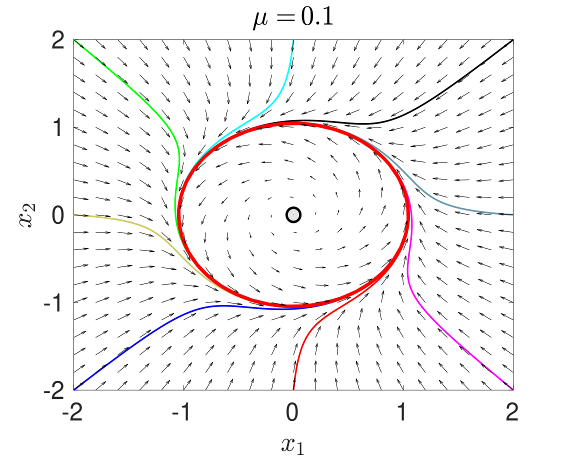

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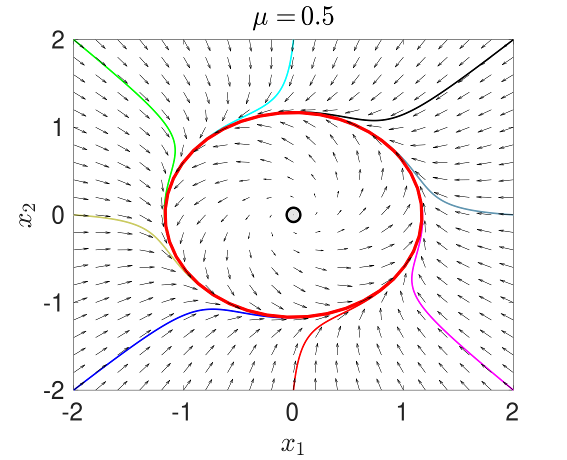

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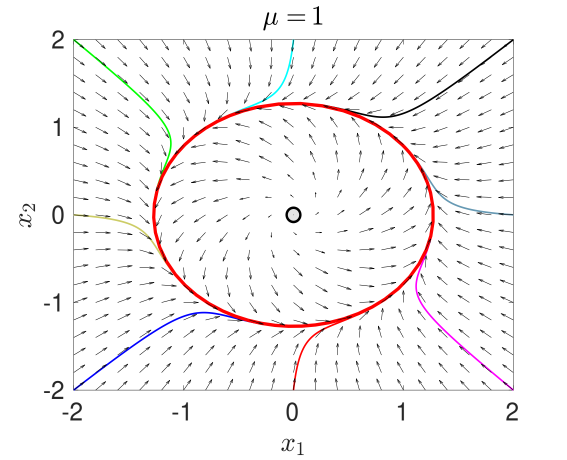

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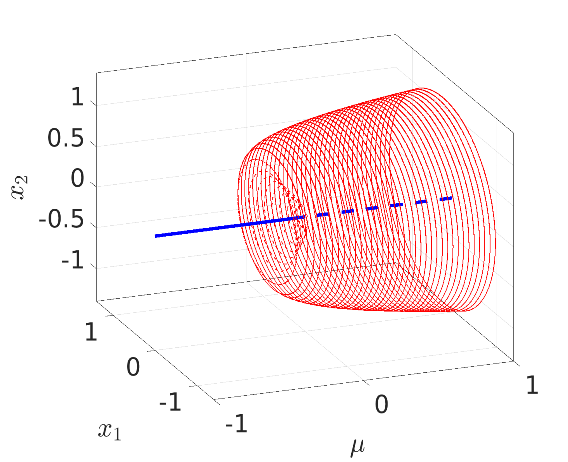

Bifurcations of limit cycles

bifurcation at $\mu = \mu_c$	amplitude	period
supercritical Hopf bifurcation	$\mathcal{O}\left(\sqrt{\lvert\mu - \mu_c\rvert}\right)$	$\mathcal{O}(1)$
subcritical Hopf bifurcation	$\mathcal{O}\left(\sqrt{\lvert\mu - \mu_c\rvert}\right)$	$\mathcal{O}(1)$
saddle-node bifurcation of cycles	$\mathcal{O}(1)$	$\mathcal{O}(1)$
infinite-period (SNIC, SNIPER)	$\mathcal{O}(1)$	$\mathcal{O}\left(\frac{1}{\sqrt{\lvert\mu - \mu_c\rvert}}\right)$
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infinite-period bifurcation: we have already presented an example on Problem Sheet 0

SNIC ... saddle-node bifurcation on invariant circle

SNIPER ... saddle-node infinite-period bifurcation

Example: infinite-period (SNIC, SNIPER) bifurcation

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$\mu \in (-1, 1)$: three critical points:

$[0, 0]$: unstable spiral eigenvalues: $1 \pm \mu i$

$\left[\sqrt{1-\mu^2}, \mu\right]$: stable node eigenvalues: $-2, -\sqrt{1-\mu^2}$

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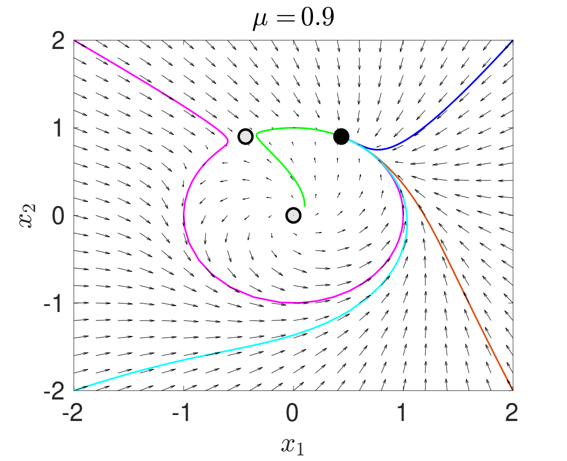

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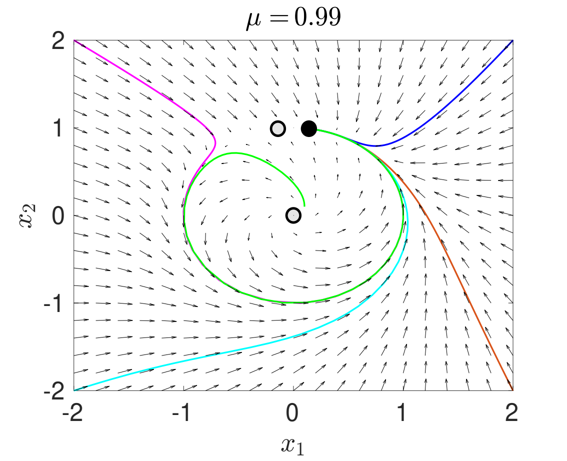

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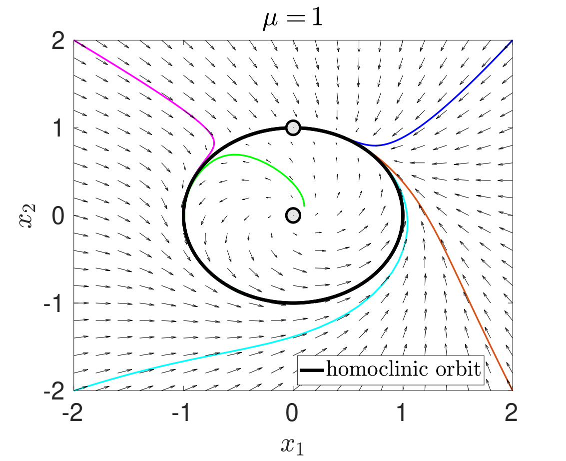

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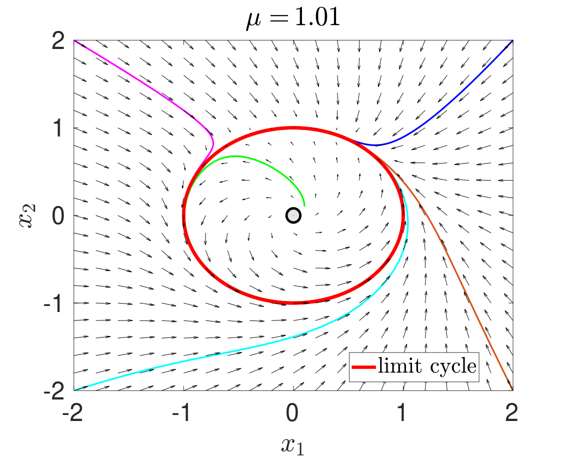

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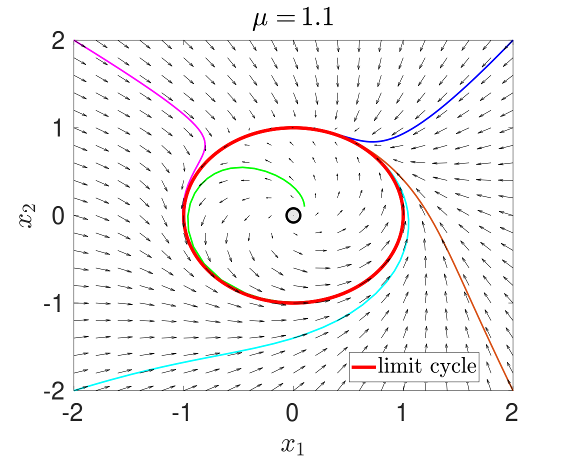

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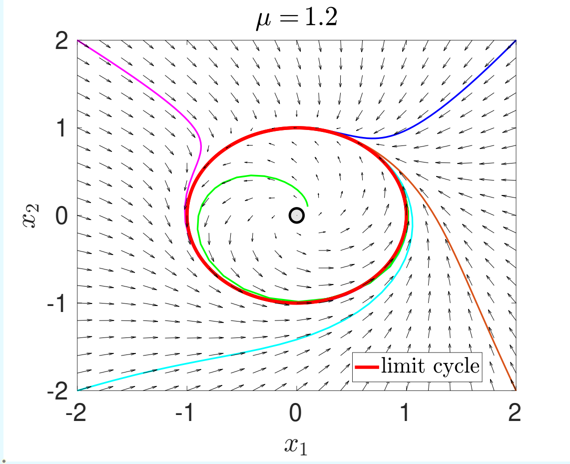

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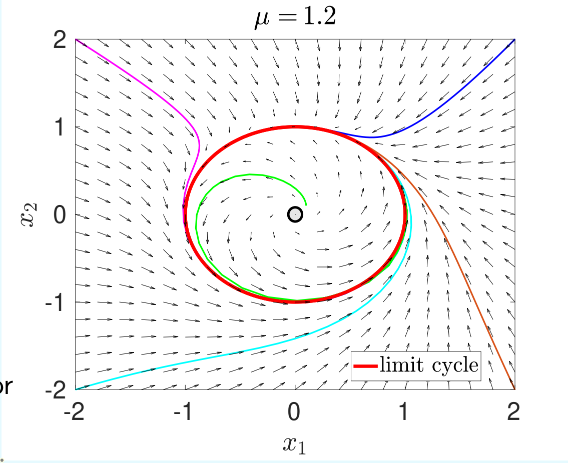

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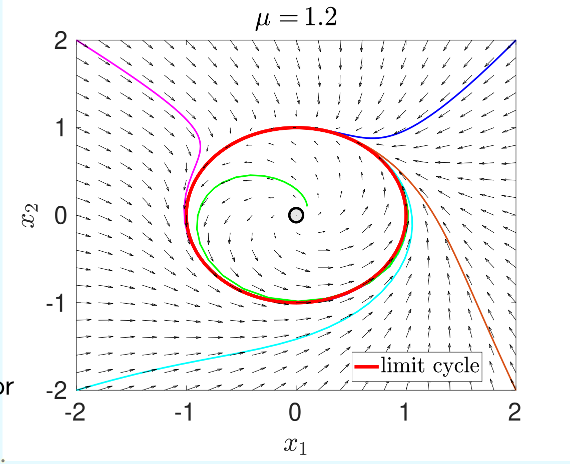

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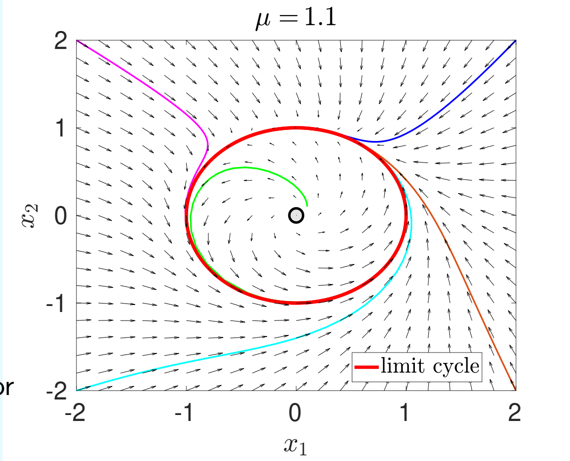

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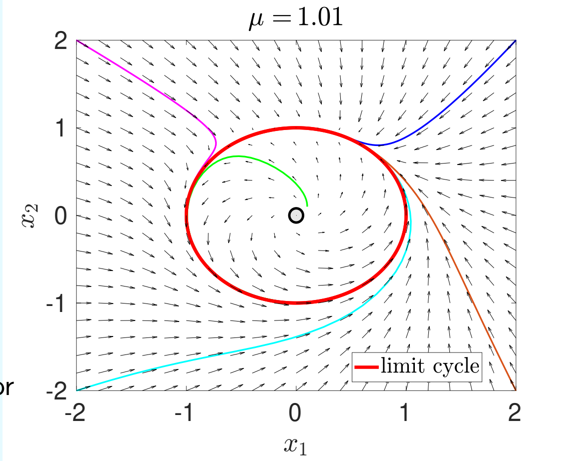

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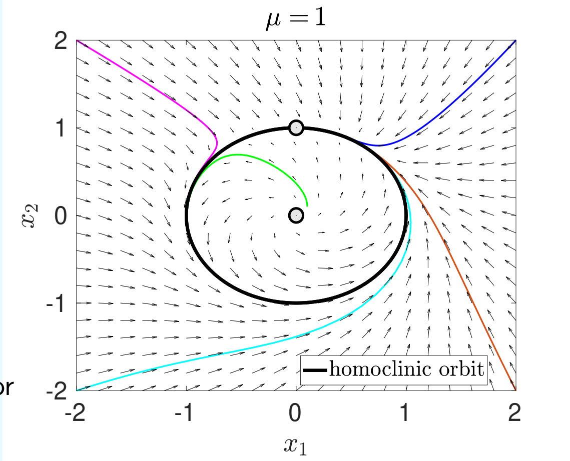

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If $\mu > 1$, then $\mathrm{d}\theta/\mathrm{d}t > \mu - 1 > 0$.

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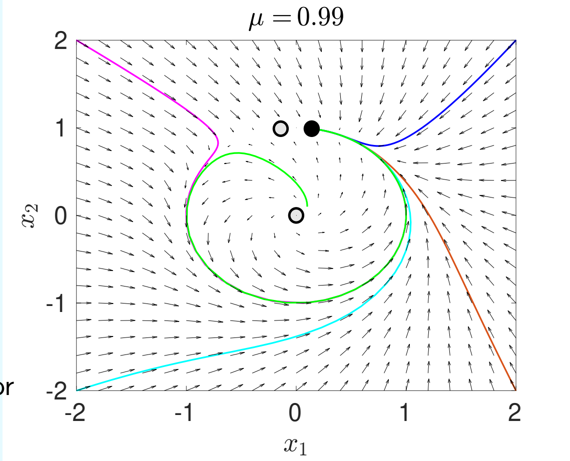

Example: infinite-period (SNIC, SNIPER) bifurcation

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1 - \mu\, x_2 + x_2^2(1 - x_1) - x_1^3$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu\, x_1 - x_1\, x_2\,(1 + x_1) + x_2 - x_2^3$$

Using variables $r(t)$ and $\theta(t)$, where $x_1(t) = r(t)\cos\theta(t)$ and $x_2(t) = r(t)\sin\theta(t)$, we obtain

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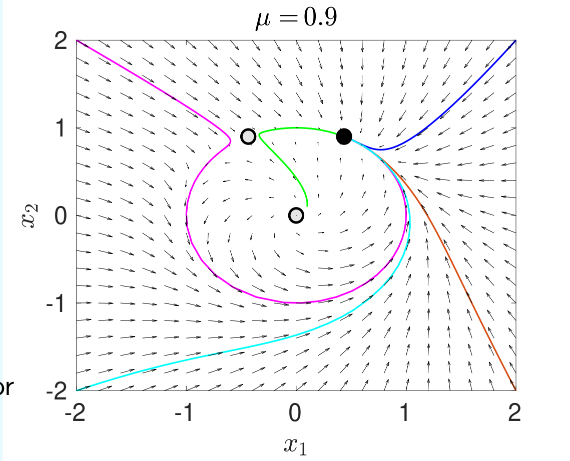

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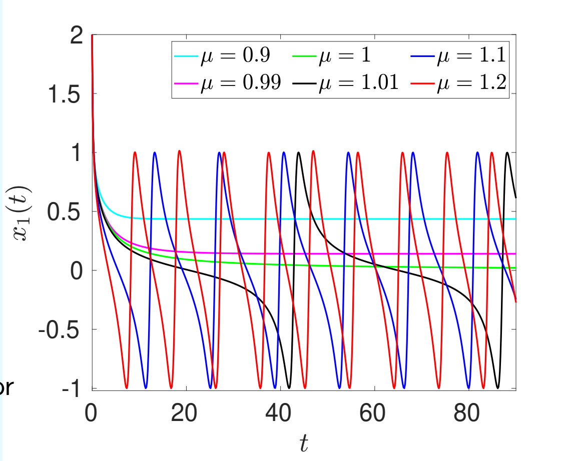

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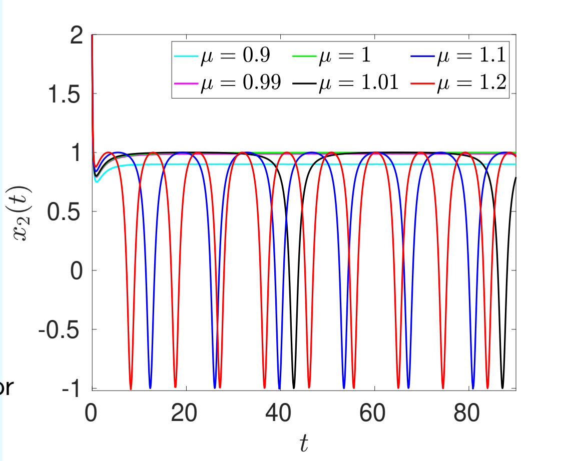

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Bifurcations of limit cycles

bifurcation at $\mu = \mu_c$	amplitude	period
supercritical Hopf bifurcation	$\mathcal{O}\left(\sqrt{\lvert\mu - \mu_c\rvert}\right)$	$\mathcal{O}(1)$
subcritical Hopf bifurcation	$\mathcal{O}\left(\sqrt{\lvert\mu - \mu_c\rvert}\right)$	$\mathcal{O}(1)$
saddle-node bifurcation of cycles	$\mathcal{O}(1)$	$\mathcal{O}(1)$
infinite-period (SNIC, SNIPER)	$\mathcal{O}(1)$	$\mathcal{O}\left(\frac{1}{\sqrt{\lvert\mu - \mu_c\rvert}}\right)$
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homoclinic (saddle-loop) bifurcation	$\mathcal{O}(1)$	$\mathcal{O}\left(\big\lvert \log\lvert\mu - \mu_c\rvert\big\rvert\right)$

homoclinic bifurcation: another bifurcation when limit cycle is born with infinite period

saddle-loop bifurcation

new example

Example: homoclinic (saddle-loop) bifurcation

$$\frac{\mathsf{d}x_1}{\mathsf{d}t} = \mu\, x_1 + x_2 - x_2^2 - x_1\, x_2$$

$$\frac{\mathsf{d}x_2}{\mathsf{d}t} = -x_1$$

Example

$$\frac{\mathsf{d}x_1}{\mathsf{d}t} = \mu\, x_1 \,+\, x_2 \,-\, x_2^2 \,-\, x_1\, x_2$$

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two critical points: $\mathbf{x}_{c1} = [0, 0]$ and $\mathbf{x}_{c2} = [0, 1]$

Jacobian matrix is $\mathsf{D}\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \mu - x_2 & 1 - 2x_2 - x_1 \\ -1 & 0 \end{pmatrix}$

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$$\frac{\mathsf{d}x_1}{\mathsf{d}t} = \mu\, x_1 + x_2 - x_2^2 - x_1\, x_2$$
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$\mathsf{D}\mathbf{f}(\mathbf{x}_{c1}) = \begin{pmatrix} \mu & 1 \\ -1 & 0 \end{pmatrix}$, eigenvalues $\lambda_\pm = \dfrac{\mu}{2} \pm \dfrac{\sqrt{\mu^2 - 4}}{2}$

$\implies \quad \mathbf{x}_{c1}$ is stable for $\mu < 0$ and unstable for $\mu > 0$

$\mathsf{D}\mathbf{f}(\mathbf{x}_{c2}) = \begin{pmatrix} \mu - 1 & -1 \\ -1 & 0 \end{pmatrix}$, eigenvalues $\lambda_\pm = \dfrac{\mu - 1}{2} \pm \dfrac{\sqrt{\mu^2 - 2\mu + 5}}{2}$

$\implies \quad \mathbf{x}_{c2}$ is an (unstable) saddle for all $\mu \in \mathbb{R}$

Example

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu\, x_1 + x_2 - x_2^2 - x_1\, x_2$$

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fixed point $\mathbf{x}_{c1} = [0, 0]$ is a stable spiral

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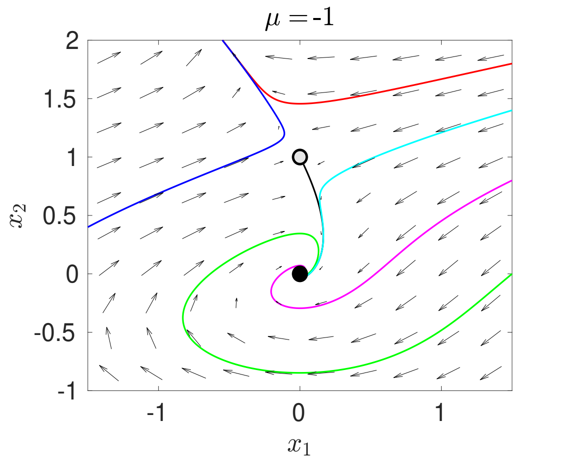

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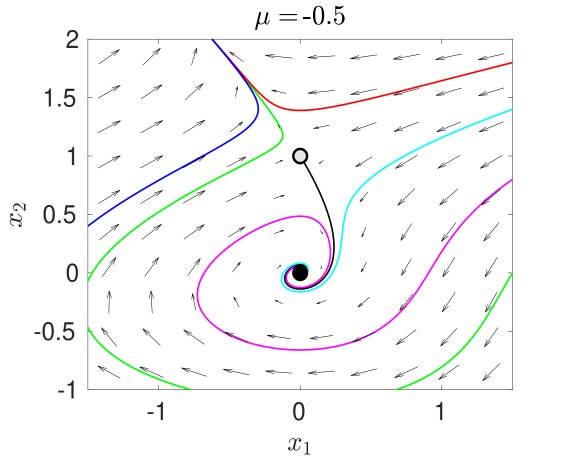

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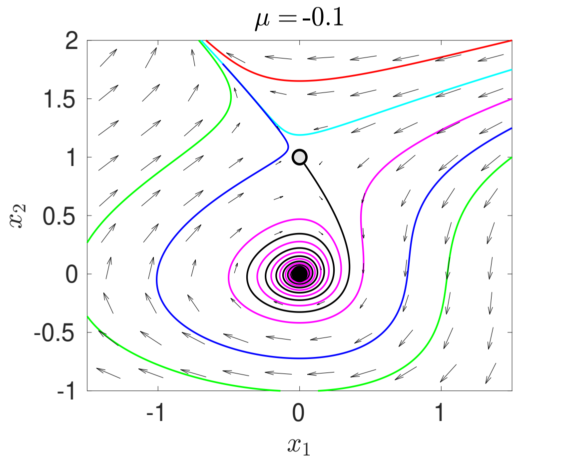

Example: supercritical Hopf bifurcation

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu\, x_1 + x_2 - x_2^2 - x_1\, x_2$$

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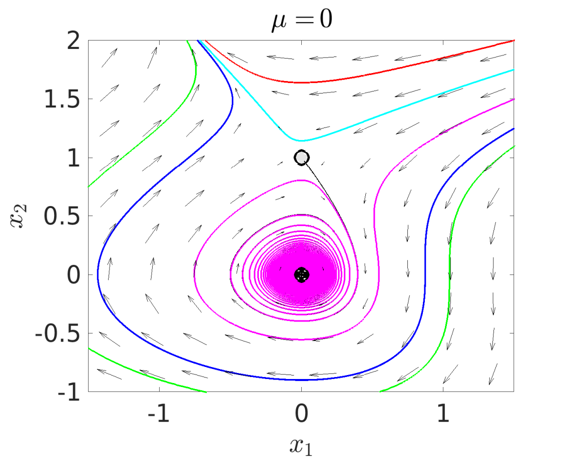


$\mu = 0$: fixed point $\mathbf{x}_{c1} = [0, 0]$ is still a stable spiral, though a very weak one

supercritical Hopf bifurcation at $\mu = 0$

the limit cycle exists in interval $\mu \in (0, 0.135454802155\ldots)$

Example: homoclinic (saddle-loop) bifurcation

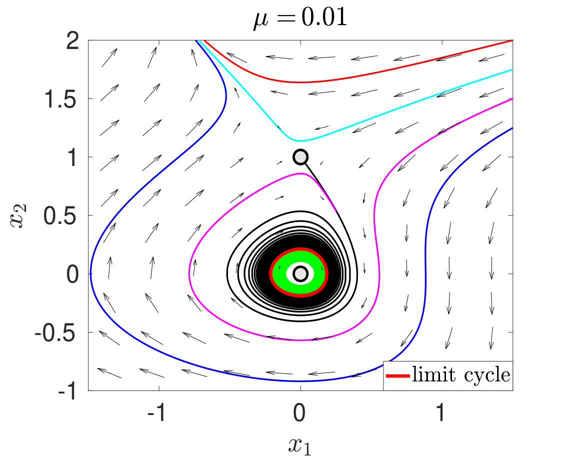
$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu\, x_1 + x_2 - x_2^2 - x_1\, x_2$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -x_1$$

$\mu > 0$: $\mathbf{x}_{c1} = [0, 0]$ is an unstable spiral

the limit cycle exists in interval

$\mu \in (0, 0.135454802155\dots)$



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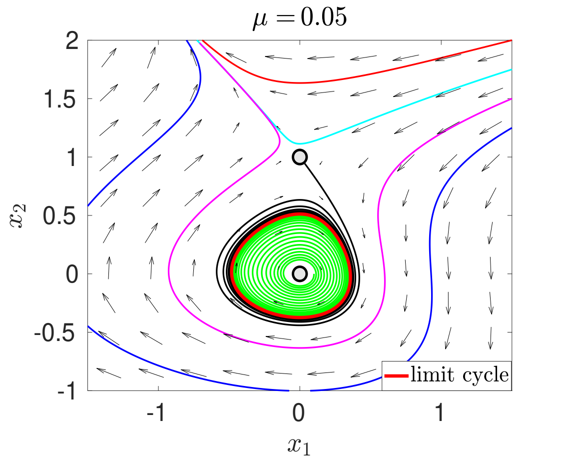

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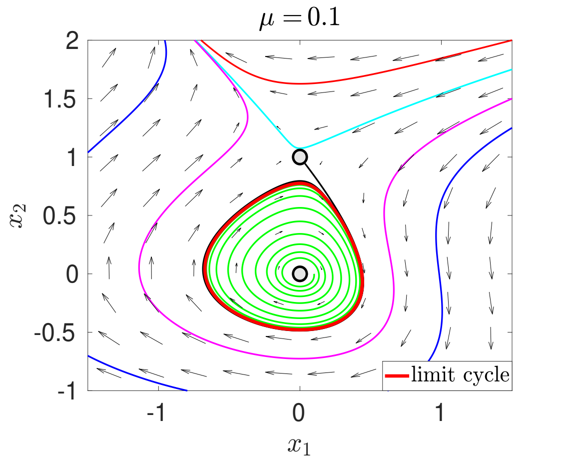

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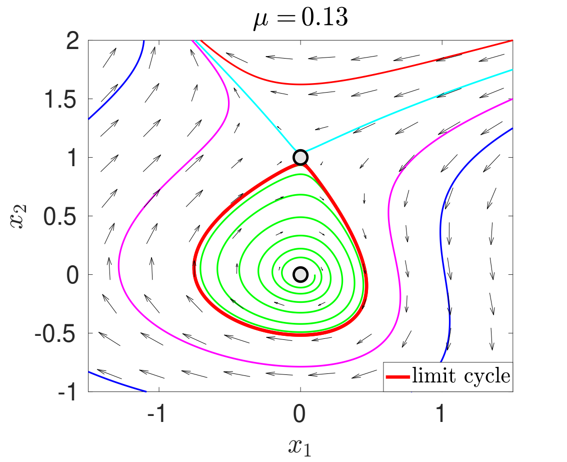

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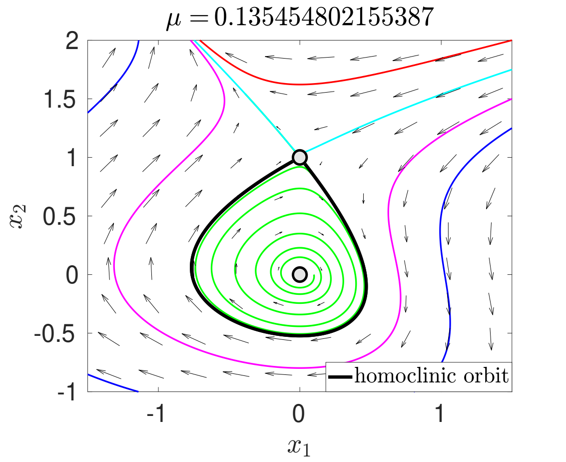
homoclinic (saddle-loop) bifurcation



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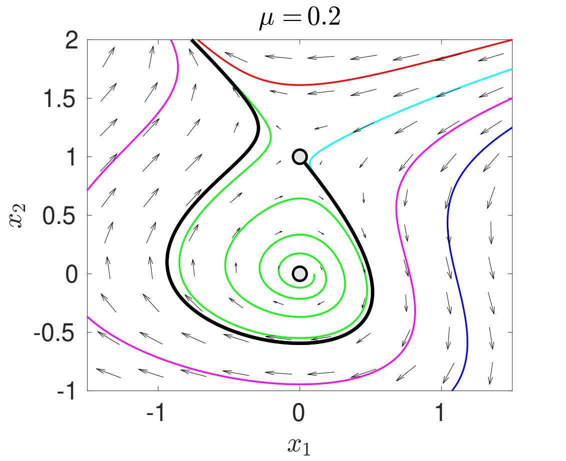
homoclinic (saddle-loop) bifurcation

$\mu > 0.135454802155\dots$: no limit cycle



Example: homoclinic bifurcation and supercritical Hopf bifurcation

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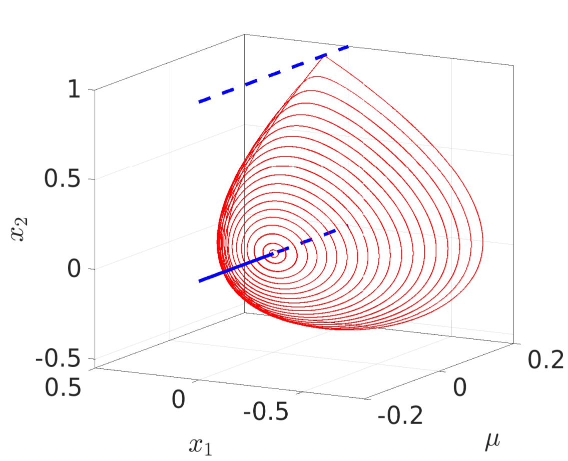
bifurcation diagram

[show 3D animation]



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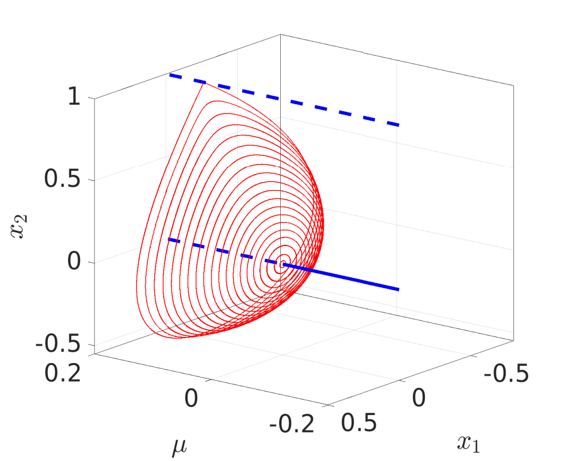

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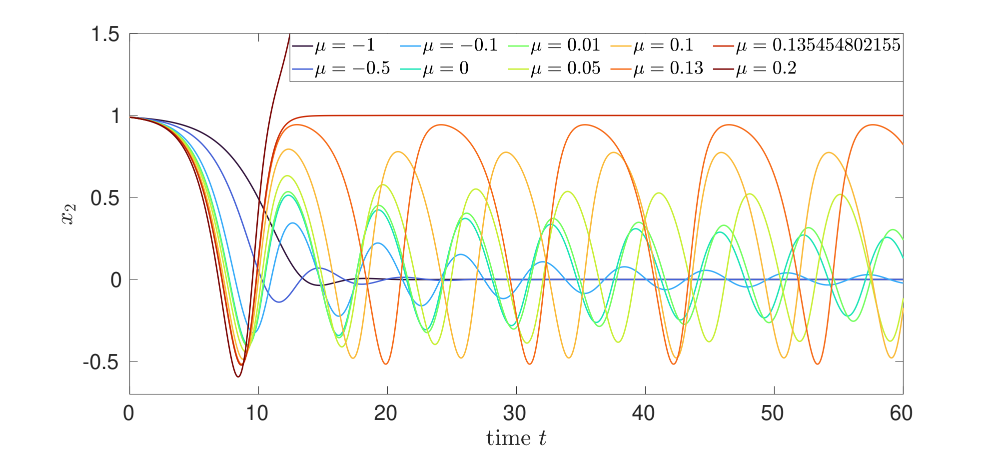

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Summary: bifurcations of limit cycles

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subcritical Hopf bifurcation	$\mathcal{O}\left(\sqrt{\lvert\mu - \mu_c\rvert}\right)$	$\mathcal{O}(1)$
saddle-node bifurcation of cycles	$\mathcal{O}(1)$	$\mathcal{O}(1)$
infinite-period (SNIC, SNIPER)	$\mathcal{O}(1)$	$\mathcal{O}\left(\dfrac{1}{\sqrt{\lvert\mu - \mu_c\rvert}}\right)$
homoclinic (saddle-loop) bifurcation	$\mathcal{O}(1)$	$\mathcal{O}\left(\big\lvert \log\lvert\mu - \mu_c\rvert\big\rvert\right)$

Additional examples: Questions 1, 4, 5 and 6 on Problem Sheet 3.
They are formulated in a way that the questions do not specify what bifurcations of limit cycles are there.

There is also Question 2 on Problem Sheet 3 which asks you to look for a Hopf bifurcation.

B5.6: Nonlinear Dynamics, Bifurcations and Chaos (Lecture 12)

- summary of Lecture 11: we discussed
  Bifurcations of limit cycles, covering saddle-node bifurcation of cycles, infinite-period (SNIC) bifurcation and homoclinic (saddle-loop) bifurcation.
- today: we will conclude our discussion of Problem Sheet 3

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- course synopsis of **Lectures 9-16**:
  Bifurcations of limit cycles, covering supercritical and subcritical Hopf bifurcations, saddle-node bifurcation of cycles, infinite-period (SNIC) bifurcation and homoclinic (saddle-loop) bifurcation. Oscillations in chemical reaction networks. Weakly nonlinear oscillators. Poincaré-Lindstedt method. Conservative and non-conservative systems. Liénard systems, van der Pol oscillator. Hilbert's 16th problem. Lorenz equations. Lorenz map. Poincaré section. Poincaré map. Converse of Sharkovsky's theorem. Bernoulli shift map, symbolic dynamics. Tent map. Dynamics on metric spaces, sensitive dependence on initial conditions, transitivity, conjugate maps, chaotic dynamics.

Weakly nonlinear-oscillators, Poincaré-Lindstedt method

- Weakly nonlinear-oscillators: $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = -x + \varepsilon\, g\left(x, \dfrac{\mathrm{d}x}{\mathrm{d}t}\right)$ where $0 < \varepsilon \ll 1$
- we will apply the Poincaré-Lindstedt method to examples of both conservative and non-conservative systems

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- we will apply the Poincaré-Lindstedt method to examples of both conservative and non-conservative systems
- conservative systems:
  - derivation on the whiteboard: $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = -x + \varepsilon\, x^3$
  - additional example $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = -x + \varepsilon\, x^2$ is analyzed in Question 3 on Problem Sheet 3 (solutions are available on the course website)
- non-conservative systems: we will consider the van der Pol oscillator
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -x + \mu\,(1 - x^2)\,\frac{\mathrm{d}x}{\mathrm{d}t}$$
which can be analyzed using the Poincaré-Lindstedt method for $\mu = \varepsilon \ll 1$

van der Pol oscillator

$$\frac{\mathsf{d}^2 x}{\mathsf{d}t^2} = -\,x \;+\; \mu\,(1 - x^2)\,\frac{\mathsf{d}x}{\mathsf{d}t}$$

van der Pol oscillator

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\,x \,+\, \mu\,(1 - x^2)\,\frac{\mathrm{d}x}{\mathrm{d}t}$$

Denoting $y_1 = x$ and $y_2 = \dfrac{\mathrm{d}x}{\mathrm{d}t}$, we can rewrite the van der Pol equation as

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- The origin $\mathbf{0} = [0, 0]$ is an unstable spiral for $0 < \mu = \varepsilon \ll 1$.

van der Pol oscillator

$$\omega^2(\varepsilon)\,\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} = -\,x \;+\; \varepsilon\,\omega(\varepsilon)(1-x^2)\,\frac{\mathrm{d}x}{\mathrm{d}\tau}$$

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- The origin $\mathbf{0} = [0,0]$ is the only critical point.
- The Jacobian matrix $D\mathbf{f}(\mathbf{0}) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}$ has eigenvalues $\lambda_{\pm} = \dfrac{\mu}{2} \pm \dfrac{\sqrt{\mu^2-4}}{2}$.
- The origin $\mathbf{0} = [0,0]$ is an unstable spiral for $0 < \mu = \varepsilon \ll 1$.
- To apply the Poincaré-Lindstedt method for $\mu = \varepsilon$, we transform the time variable as $\tau = \omega(\varepsilon)\,t$ where $2\pi/\omega(\varepsilon)$ is the period of the periodic solution.

van der Pol oscillator

$$\omega^2(\varepsilon)\,\frac{\mathsf{d}^2x}{\mathsf{d}\tau^2} = -\,x\;+\;\varepsilon\,\omega(\varepsilon)(1-x^2)\,\frac{\mathsf{d}x}{\mathsf{d}\tau}$$

Substituting

$$x(\tau;\varepsilon) = x_0(\tau)\;+\;\varepsilon\,x_1(\tau)\;+\;\varepsilon^2\,x_2(\tau)\;+\;\ldots\quad\text{and}\quad\omega(\varepsilon) = \omega_0\;+\;\varepsilon\,\omega_1\;+\;\varepsilon^2\,\omega_2\;+\;\ldots$$

and equating coefficients of ε^0 and ε^1, we obtain $\omega_0 = 1$, $x_0(\tau) = A\,\cos(\tau)$ and

$$\frac{\mathsf{d}^2x_1}{\mathsf{d}\tau^2}+x_1 = -2\,\omega_1\,\frac{\mathsf{d}^2x_0}{\mathsf{d}\tau^2}+(1-x_0^2)\,\frac{\mathsf{d}x}{\mathsf{d}\tau} = 2\,\omega_1\,A\,\cos(\tau)+\left(\frac{A^3}{4}-A\right)\sin(\tau)+\frac{A^3}{4}\,\sin(3\tau)$$

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Eliminating the secular terms gives $\omega_1 = 0$ and $A = 2$.

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We have $x(\tau;\varepsilon) = 2\,\cos(\omega t)\;+\;\varepsilon\,\sin^3(\omega t)\;+\ldots$ with $\omega = 1 - \varepsilon^2/16 + \ldots$

$\Rightarrow$ the limit cycle is approximately circular with radius 2 for $\mu = \varepsilon \ll 1$

van der Pol oscillator

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\,x \,+\, \mu\,(1 - x^2)\,\frac{\mathrm{d}x}{\mathrm{d}t}$$

analysis for $\mu \ll 1$:

Poincaré-Lindstedt method implies that the limit cycle is approximately circular with radius 2 and period $\dfrac{2\pi}{1 - \varepsilon^2/16 + \ldots}$

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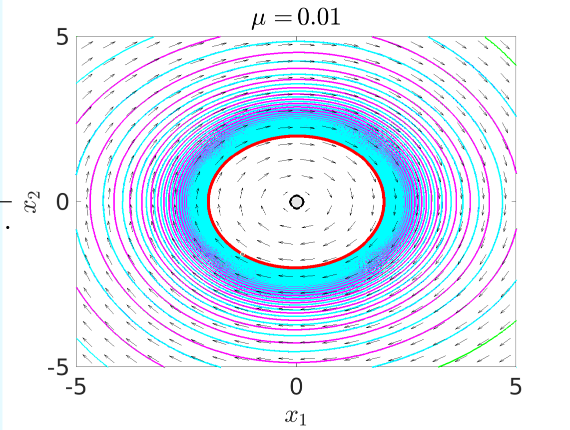

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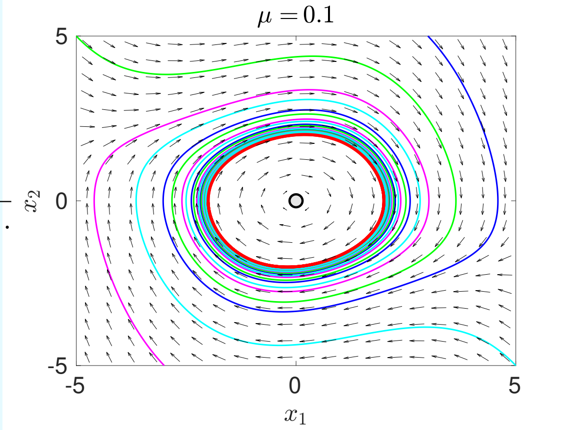

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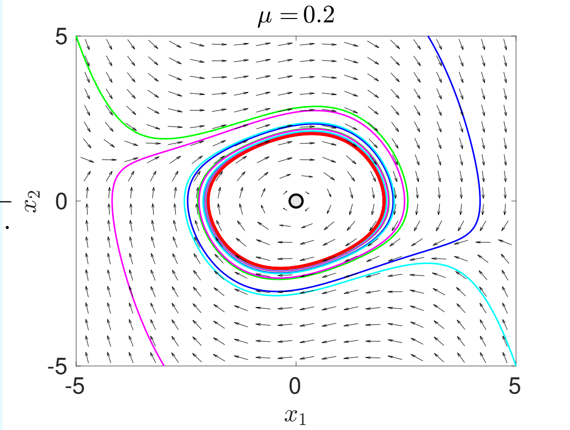

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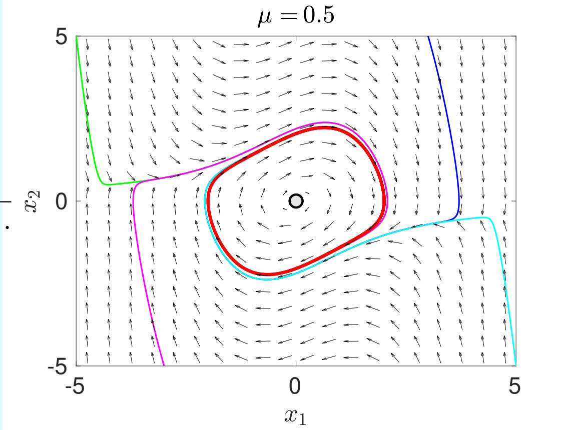

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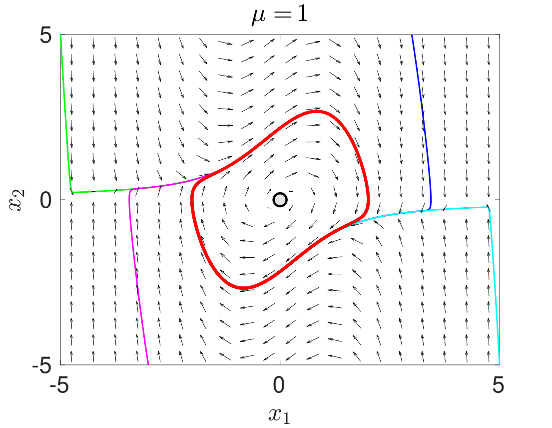

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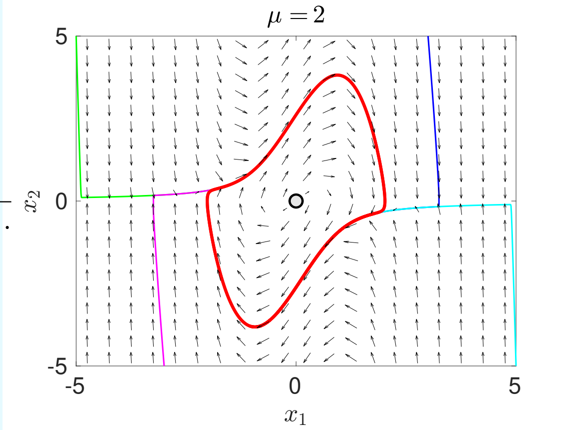

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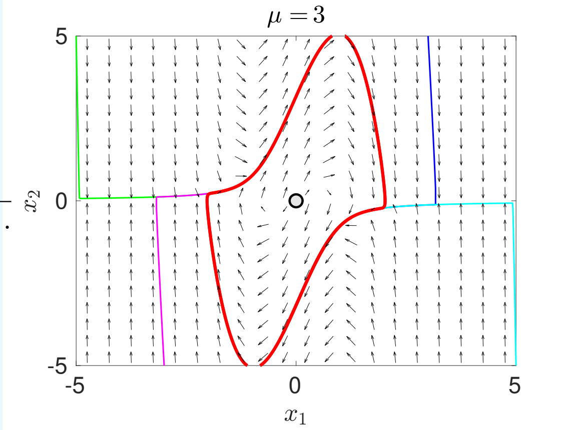

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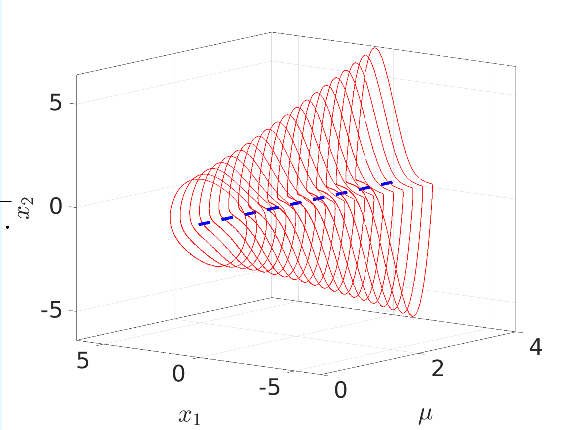

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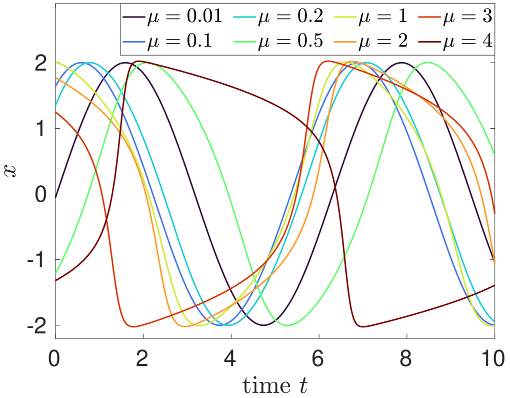

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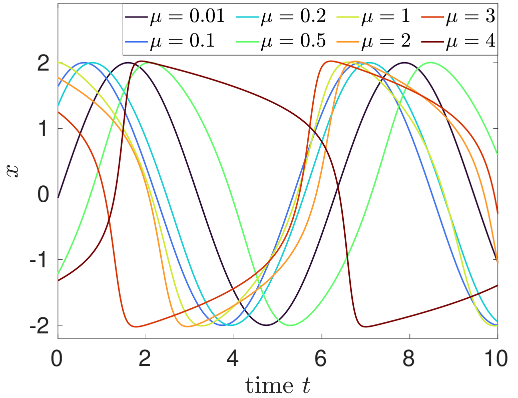

the van der Pol equation is a special case of the Liénard equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -g(x) - f(x)\,\frac{\mathrm{d}x}{\mathrm{d}t} \qquad \text{for} \quad g(x) = x \quad \text{and} \quad f(x) = \mu\,(x^2 - 1)$$

B5.6: Nonlinear Dynamics, Bifurcations and Chaos (Lecture 13)

- summary of Lecture 12: we discussed
  Weakly nonlinear oscillators. Poincaré-Lindstedt method. Conservative and non-conservative systems. Liénard systems, van der Pol oscillator.

- today: we will start our discussion of Problem Sheet 4

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- course synopsis of **Lectures 9-16**:
  Bifurcations of limit cycles, covering supercritical and subcritical Hopf bifurcations, saddle-node bifurcation of cycles, infinite-period (SNIC) bifurcation and homoclinic (saddle-loop) bifurcation. Oscillations in chemical reaction networks. Weakly nonlinear oscillators. Poincaré-Lindstedt method. Conservative and non-conservative systems. Liénard systems, van der Pol oscillator. Hilbert's 16th problem. Lorenz equations. Lorenz map. Poincaré section. Poincaré map. Converse of Sharkovsky's theorem. Bernoulli shift map, symbolic dynamics. Tent map. Dynamics on metric spaces, sensitive dependence on initial conditions, transitivity, conjugate maps, chaotic dynamics.

Planar ODEs with polynomial right-hand sides ($n = 2$)

Consider the planar autonomous system of ODEs:

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= f_1(x_1, x_2)\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= f_2(x_1, x_2)\end{aligned}$$

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Poincaré-Bendixson theorem (Lecture 3, Problem Sheet 1):

Suppose that R is compact (*i.e.* closed and bounded) and it does not contain any fixed points. Suppose that there exists $\mathbf{x}_0 \in R$ such that $\phi_t(\mathbf{x}_0) \in R$ for all $t \geq 0$, *i.e.* the trajectory is confined in R for $t \geq 0$.
Then either $\Gamma_{\mathbf{x}_0}$ is a closed orbit, or $\phi_t(\mathbf{x}_0)$ spirals toward a closed orbit as $t \to \infty$.

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There is no chaotic behaviour of planar ($n = 2$) polynomial ODE systems, but there could still be relatively complicated dynamics (multiple limit cycles) and there are a number of unsolved problems.

Hilbert's 16th problem

Consider the planar autonomous system of ODEs:

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Question 7 on Problem Sheet 4: Show that $H(2) \geq 2$.

This is not the best known lower bound on $H(2)$: one can find quadratic systems with four limit cycles, giving $H(2) \geq 4$.

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Find: $$\frac{dx_1}{dt} = f_1(x_1, x_2; \mu)$$
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where $f_1(x_1, x_2; \mu)$ and $f_2(x_1, x_2; \mu)$ are quadratic polynomials in variables x_1 and x_2 for any value of the parameter $\mu \in \mathbb{R}$ and there exist two limit cycles in the phase plane for some parameter values.

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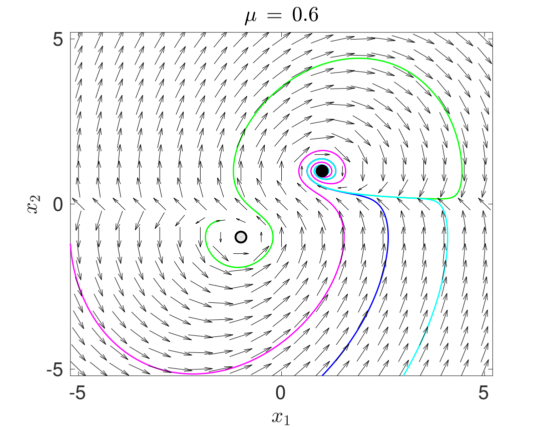

You are asked to find f_1 and f_2 such that there are two critical points $\mathbf{x}_{c1}$ and $\mathbf{x}_{c2}$ for all values of parameter $\mu \in \mathbb{R}$ and the ODE system undergoes a Hopf bifurcation at each critical point, $\mathbf{x}_{c1}$ and $\mathbf{x}_{c2}$, as μ passes through the bifurcation point $\mu = \mu_c$.

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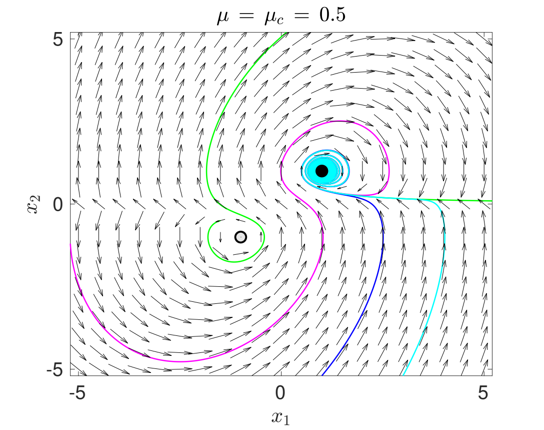

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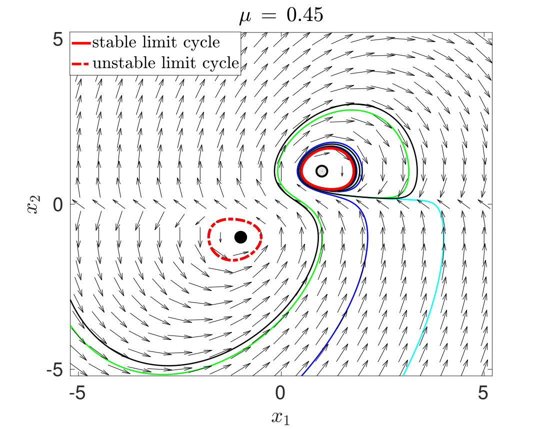

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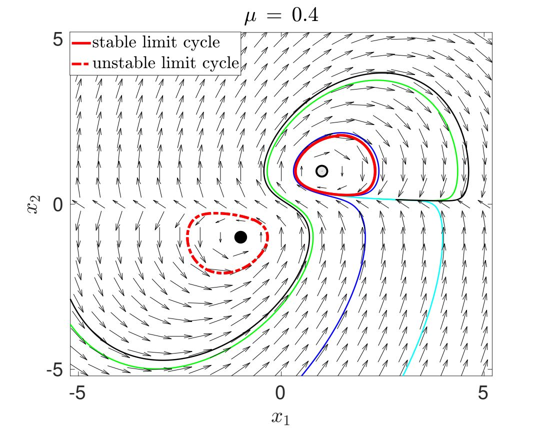

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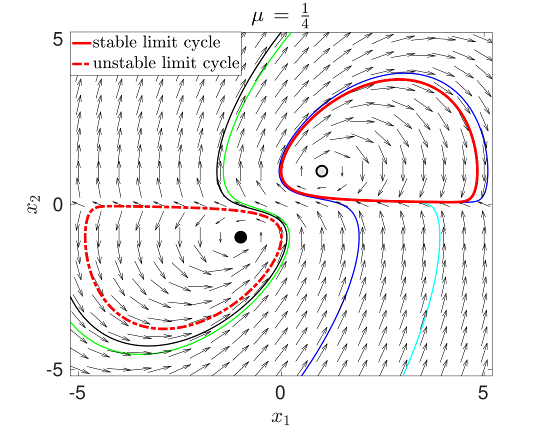

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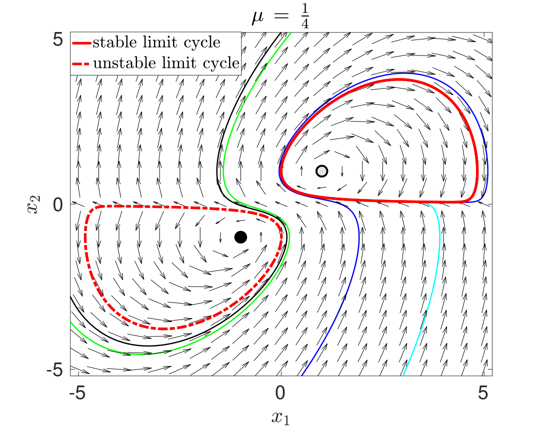

You are asked to find f_1 and f_2 such that there are two critical points $\mathbf{x}_{c1}$ and $\mathbf{x}_{c2}$ for all values of parameter $\mu \in \mathbb{R}$ and the ODE system undergoes a Hopf bifurcation at each critical point, $\mathbf{x}_{c1}$ and $\mathbf{x}_{c2}$, as μ passes through the bifurcation point $\mu = \mu_c$.

Question 2 on Problem Sheet 4: another question to find a planar quadratic system undergoing a specific bifurcation (SNIC, SNIPER)

Designing chemical systems with prescribed dynamical behaviour

applications: synthetic biology, DNA computing, engineering artificial networks

Questions: Can we design a chemical system which oscillates? Can we design a chemical system such that its ODE description has one (two, three, ...) limit cycle(s)?

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$$\alpha_i X_1 + \beta_i X_2 \xrightarrow{k_i} \gamma_i X_1 + \delta_i X_2 \,, \qquad \text{for} \quad i = 1, 2, \dots, \ell,$$

where each reaction has at most two reactants and at most two products, *i.e.*

$$\max_{i=1,2,\dots,\ell} \max \big\{ (\alpha_i + \beta_i), (\gamma_i + \delta_i) \big\} \, \leq \, 2 \,.$$

Such chemical systems are sometimes called bimolecular. In this question, you will show that there is no bimolecular chemical system which would have a limit cycle.

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$\Longrightarrow$ we need more complex reaction networks to get one, two, three, ... limit cycles

Polynomial ODEs with multiple limit cycles

consider Liénard systems in the form:

$$\frac{\mathsf{d}^2 x}{\mathsf{d}t^2} = -\,x \,+\, \mu\, h(x)\, \frac{\mathsf{d}x}{\mathsf{d}t}$$

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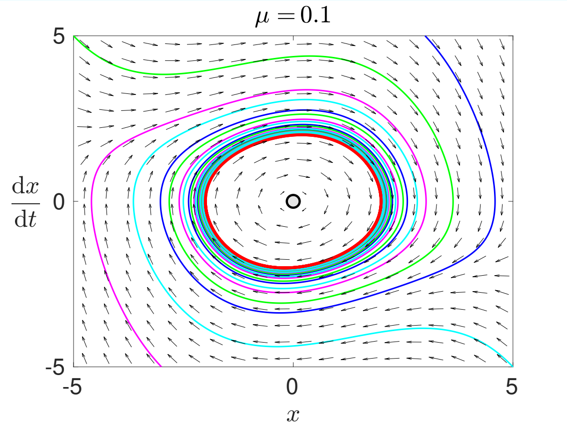

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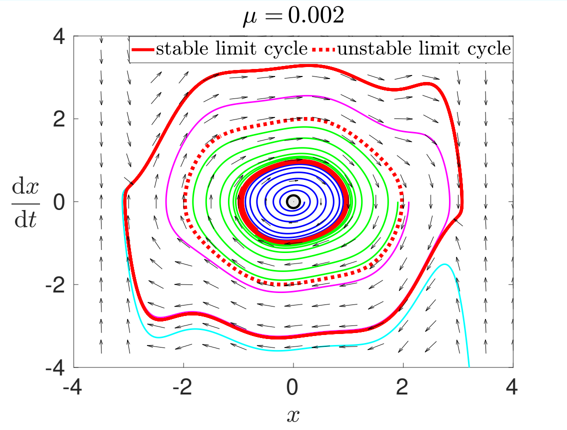
6-th order polynomial:
$h(x) = 72 \;-\; 392\, x^2 \;+\; 224\, x^4 \;-\; 25\, x^6$

three limit cycles: two limit cycles are stable and one limit cycle is unstable



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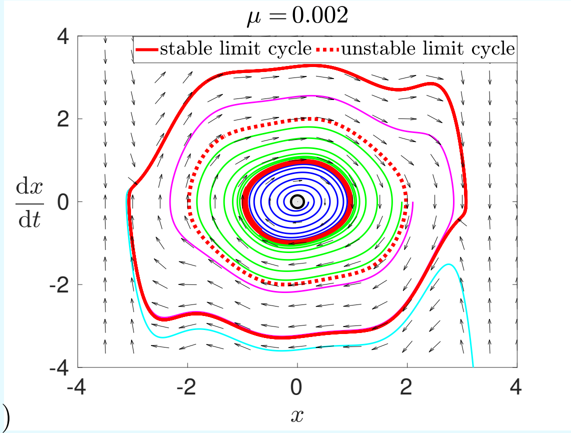

we could also start in polar coordinates

$$\frac{\mathrm{d}r}{\mathrm{d}t} = r(r^2-1)(r^2-4)(r^2-9)(r^2-16)$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1$$

giving us a polynomial system of degree $d = 9$ with four limit cycles at $r = 1,\ 2,\ 3$ and 4 (this is not the most optimal approach: one can find quadratic systems with four limit cycles, giving $H(2) \geq 4$)

Lorenz equations: summary of Lecture 8

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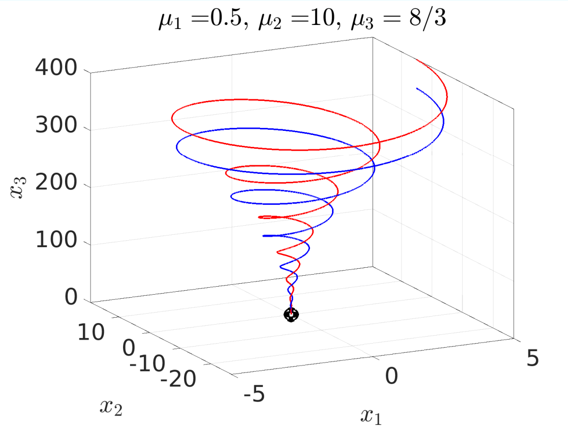

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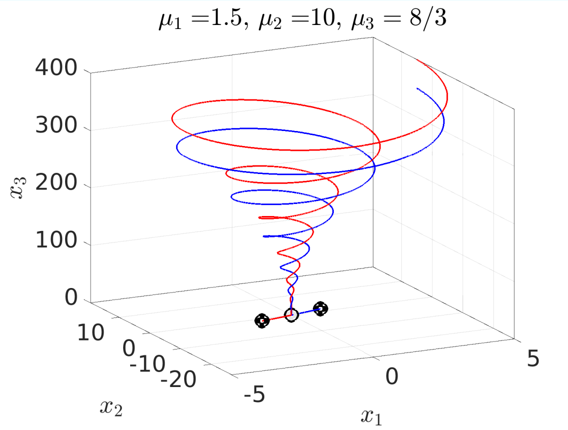

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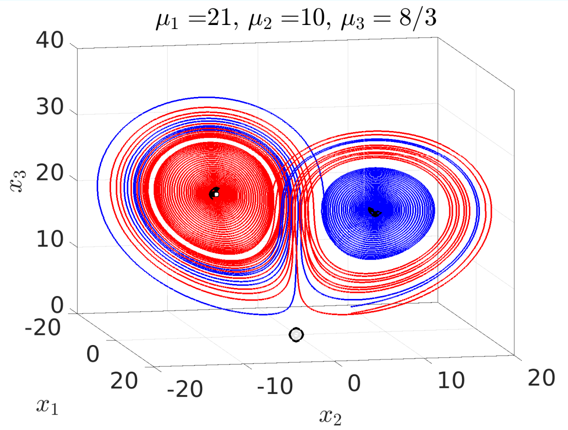

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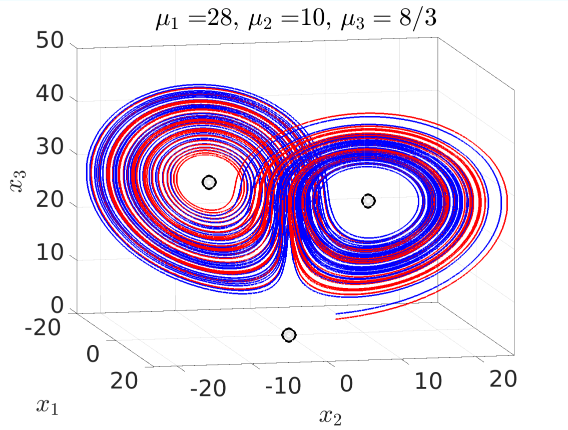

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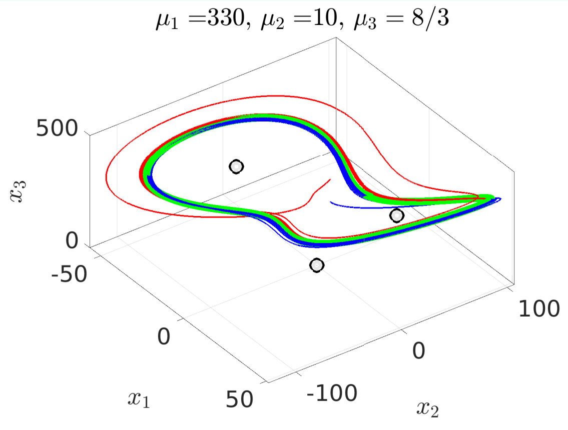

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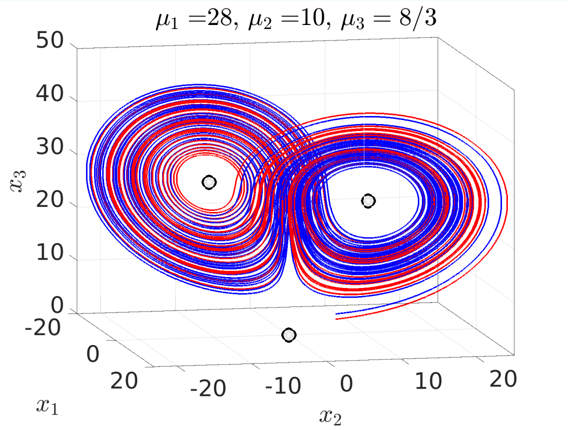
Problem Sheet 2: we used the Lorenz system to further practice techniques studied in Lectures 1–8



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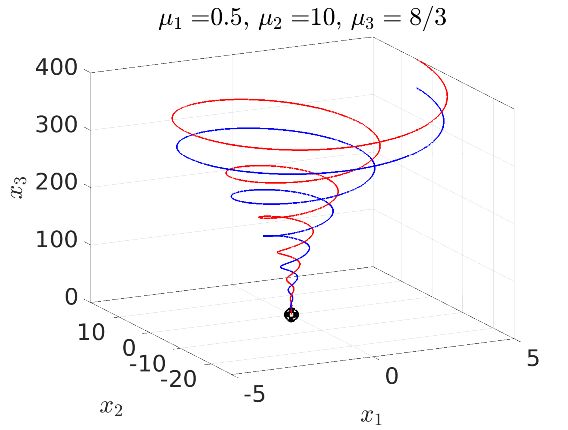

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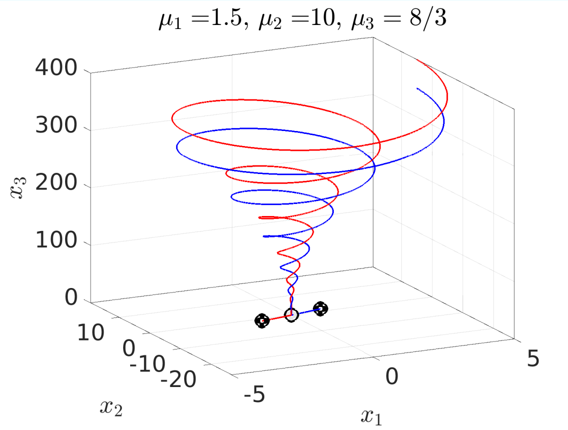

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- finding the Lyapunov function to prove the global stability of the fixed point at the origin $\mathbf{0} = [0, 0, 0]$ for $\mu_1 < 1$
- using the extended center manifold theory to analyze the supercritical pitchfork bifurcation at $\mu_1 = 1$, calculating the center manifold and the dynamics on it

Lorenz equations: Question 6 on Problem Sheet 2

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 3\,(x_2 - x_1)$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu_1\,x_1 - x_2 - x_1\,x_3$$

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- fixed points $\mathbf{x}_{c1} = \mathbf{0} = [0, 0, 0]$

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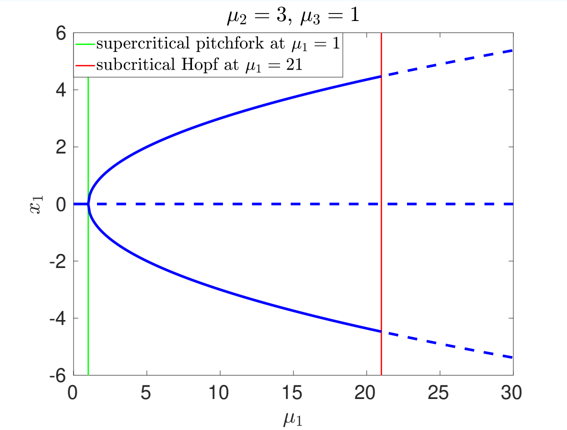

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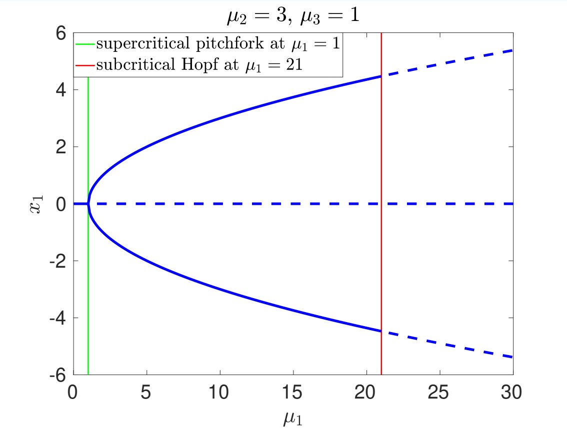

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Lorenz equations: Question 6 on Problem Sheet 2

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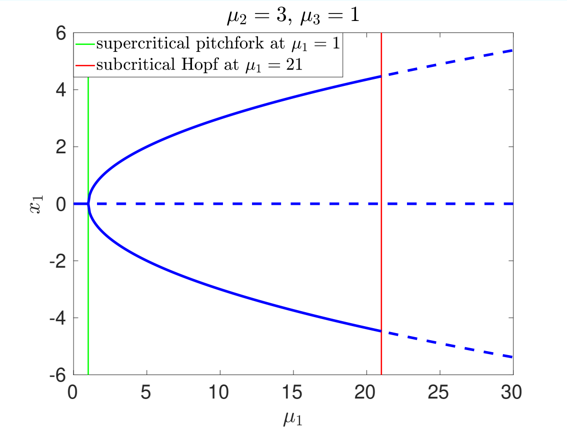

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  at $\mu_1 = 1$ ($\mathbf{x}_{c1} = \mathbf{0}$ is stable for $\mu_1 < 1$ and unstable for $\mu_1 > 1$)
- subcritical Hopf bifurcation at $\mu_1 = 21$

  $\mathbf{x}_{c2}$ and $\mathbf{x}_{c2}$ are stable for $\mu_1 < 21$ and unstable for $\mu_1 > 21$

Lorenz equations: general case

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2\,(x_2 - x_1)$$
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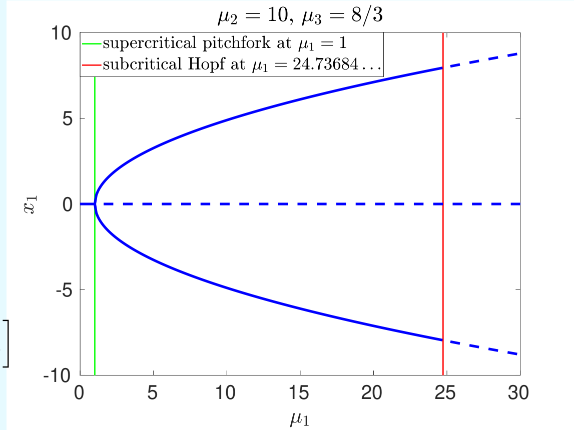

- fixed points $\mathbf{x}_{c1} = \mathbf{0} = [0, 0, 0]$

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- supercritical pitchfork bifurcation
  at $\mu_1 = 1$ $(\mathbf{x}_{c1} = \mathbf{0}$ is stable for $\mu_1 < 1$ and unstable for $\mu_1 > 1)$
- subcritical Hopf bifurcation at $\mu_1 = \mu_c = \mu_2(\mu_2 + \mu_3 + 3)/(\mu_2 - \mu_3 - 1)$
  $\mathbf{x}_{c2}$ and $\mathbf{x}_{c2}$ are stable for $\mu_1 < \mu_c$ and unstable for $\mu_1 > \mu_c$

Lorenz equations: trapping region

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2\,(x_2 - x_1)$$
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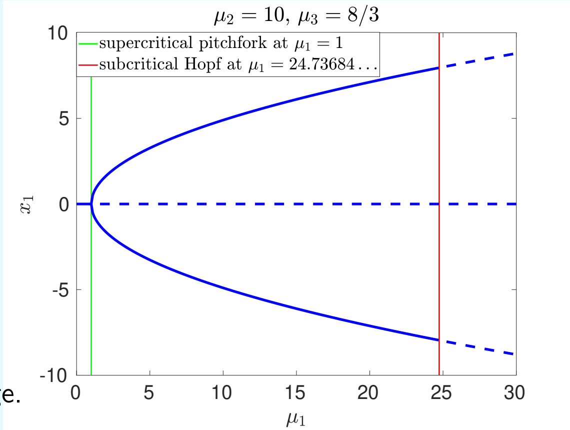
Questions 6 on Problem Sheet 2:
All trajectories eventually enter and remain inside a large sphere of the form
$x_1^2 + x_2^2 + (x_3 - \mu_1 - 3)^2 = C(\mu_1)$
where constant $C(\mu_1)$ is sufficiently large.



Lorenz equations: trapping region and volume contraction

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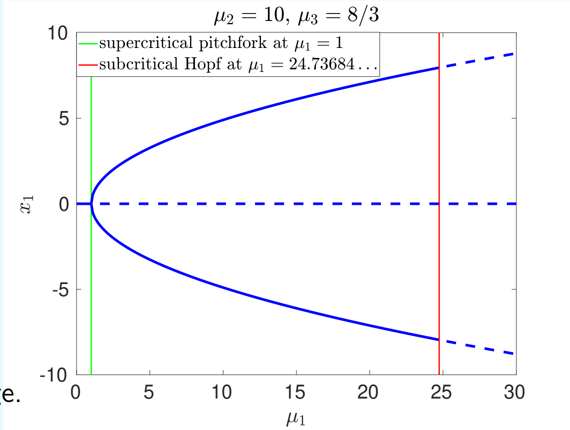

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where constant $C(\mu_1)$ is sufficiently large.

Let $U \equiv U(0) \subset \mathbb{R}^3$ be a compact connected subset of initial conditions.

Let $U(t) = \phi_t(U)$ and $v(t) = |U(t)| = |\phi_t(U)|$ be the volume of $U(t)$. Then

$$\lim_{t\to\infty} v(t) = 0$$

Lorenz equations: Lorenz map

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2\,(x_2 - x_1)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu_1\,x_1 - x_2 - x_1\,x_3$$
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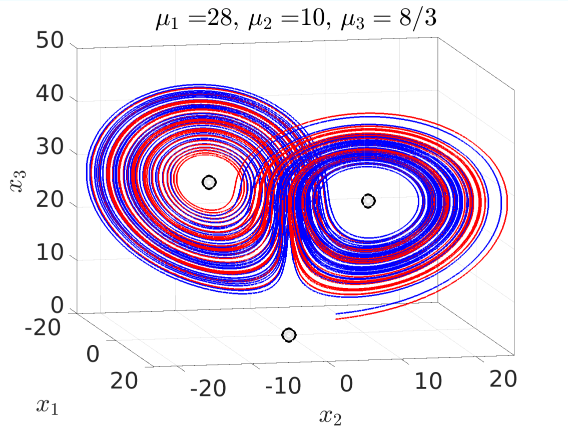
Lorenz map: we investigate chaos using a discrete-time dynamical system



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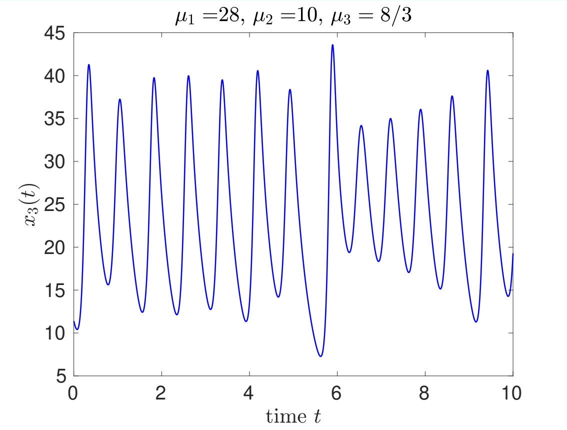

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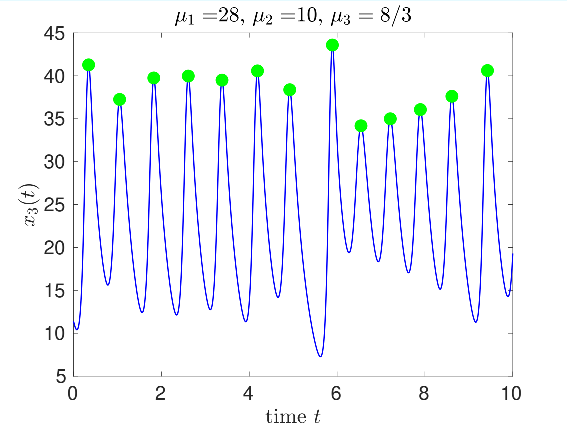


Lorenz map: we investigate chaos using a discrete-time dynamical system

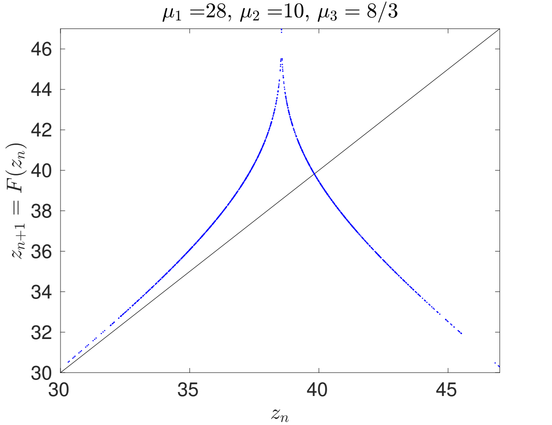
Consider local maxima z_n, $n = 1, 2, \ldots$ of $x_3(t)$ and define Lorenz map by:

$$z_{n+1} = F(z_n)$$

Then a closed orbit corresponds to an N-cycle $\{z_0, z_1, z_2, \ldots, z_{N-1}\}$ of the Lorenz map.

Lorenz equations: Lorenz map

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2\,(x_2 - x_1)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu_1\,x_1 - x_2 - x_1\,x_3$$
$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = x_1\,x_2 - \mu_3\,x_3$$



Lorenz map: we investigate chaos using a discrete-time dynamical system

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Then a closed orbit corresponds to an N-cycle $\{z_0, z_1, z_2, \ldots, z_{N-1}\}$ of the Lorenz map.

Lecture 7: N-cycle is *unstable* if $\left|F'(z_0)\,F'(z_1)\,\ldots\,F'(z_{N-1})\right| > 1$

There are no stable fixed points or limit cycles for: $\mu > \mu_c = \dfrac{\mu_2(\mu_2 + \mu_3 + 3)}{\mu_2 - \mu_3 - 1}$

Lorenz equations: Lorenz map

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2 \, (x_2 \; - \; x_1)$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu_1 \, x_1 \; - \; x_2 \; - \; x_1 \, x_3$$

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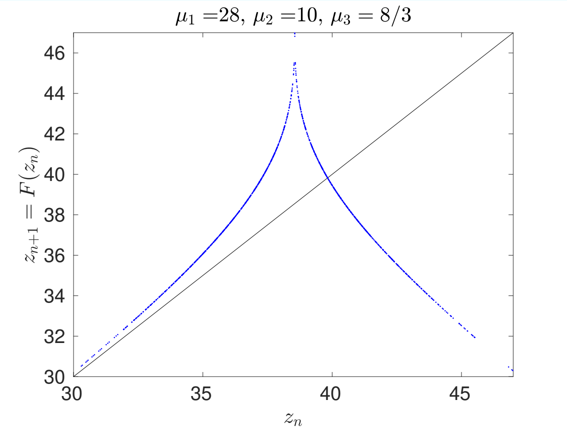
Lorenz map: we investigate chaos using a discrete-time dynamical system



NEXT LECTURE:

Poincaré map: we investigate ODEs using a discrete-time dynamical system

B5.6: Nonlinear Dynamics, Bifurcations and Chaos (Lecture 14)

- summary of Lecture 13: we discussed
  Hilbert's 16th problem. Oscillations in chemical reaction networks. Lorenz equations. Lorenz map. Questions 2, 6 and 7 on Problem Sheet 4.

- today: we will continue in our discussion of Problem Sheet 4

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- course synopsis of **Lectures 9-16**:
  Bifurcations of limit cycles, covering supercritical and subcritical Hopf bifurcations, saddle-node bifurcation of cycles, infinite-period (SNIC) bifurcation and homoclinic (saddle-loop) bifurcation. Oscillations in chemical reaction networks. Weakly nonlinear oscillators. Poincaré-Lindstedt method. Conservative and non-conservative systems. Liénard systems, van der Pol oscillator. Hilbert's 16th problem. Lorenz equations. Lorenz map. Poincaré section. Poincaré map. Converse of Sharkovsky's theorem. Bernoulli shift map, symbolic dynamics. Tent map. Dynamics on metric spaces, sensitive dependence on initial conditions, transitivity, conjugate maps, chaotic dynamics.

Poincaré section and Poincaré map

consider an illustrative example:

$$\frac{\mathsf{d}x_1}{\mathsf{d}t} = -x_2 + \frac{x_1}{10}\left(1 - x_1^2 - x_2^2\right)$$

$$\frac{\mathsf{d}x_2}{\mathsf{d}t} = x_1 + \frac{x_2}{10}\left(1 - x_1^2 - x_2^2\right)$$

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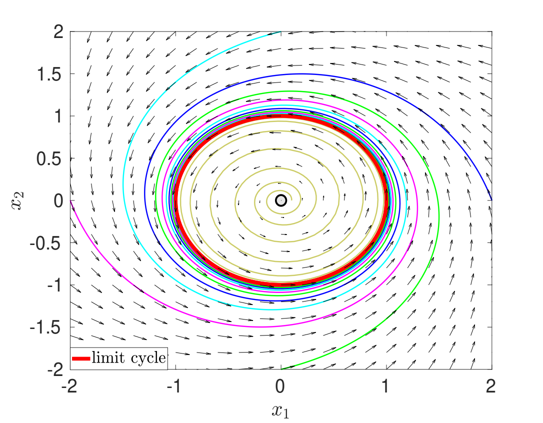
transform to polar coordinates to get:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{r\left(1 - r^2\right)}{10} \quad \text{and} \quad \frac{\mathrm{d}\theta}{\mathrm{d}t} = 1$$

$\Longrightarrow$ stable limit cycle at $r = 1$



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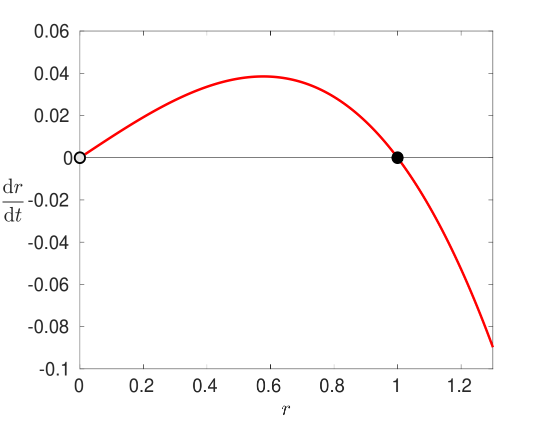

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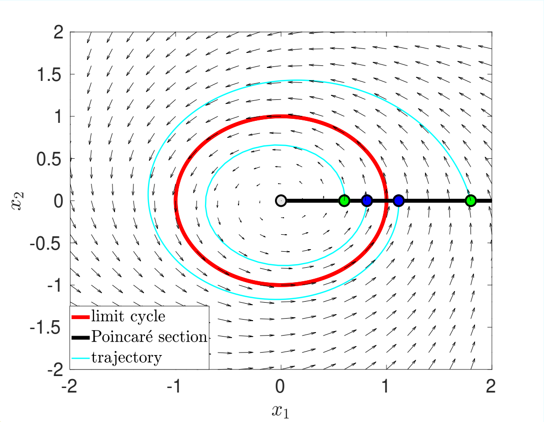

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Poincaré map: $P : \Sigma \to \Sigma$, where $P(x_1)$ is defined such that the positive semi-orbit of $[x_1, 0]$ intersects Σ for the first time at $[P(x_1), 0]$

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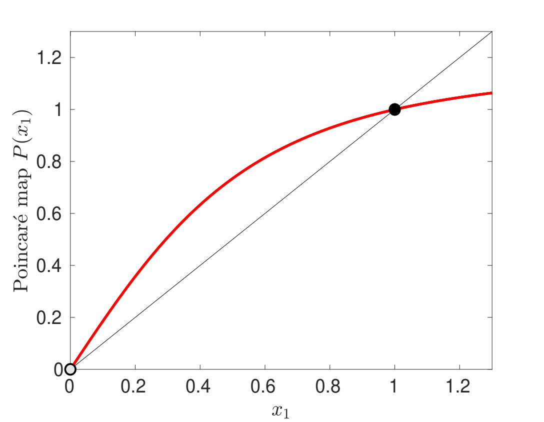

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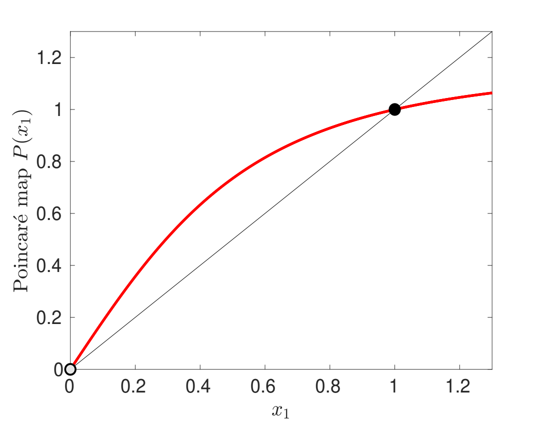

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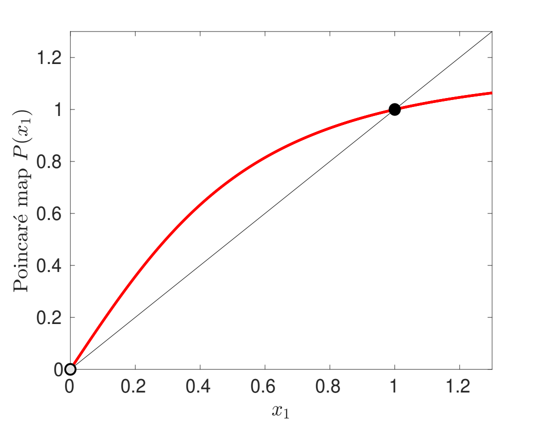

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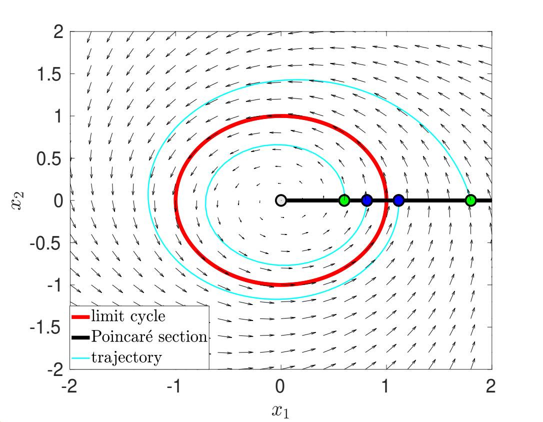

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Poincaré map : saddle-node bifurcation of cycles example

saddle-node bifurcation of cycles at $\mu = -1/4$: a half-stable cycle appears, it splits into a pair of limit cycles for $\mu > -1/4$, one stable, one unstable

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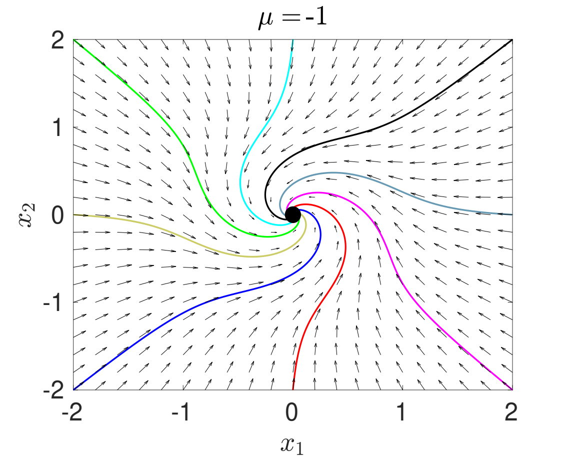

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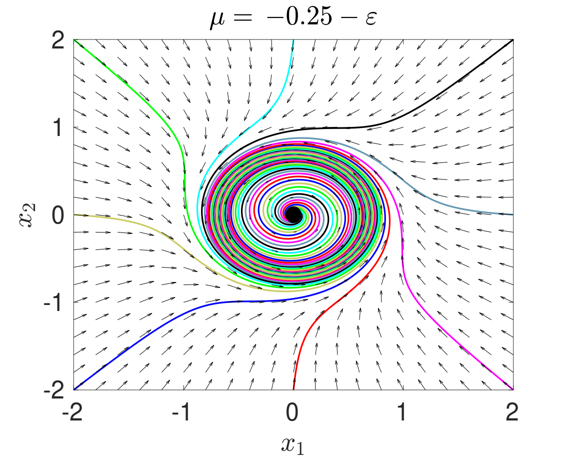

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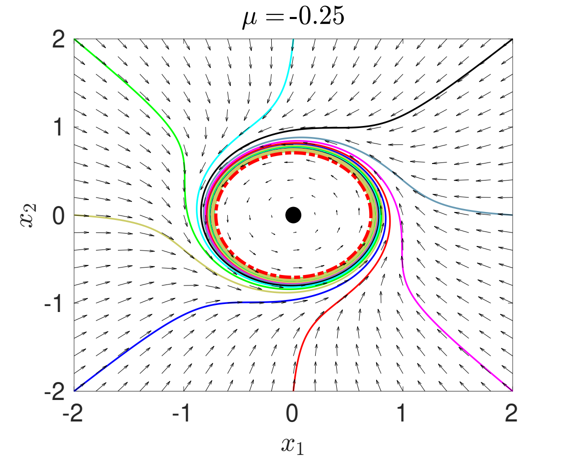

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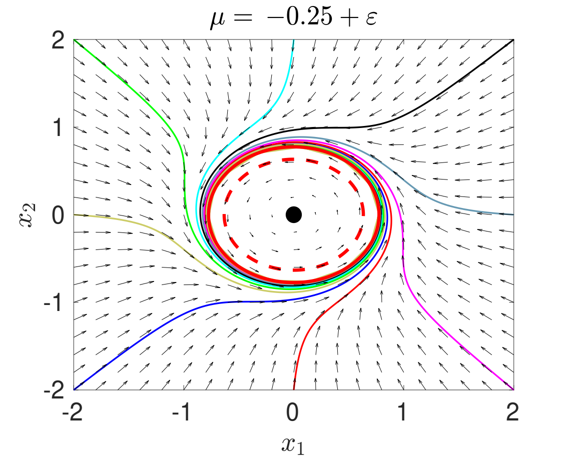

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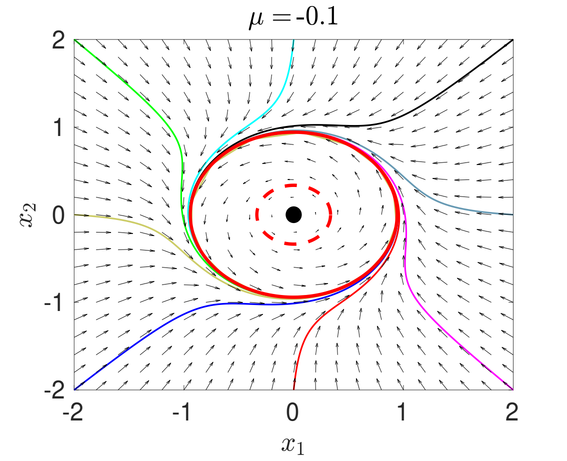

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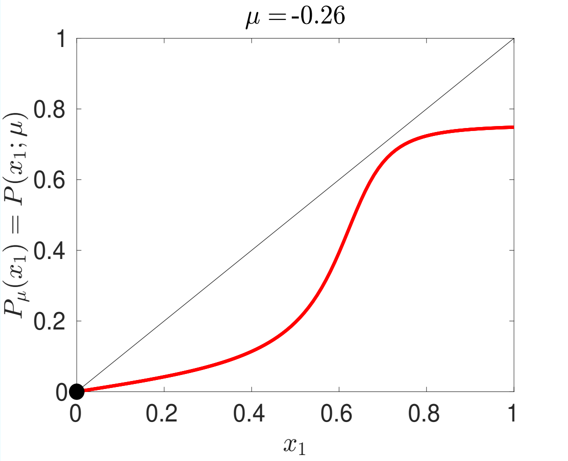

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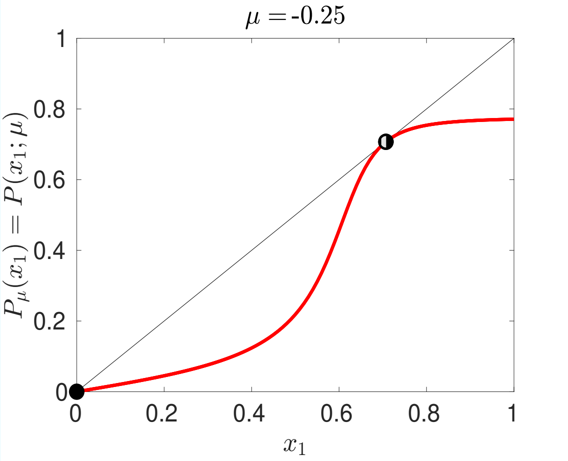

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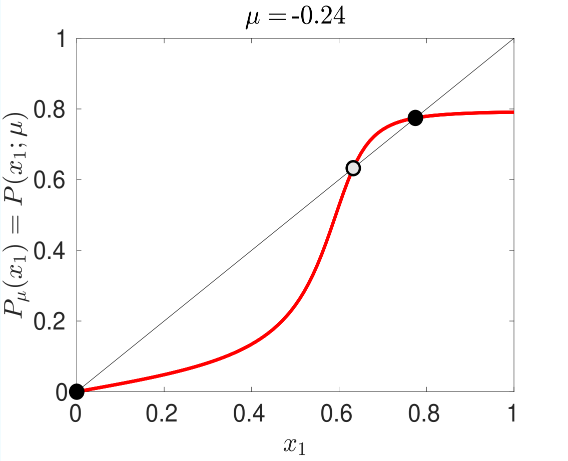

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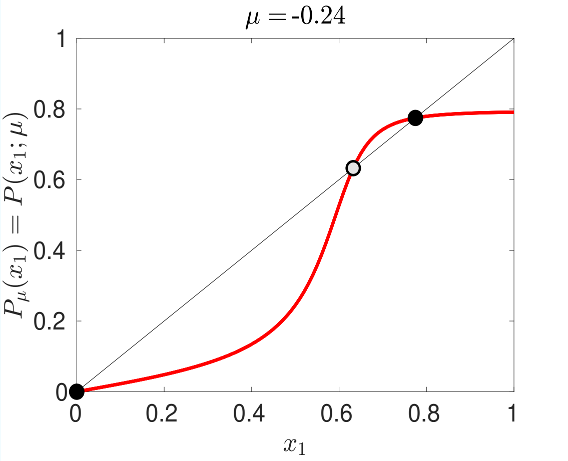

Another example: Question 1 on Problem Sheet 4

Summary of our whiteboard derivations

- we discussed chaos, symbolic dynamics and the Bernoulli shift map
- we studied dynamical systems associated with function $F : \mathbb{M} \to \mathbb{M}$, where $\mathbb{M}$ is a metric space, *i.e.* a set with metric (distance) $d : \mathbb{M} \times \mathbb{M} \to [0, \infty)$

Summary – general theory

- we discussed chaos, symbolic dynamics and the Bernoulli shift map
- we studied dynamical systems associated with function $F : \mathbb{M} \to \mathbb{M}$, where $\mathbb{M}$ is a metric space, *i.e.* a set with metric (distance) $d : \mathbb{M} \times \mathbb{M} \to [0, \infty)$
- $F : \mathbb{M} \to \mathbb{M}$ is called *transitive* if there exists $x_0 \in \mathbb{M}$ such that orbit $O(x_0) = \big\{x_0, F(x_0), F^{(2)}(x_0), F^{(3)}(x_0), \dots\big\}$ is a dense subset of $\mathbb{M}$ (a *transitive point* of F is a point $x_0 \in \mathbb{M}$ which has a dense orbit $O(x_0)$ under F)
- $F : \mathbb{M} \to \mathbb{M}$ has *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that for all $x \in \mathbb{M}$ and any open set $U \subset \mathbb{M}$ satisfying $x \in U$, there is a point $y \in U$ and $j \in \mathbb{N}$ with $d\big(F^{(j)}(x), F^{(j)}(y)\big) > \delta$
- $F : \mathbb{M} \to \mathbb{M}$ is said to be *chaotic* if:
  - (i) the set of all periodic points is dense in $\mathbb{M}$
  - (ii) F is transitive
  - (iii) F has sensitive dependence on initial conditions
- if $F : \mathbb{M} \to \mathbb{M}$ is continuous and $\mathbb{M}$ is not a finite set, then (i) and (ii) imply (iii)

Summary – Bernoulli shift map

$\mathbb{M}_{01} = \Big\{ (a_1, a_2, a_3, a_4, \dots) \,\Big|\, \text{ such that } a_j = 0 \text{ or } a_j = 1 \text{ for } j = 1, 2, 3, 4, \dots \Big\}$

$\mathbb{M}_{01}$ is a metric space with metric defined by

$$d(x, y) = \sum_{j=1}^{\infty} \frac{|a_j - b_j|}{2^j} \text{ for } x = (a_1, a_2, a_3, a_4, \dots) \text{ and } y = (b_1, b_2, b_3, b_4, \dots)$$

Bernoulli shift map: $\sigma : \mathbb{M}_{01} \to \mathbb{M}_{01}$ where $\sigma((a_1, a_2, a_3, a_4, \dots)) = (a_2, a_3, a_4, a_5 \dots)$

we stated and proved some of properties of the shift map, namely:

- fixed points are $(0, 0, 0, 0, \dots)$ and $(1, 1, 1, 1, \dots)$
  2-cycle is $\big\{ (0, 1, 0, 1, 0, 1, 0, 1, \dots), (1, 0, 1, 0, 1, 0, 1, 0, \dots) \big\}$
- shift map $\sigma : \mathbb{M}_{01} \to \mathbb{M}_{01}$ is continuous
- shift map $\sigma : \mathbb{M}_{01} \to \mathbb{M}_{01}$ is transitive

NEXT LECTURE: we will prove that shift map $\sigma : \mathbb{M}_{01} \to \mathbb{M}_{01}$ is chaotic, *i.e.*

(i) the set of all periodic points is dense in $\mathbb{M}_{01}$
(ii) σ is transitive
(iii) σ has sensitive dependence on initial conditions

B5.6: Nonlinear Dynamics, Bifurcations and Chaos (Lecture 15)

- summary of Lecture 14: we discussed
  Poincaré section. Poincaré map. Bernoulli shift map, symbolic dynamics. Dynamics on metric spaces, sensitive dependence on initial conditions, transitivity, chaotic dynamics. Questions 1, 3, 4 and 5 on Problem Sheet 4.
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- today: we will continue in our discussion of Problem Sheet 4

- course synopsis of **Lectures 9-16**:
  Bifurcations of limit cycles, covering supercritical and subcritical Hopf bifurcations, saddle-node bifurcation of cycles, infinite-period (SNIC) bifurcation and homoclinic (saddle-loop) bifurcation. Oscillations in chemical reaction networks. Weakly nonlinear oscillators. Poincaré-Lindstedt method. Conservative and non-conservative systems. Liénard systems, van der Pol oscillator. Hilbert's 16th problem. Lorenz equations. Lorenz map. Poincaré section. Poincaré map. Converse of Sharkovsky's theorem. Bernoulli shift map, symbolic dynamics. Tent map. Dynamics on metric spaces, sensitive dependence on initial conditions, transitivity, conjugate maps, chaotic dynamics.

Summary of Lecture 14 – general theory

- we discussed chaos, symbolic dynamics and the Bernoulli shift map
- we studied dynamical systems associated with function $F : \mathbb{M} \to \mathbb{M}$, where $\mathbb{M}$ is a metric space, *i.e.* a set with metric (distance) $d : \mathbb{M} \times \mathbb{M} \to [0, \infty)$
- $F : \mathbb{M} \to \mathbb{M}$ is called *transitive* if there exists $x_0 \in \mathbb{M}$ such that orbit $O(x_0) = \{x_0, F(x_0), F^{(2)}(x_0), F^{(3)}(x_0), \dots\}$ is a dense subset of $\mathbb{M}$ (a *transitive point* of F is a point $x_0 \in \mathbb{M}$ which has a dense orbit $O(x_0)$ under F)

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- $F : \mathbb{M} \to \mathbb{M}$ has *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that for all $x \in \mathbb{M}$ and any open set $U \subset \mathbb{M}$ satisfying $x \in U$, there is a point $y \in U$ and $j \in \mathbb{N}$ with $d\big(F^{(j)}(x), F^{(j)}(y)\big) > \delta$
- $F : \mathbb{M} \to \mathbb{M}$ is said to be *chaotic* if:

  (i) the set of all periodic points is dense in $\mathbb{M}$

  (ii) F is transitive

  (iii) F has sensitive dependence on initial conditions
- if $F : \mathbb{M} \to \mathbb{M}$ is continuous and $\mathbb{M}$ is not a finite set, then (i) and (ii) imply (iii)

Summary of Lecture 14 – Bernoulli shift map

$\mathbb{M}_{01} = \left\{ (a_1, a_2, a_3, a_4, \dots) \,\middle|\, \text{such that } a_j = 0 \text{ or } a_j = 1 \text{ for } j = 1, 2, 3, 4, \dots \right\}$

$\mathbb{M}_{01}$ is a metric space with metric defined by

$$d(x, y) = \sum_{j=1}^{\infty} \frac{|a_j - b_j|}{2^j} \text{ for } x = (a_1, a_2, a_3, a_4, \dots) \text{ and } y = (b_1, b_2, b_3, b_4, \dots)$$

Lemma 1: If $x = (a_1, a_2, a_3, a_4, \dots) \in \mathbb{M}_{01}$ and $y = (b_1, b_2, b_3, b_4, \dots) \in \mathbb{M}_{01}$ with $a_1 = b_1$, $a_2 = b_2$, $\dots$, $a_n = b_n$. Then $d(x, y) \leq 2^{-n}$.

Lemma 2: If $d(x, y) < 2^{-n}$. Then $a_1 = b_1$, $a_2 = b_2$, $\dots$, $a_n = b_n$.

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Bernoulli shift map: $\sigma : \mathbb{M}_{01} \to \mathbb{M}_{01}$ where $\sigma((a_1, a_2, a_3, a_4, \dots)) = (a_2, a_3, a_4, a_5 \dots)$

fixed points are $(0, 0, 0, 0, \dots)$ and $(1, 1, 1, 1, \dots)$

2-cycle is $\{(0, 1, 0, 1, 0, 1, 0, 1, \dots), (1, 0, 1, 0, 1, 0, 1, 0, \dots)\}$

3-cycles are $\{(0, 0, 1, 0, 0, 1, \dots), (0, 1, 0, 0, 1, 0, \dots), (1, 0, 0, 1, 0, 0, \dots)\}$,
$\{(0, 1, 1, 0, 1, 1, \dots), (1, 1, 0, 1, 1, 0, \dots), (1, 0, 1, 1, 0, 1, \dots)\}$

Lemma 3: The Bernoulli shift map $\sigma : \mathbb{M}_{01} \to \mathbb{M}_{01}$ is continuous.

Lemma 4: The Bernoulli shift map $\sigma : \mathbb{M}_{01} \to \mathbb{M}_{01}$ is transitive.

Bernoulli shift map

Definition: $F : \mathbb{M} \to \mathbb{M}$ has *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that for all $x \in \mathbb{M}$ and any open set $U \subset \mathbb{M}$ satisfying $x \in U$, there is a point $y \in U$ and $j \in \mathbb{N}$ with $d\big(F^{(j)}(x), F^{(j)}(y)\big) > \delta$.

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Lemma 6: The Bernoulli shift map $\sigma : \mathbb{M}_{01} \to \mathbb{M}_{01}$ is chaotic.

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Lemma 6: The Bernoulli shift map $\sigma : \mathbb{M}_{01} \to \mathbb{M}_{01}$ is chaotic.

Remark: we could obtain the same properties if we worked with the metric space of bi-infinite sequences of 0's and 1's, *i.e.* where
$x = \big(\ldots, a_{-j}, \ldots, a_{-2}, a_{-1} \,|\, a_0, a_1, a_2, \ldots, a_j, \ldots\big)$ and
$y = \big(\ldots, b_{-j}, \ldots, b_{-2}, b_{-1} \,|\, b_0, b_1, b_2, \ldots, b_j, \ldots\big)$ have distance $d(x,y) = \sum_{j=-\infty}^{\infty} \frac{|a_j - b_j|}{2^{|j|}}$

Question 3 on Problem Sheet 4

Let $x_0 \in [0,1)$ and $F : [0,1) \to [0,1)$. Define sequence $x_k \in [0,1)$, $k = 0,1,2,\ldots,$ iteratively by $x_{k+1} = F(x_k)$, where

$$F(x) = \begin{cases} 2x & \text{for } x \in [0,1/2) \\ 2x-1 & \text{for } x \in [1/2,1) \end{cases}$$

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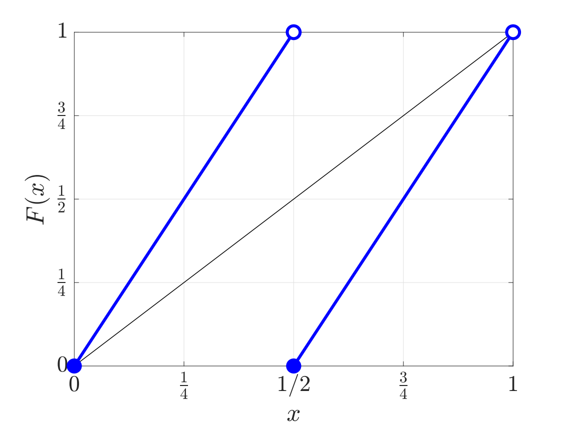

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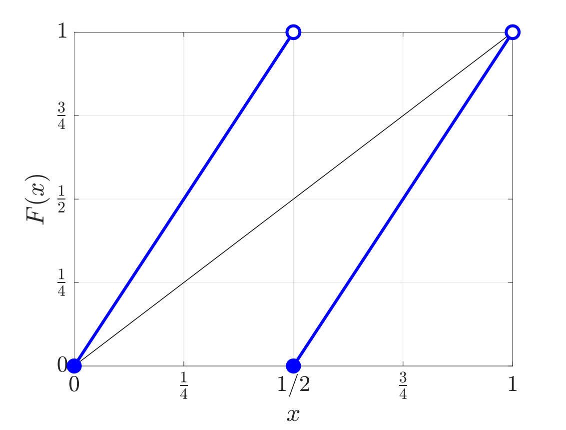

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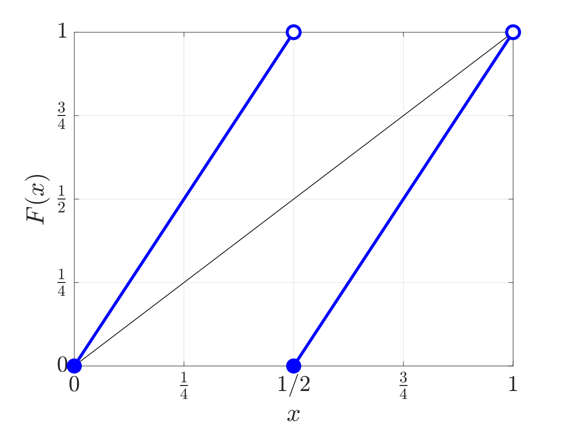

Q3(a): if $x_0 \in [0,1)$ is not a dyadic rational, then $F^{(k)}(x) = 0.a_{k+1}a_{k+2}a_{k+3}a_{k+4}\ldots$

F satisfies properties (i)–(iii) in our definition of chaotic maps $\implies$ F is chaotic

Question 5 on Problem Sheet 4

Let $x_0 \in [0,1]$ and $\mu \in [0,1]$. Define sequence $x_k \in [0,1]$, $k = 0,1,2,\ldots,$ iteratively by $x_{k+1} = F_\mu(x_k)$, where
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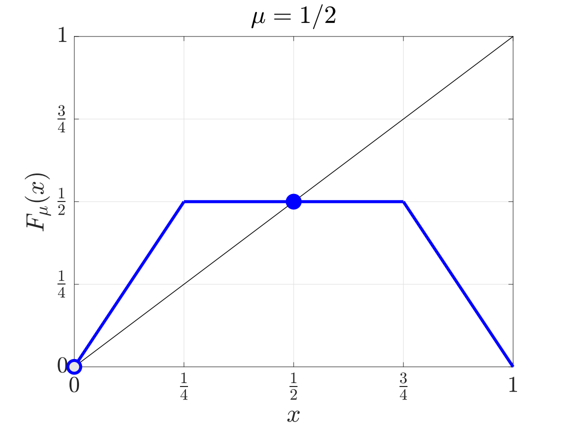

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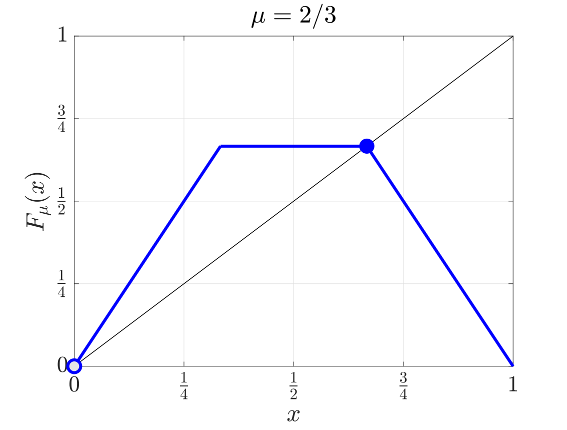

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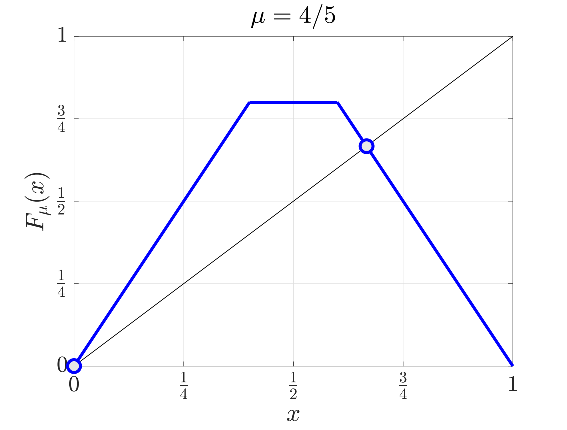

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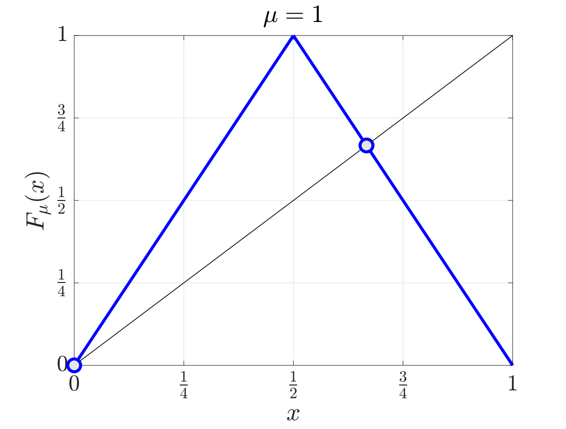

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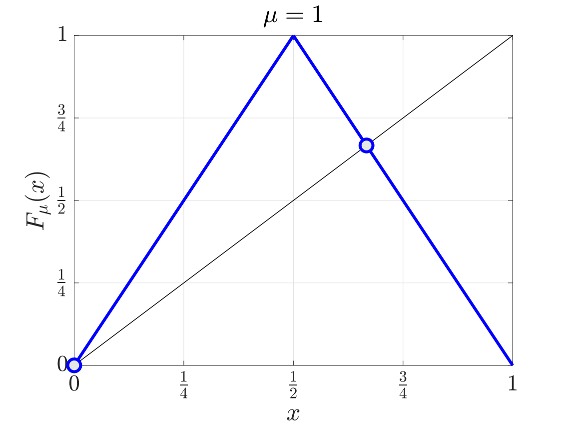

If $x_0 \in [0,1)$ has a binary expansion

$$x_0 = 0.a_1a_2a_3a_4\cdots = \sum_{j=1}^{\infty} \frac{a_j}{2^j}$$

where $a_j \in \{0,1\}$ for $j = 1,2,3\dots,$ then

$$F_1^{(k)}(x) = \begin{cases} 0.a_{k+1}a_{k+2}a_{k+3}\dots \text{if } a_k = 0 \\ 0.a'_{k+1}a'_{k+2}a'_{k+3}\dots \text{if } a_k = 1 \end{cases}$$

where $a'_j = 1$ if $a_j = 0$, and $a'_j = 0$ if $a_j = 1$

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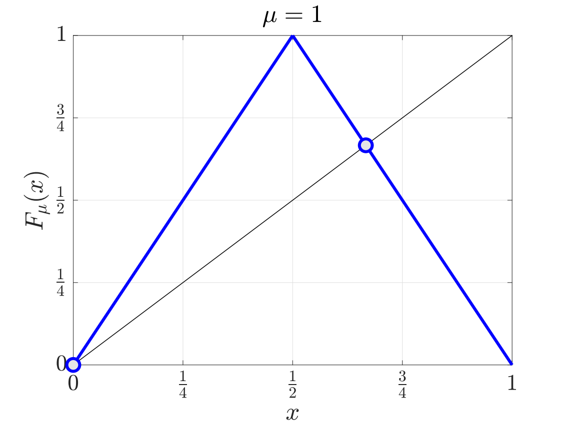

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F_1 satisfies properties (i)–(iii) in our definition of chaotic maps $\Longrightarrow$ F_1 is chaotic

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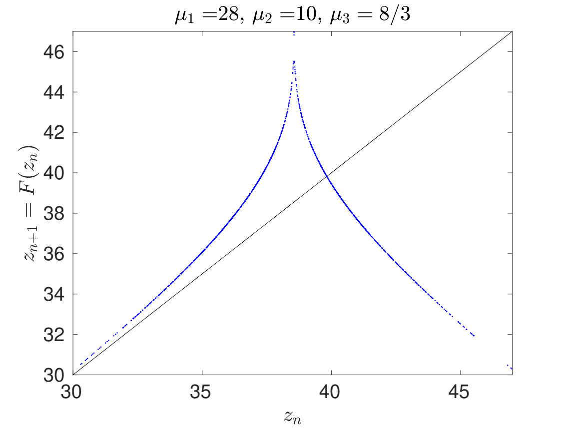

some maps look ‘similar’ to the tent map F_1: Lorenz map

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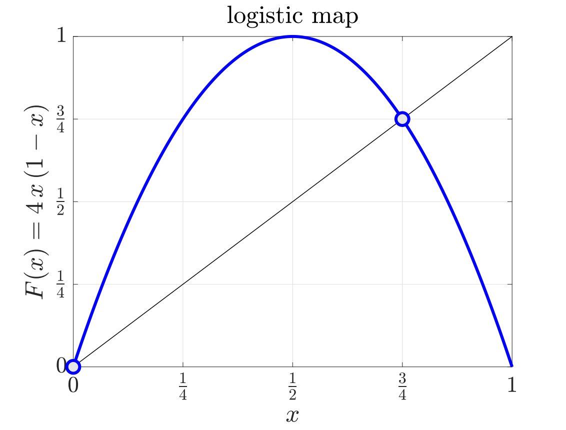

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some maps look 'similar' to the tent map F_1: logistic map $F(x) = 4\,x\,(1-x)$

Question 5 on Problem Sheet 4

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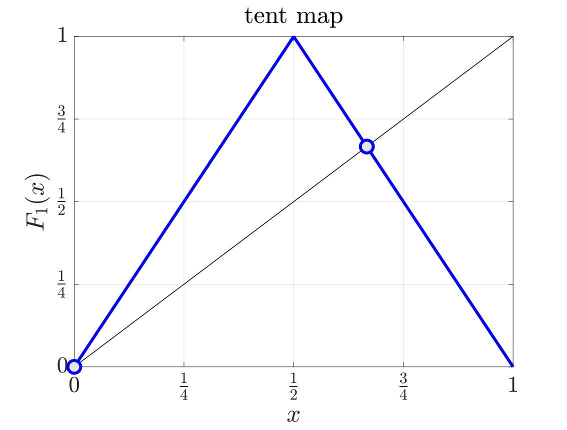

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some maps look 'similar' to the tent map F_1 and tent map $F_1(x)$ is chaotic

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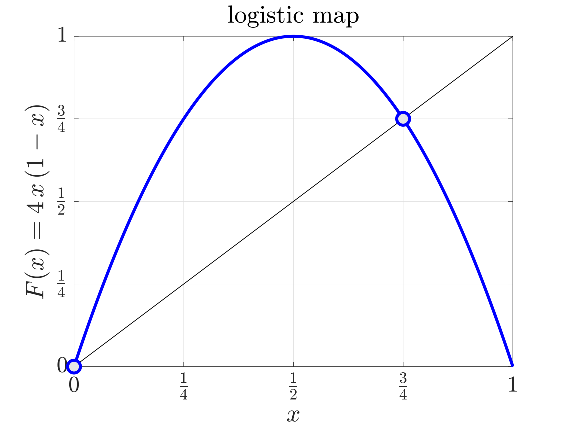

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where $a_j \in \{0,1\}$ for $j = 1,2,3\dots$, then

$$F_1^{(k)}(x) = \begin{cases} 0.a_{k+1}a_{k+2}a_{k+3}\dots \text{ if } a_k = 0 \\ 0.a'_{k+1}a'_{k+2}a'_{k+3}\dots \text{ if } a_k = 1 \end{cases}$$

where $a'_j = 1$ if $a_j = 0$, and $a'_j = 0$ if $a_j = 1$

NEXT LECTURE: we will show that maps ‘similar’ to chaotic maps are chaotic (in particular, we will show that the logistic map $F(x) = 4\,x\,(1-x)$ is chaotic)

B5.6: Nonlinear Dynamics, Bifurcations and Chaos (Lecture 16)

- summary of Lecture 15: we discussed
  Bernoulli shift map, symbolic dynamics. Tent map. Dynamics on metric spaces, sensitive dependence on initial conditions, transitivity, chaotic dynamics.
  Questions 3, 4 and 5 on Problem Sheet 4
- today: we will conclude our discussion of Problem Sheet 4

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- today: we will conclude our discussion of Problem Sheet 4
- course synopsis of **Lectures 9-16**:
  Bifurcations of limit cycles, covering supercritical and subcritical Hopf bifurcations, saddle-node bifurcation of cycles, infinite-period (SNIC) bifurcation and homoclinic (saddle-loop) bifurcation. Oscillations in chemical reaction networks. Weakly nonlinear oscillators. Poincaré-Lindstedt method. Conservative and non-conservative systems. Liénard systems, van der Pol oscillator. Hilbert's 16th problem. Lorenz equations. Lorenz map. Poincaré section. Poincaré map. Converse of Sharkovsky's theorem. Bernoulli shift map, symbolic dynamics. Tent map. Dynamics on metric spaces, sensitive dependence on initial conditions, transitivity, conjugate maps, chaotic dynamics.

Summary of Lecture 15 – general theory

- we studied dynamical systems associated with function $F : \mathbb{M} \to \mathbb{M}$, where $\mathbb{M}$ is a metric space, *i.e.* a set with metric (distance) $d : \mathbb{M} \times \mathbb{M} \to [0, \infty)$

- $F : \mathbb{M} \to \mathbb{M}$ is called *transitive* if there exists $x_0 \in \mathbb{M}$ such that orbit $O(x_0) = \left\{ x_0, F(x_0), F^{(2)}(x_0), F^{(3)}(x_0), \dots \right\}$ is a dense subset of $\mathbb{M}$ (a *transitive point* of F is a point $x_0 \in \mathbb{M}$ which has a dense orbit $O(x_0)$ under F)

- $F : \mathbb{M} \to \mathbb{M}$ has *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that for all $x \in \mathbb{M}$ and any open set $U \subset \mathbb{M}$ satisfying $x \in U$, there is a point $y \in U$ and $j \in \mathbb{N}$ with $d\big(F^{(j)}(x), F^{(j)}(y)\big) > \delta$

- $F : \mathbb{M} \to \mathbb{M}$ is said to be *chaotic* if:
  - (i) the set of all periodic points is dense in $\mathbb{M}$
  - (ii) F is transitive
  - (iii) F has sensitive dependence on initial conditions

- if $F : \mathbb{M} \to \mathbb{M}$ is continuous and $\mathbb{M}$ is not a finite set, then (i) and (ii) imply (iii)

Summary of Lecture 15 – chaotic maps

Bernoulli shift map: $\sigma : \mathbb{M}_{01} \to \mathbb{M}_{01}$ where $\sigma((a_1, a_2, a_3, a_4, \dots)) = (a_2, a_3, a_4, a_5 \dots)$
and $\mathbb{M}_{01} = \{(a_1, a_2, a_3, a_4, \dots) \mid$ such that $a_j = 0$ or $a_j = 1$ for $j = 1, 2, 3, 4, \dots\}$

$\mathbb{M}_{01}$ is a metric space with metric defined by

$d(x, y) = \sum_{j=1}^{\infty} \frac{|a_j - b_j|}{2^j}$ for $x = (a_1, a_2, a_3, a_4, \dots)$ and $y = (b_1, b_2, b_3, b_4, \dots)$

Lemma 6: The Bernoulli shift map $\sigma : \mathbb{M}_{01} \to \mathbb{M}_{01}$ is chaotic.

Summary of Lecture 15 – chaotic maps

Bernoulli shift map: $\sigma : \mathbb{M}_{01} \to \mathbb{M}_{01}$ where $\sigma((a_1, a_2, a_3, a_4, \dots)) = (a_2, a_3, a_4, a_5 \dots)$ and $\mathbb{M}_{01} = \{(a_1, a_2, a_3, a_4, \dots) \,|\, \text{such that } a_j = 0 \text{ or } a_j = 1 \text{ for } j = 1, 2, 3, 4, \dots\}$

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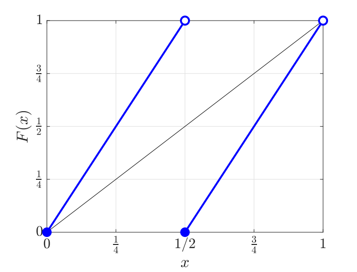
Lemma 6: The Bernoulli shift map $\sigma : \mathbb{M}_{01} \to \mathbb{M}_{01}$ is chaotic.

Doubling map: (Question 3 on Problem Sheet 4)

$F : [0, 1) \to [0, 1)$ where

$$F(x) = \begin{cases} 2x & \text{for } x \in [0, 1/2) \\ 2x - 1 & \text{for } x \in [1/2, 1) \end{cases}$$

$\mathbb{M} = [0, 1)$ with $d(x, y) = |x - y|$



F satisfies properties (i)–(iii) in our definition of chaotic maps $\Longrightarrow$ F is chaotic

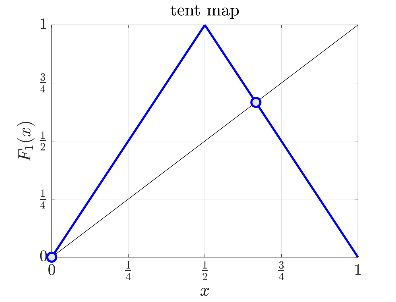
Summary of Lecture 15 – chaotic maps

Tent map: (Question 5 on Problem Sheet 4)

$F_1 : [0,1] \to [0,1]$ where

$$F_1(x) = \begin{cases} 2x & \text{for } \ x \in [0, 1/2] \\ 2 - 2x & \text{for } \ x \in [1/2, 1] \end{cases}$$

$\mathbb{M} = [0,1]$ with $d(x,y) = |x - y|$



Tent map F_1 is chaotic

Tent map: (Question 5 on Problem Sheet 4)

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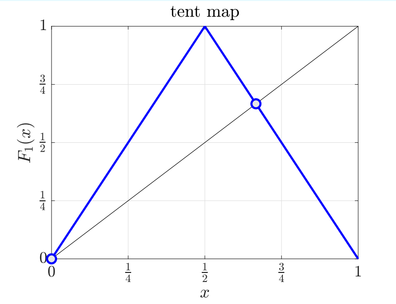
$\mathbb{M} = [0,1]$ with $d(x,y) = |x-y|$

F_1 satisfies properties (i)–(iii) in our definition of chaotic maps $\implies$ F_1 is chaotic



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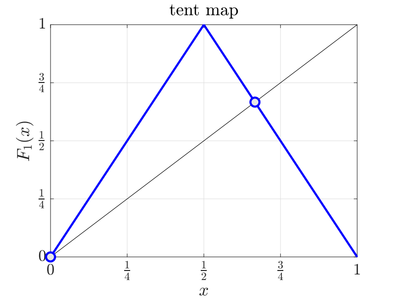
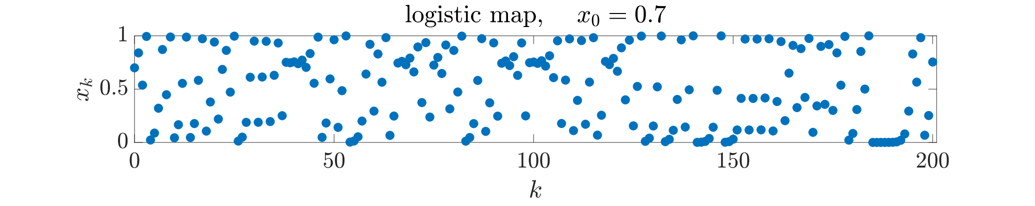


Another 'intuitive definition' of chaos: Question 3 on Problem Sheet 0

Starting with $x_0 = 0.7$, we plot $x_{k+1} = F(x_k)$, for the logistic map $F(x) = 4\,x\,(1 - x)$



Tent map F_1 is chaotic

Tent map: (Question 5 on Problem Sheet 4)

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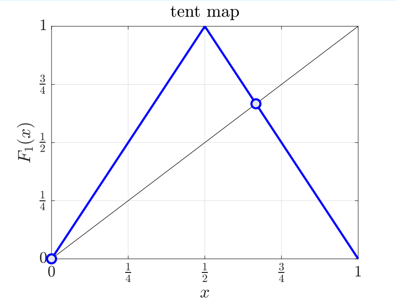

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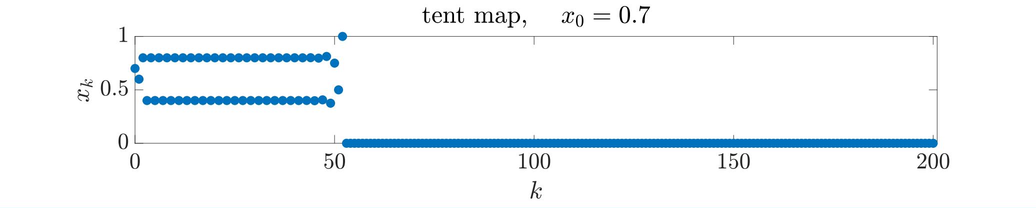

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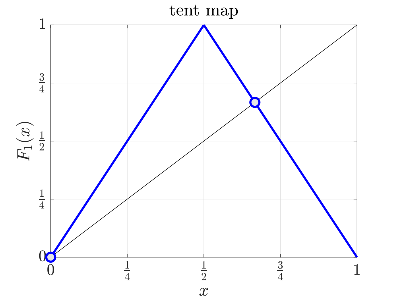
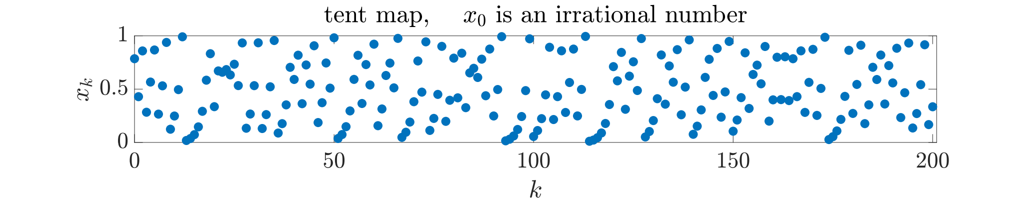


Another 'intuitive definition' of chaos: Question 3 on Problem Sheet 0

Starting with $x_0 \notin \mathbb{Q}$, we plot $x_{k+1} = F(x_k)$, for the tent map $F_1(x)$



Tent map F_1 is chaotic

Tent map: (Question 5 on Problem Sheet 4)

$F_1 : [0,1] \to [0,1]$ where

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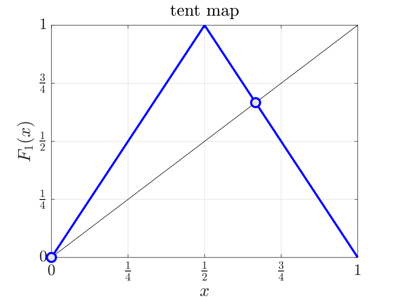

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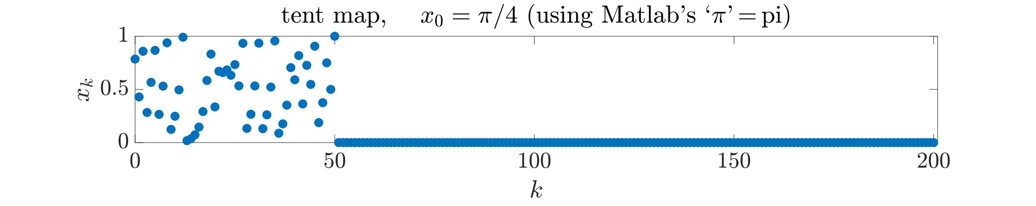

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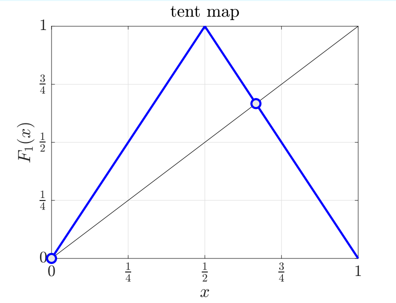

Conclusions:

- the tent map F_1 is chaotic using our definition of chaos – we can prove it
- invariant distribution is the uniform distribution on $[0,1]$ – we can prove it

but the tent map F_1 is not a good random number generator:

- very small numbers $x_k \approx 0$ are followed by very small numbers $2x_k \approx 0$
- x_k becomes 0 after 50 iterations for $x_0 = \pi/4$ in Matlab
  (in computers, irrational numbers are represented as rational numbers, $2^{50} \approx 10^{15}$)

Tent map F_1 is chaotic

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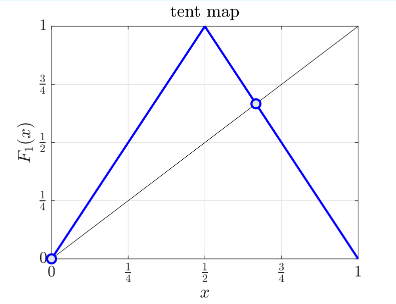
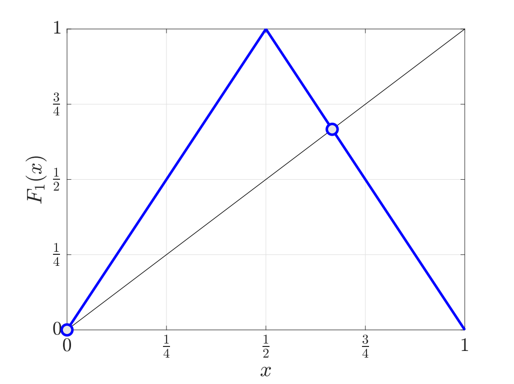
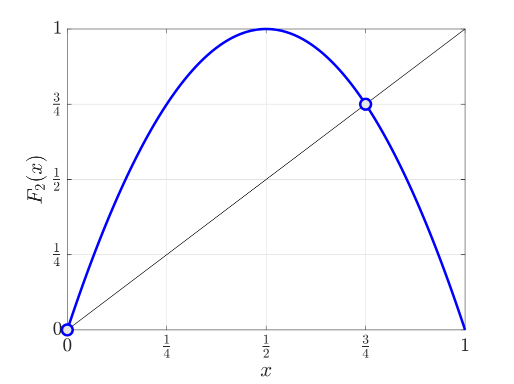


NEXT: the property that the tent map is chaotic can be used to prove the chaotic behaviour of other dynamical systems

Tent map vs. logistic map

tent map $F_1(x) = \begin{cases} 2x & \text{for } x \in [0, 1/2] \\ 2 - 2x & \text{for } x \in [1/2, 1] \end{cases}$ logistic map $F_2(x) \; = \; 4\,x\,(1 - x)$





Lemma: We have $h \circ F_1 = F_2 \circ h$ where $h : [0, 1] \to [0, 1]$ is $h(x) = \sin^2(\pi x/2)$.

Tent map vs. logistic map

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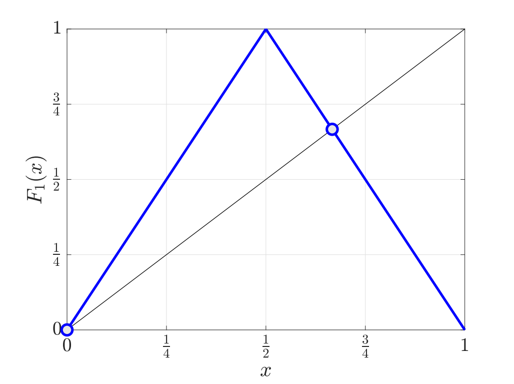

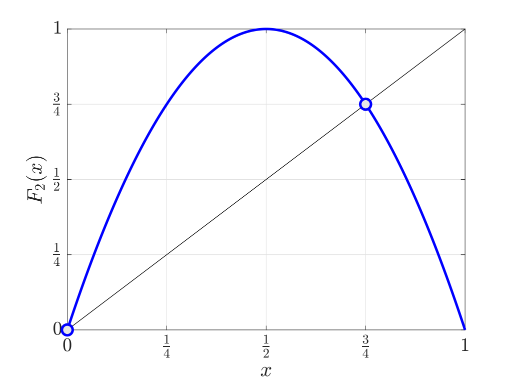

Lemma: We have $h \circ F_1 = F_2 \circ h$ where $h : [0,1] \to [0,1]$ is $h(x) = \sin^2(\pi x/2)$.

Since h has inverse $h^{-1} = \dfrac{2}{\pi} \arcsin \sqrt{y}$, we can rewrite this as $F_1 = h^{-1} \circ F_2 \circ h$.

General definition: homeomorphism

Definition: Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be two metric spaces.
A function $h : \mathbb{M}_1 \to \mathbb{M}_2$ is a *homeomorphism* if:

(i) h is continuous;

(ii) h is one-to-one, *i.e.* if $h(x) = h(y)$, then $x = y$;

(iii) h is onto, *i.e.* $\forall y \in \mathbb{M}_2$ there exists $x \in \mathbb{M}_1$ such that $h(x) = y$;

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Definition: Let $F_1 : \mathbb{M}_1 \to \mathbb{M}_1$ and $F_2 : \mathbb{M}_2 \to \mathbb{M}_2$ be maps of metric spaces $\mathbb{M}_1$ and $\mathbb{M}_2$, respectively. Then F_1 and F_2 are said to be *conjugate* if there is a homeomorphism $h : \mathbb{M}_1 \to \mathbb{M}_2$ such that $h \circ F_1 = F_2 \circ h$.

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Example: tent map $F_1(x) = \begin{cases} 2x & \text{for } x \in [0, 1/2] \\ 2 - 2x & \text{for } x \in [1/2, 1] \end{cases}$ is conjugate to the logistic map $F_2(x) = 4\,x\,(1 - x)$ with conjugacy $h : [0, 1] \to [0, 1]$ given as $h(x) = \sin^2(\pi x/2)$

Conjugate maps and chaos

Theorem: Let $F_1 : \mathbb{M}_1 \to \mathbb{M}_1$ and $F_2 : \mathbb{M}_2 \to \mathbb{M}_2$ be continuous maps of metric spaces $\mathbb{M}_1$ and $\mathbb{M}_2$, respectively, and assume that there is a conjugacy $h : \mathbb{M}_1 \to \mathbb{M}_2$ with $h \circ F_1 = F_2 \circ h$. Then F_1 is chaotic if and only if F_2 is chaotic.

Conjugate maps and chaos

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Corollary: The logistic map $F_2(x) = 4\,x\,(1-x)$ is chaotic,

because the tent map $F_1(x) = \begin{cases} 2x & \text{for } x \in [0, 1/2] \\ 2-2x & \text{for } x \in [1/2, 1] \end{cases}$ is conjugate to the logistic map $F_2(x) = 4\,x\,(1-x)$ with conjugacy $h : [0,1] \to [0,1]$ given as $h(x) = \sin^2(\pi x/2)$.

Conjugate maps and chaos

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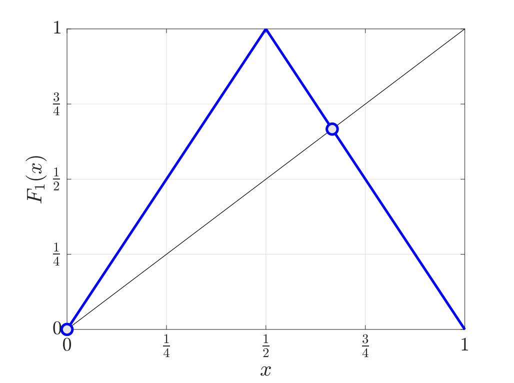
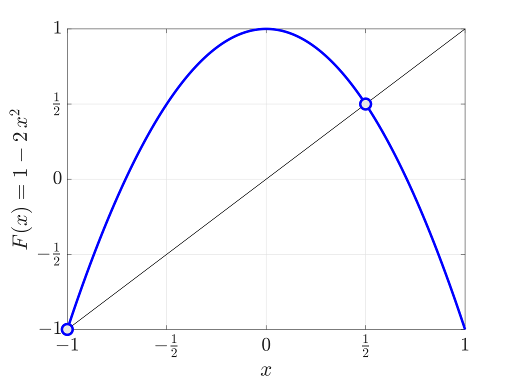
Remark: Different variants of our Theorem also hold under weaker assumptions. For example, if h is not a homeorphism, but h is only continuous and onto satisfying $h \circ F_1 = F_2 \circ h$. Then, assuming that both $F_1 : \mathbb{M}_1 \to \mathbb{M}_1$ and $F_2 : \mathbb{M}_2 \to \mathbb{M}_2$ are continuous maps of metric spaces $\mathbb{M}_1$ and $\mathbb{M}_2$, respectively, we have:
If F_1 is chaotic, then F_2 is chaotic.

Map $F(x) = 1 - 2x^2$ in Question 1 on the 2024 Exam Paper

tent map $F_1(x) = \begin{cases} 2x & \text{for } x \in [0, 1/2] \\ 2 - 2x & \text{for } x \in [1/2, 1] \end{cases}$ map $F(x) = 1 - 2x^2$ for $x \in [-1, 1]$



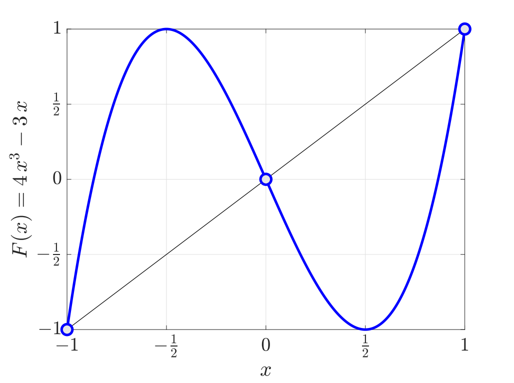


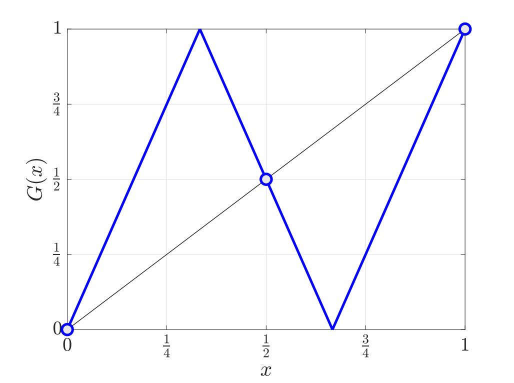
Map $F(x) = 1 - 2x^2$ is chaotic because we have $h \circ F_1 = F \circ h$ where $h : [0, 1] \to [-1, 1]$ is $h(x) = -\cos(\pi x)$ and $F_1(x)$ is chaotic.

Map $F(x) = 4x^3 - 3x$ in Question 7 on Problem Sheet 2

map $F(x) = 4x^3 - 3x$ for $x \in [-1, 1]$

map $G(x) = \begin{cases} 3x & \text{for } x \in [0, 1/3] \\ 2 - 3x & \text{for } x \in [1/3, 2/3] \\ 3x - 2 & \text{for } x \in [2/3, 1] \end{cases}$

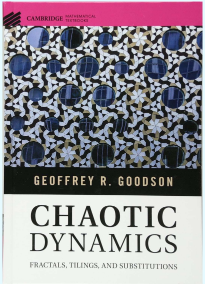


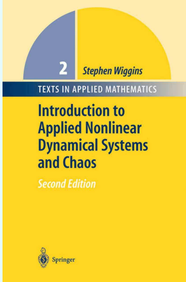


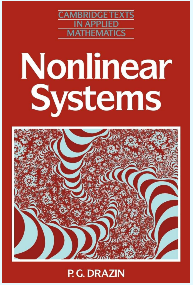
Map $F(x) = 4x^3 - 3x$ is chaotic, because we have $h \circ G = F \circ h$ with conjugacy $h : [0, 1] \to [-1, 1]$ given by $h(x) = \cos(\pi x)$ and $G(x)$ is chaotic.

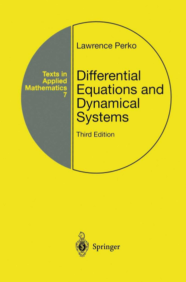
End of course. This is our last slide.

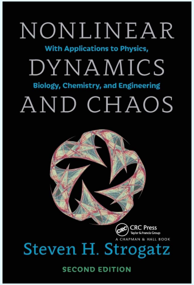
Further examples and additional discussions can be found in the reading list:

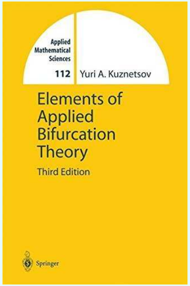












Exams: see slide 3 (discussed in Lecture 1)

Thank you for your attention!