B5.6 Nonlinear Dynamics, Bifurcations and Chaos

Sheet 3 — HT 2025

Solutions to all problems in Sections A and C

Section A: Problems 1, 2 and 3

1. Consider the ODE system

$$\frac{dx_1}{dt} = \mu\, x_1\,(1-x_1) + x_1^2 - x_1^3 - 2\,x_1\,x_2$$
$$\frac{dx_2}{dt} = x_1\,x_2 - \mu\,x_2$$

where $\mu \in (0,1)$ is a parameter.

(a) Find and classify all bifurcations of the ODE system for $0 < \mu < 1$.

(b) Sketch the phase plane for $\mu = 1/2$ and $\mu = 1/4$.

Solution:

(a) The equilibrium points satisfy

$$0 = \mu\, x_1\,(1-x_1) + x_1^2 - x_1^3 - 2\,x_1\,x_2$$
$$0 = x_1\,x_2 - \mu\,x_2$$

Consequently, the second equation implies $x_1 = \mu$ or $x_2 = 0$. Substituting into the first equation, we obtain the critical points

$$\mathbf{x}_{c1} = \left[\mu, \mu - \mu^2\right], \qquad \mathbf{x}_{c2} = [0,0], \qquad \mathbf{x}_{c3} = [1,0] \qquad \text{and} \qquad \mathbf{x}_{c4} = [-\mu, 0].$$

The Jacobian matrix is

$$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \mu + 2(1-\mu)x_1 - 3x_1^2 - 2x_2 & -2x_1 \\ x_2 & x_1 - \mu \end{pmatrix},$$

giving

$$D\mathbf{f}(\mathbf{x}_{c1}) = \begin{pmatrix} \mu - 3\mu^2 & -2\mu \\ \mu - \mu^2 & 0 \end{pmatrix}, \qquad D\mathbf{f}(\mathbf{x}_{c2}) = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix},$$

$$D\mathbf{f}(\mathbf{x}_{c3}) = \begin{pmatrix} -\mu - 1 & -2 \\ 0 & 1-\mu \end{pmatrix} \qquad \text{and} \qquad D\mathbf{f}(\mathbf{x}_{c4}) = \begin{pmatrix} -\mu - \mu^2 & 2\mu \\ 0 & -2\mu \end{pmatrix}.$$

Consequently, $\mathbf{x}_{c2}$ and $\mathbf{x}_{c3}$ are saddles and $\mathbf{x}_{c4}$ is a stable node for all considered values of parameter μ, *i.e.* for $0 < \mu < 1$. The eigenvalues corresponding to matrix $D\mathbf{f}(\mathbf{x}_{c1})$ satisfy

$$\lambda^2 + (3\mu^2 - \mu)\,\lambda + 2\mu\,(\mu - \mu^2) = 0$$

giving

$$\lambda_\pm = \frac{\mu - 3\mu^2 \pm \mu\sqrt{9\mu^2 + 2\mu - 7}}{2}.$$

If $\mu < 7/9$, then we have two complex conjugate eigenvalues

$$\lambda_\pm = \frac{\mu - 3\mu^2}{2} \pm i\,\frac{\mu\sqrt{|9\mu^2 + 2\mu - 7|}}{2}.$$

The real part is positive for $\mu < 1/3$ and negative for $\mu > 1/3$. We have a pair of purely imaginary eigenvalues at the bifurcation point $\mu = 1/3$, when

$$\lambda_\pm = \pm\frac{2}{3\sqrt{3}}\,i$$

and the stability of the critical point $\mathbf{x}_{c1}$ changes at the bifurcation point $\mu = 1/3$. We introduce new variables

$$y_1 = x_1 - \mu\,, \qquad y_2 = \sqrt{3}\left(x_2 - \mu + \mu^2\right), \qquad \overline{\mu} = \mu - \frac{1}{3}\,.$$

Then the ODE system transforms to

$$\begin{aligned}
\frac{\mathrm{d}y_1}{\mathrm{d}t} &= \left(-\overline{\mu} - 3\,\overline{\mu}^2\right) y_1 - \frac{6\,\overline{\mu} + 2}{3\sqrt{3}}\, y_2 - \frac{2}{\sqrt{3}}\, y_1\, y_2 - \left(4\,\overline{\mu} + \frac{1}{3}\right) y_1^2 - y_1^3 \\
\frac{\mathrm{d}y_2}{\mathrm{d}t} &= \frac{2 + 3\,\overline{\mu} - 9\,\overline{\mu}^2}{3\sqrt{3}}\, y_1 + y_1\, y_2
\end{aligned}$$

which can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = M(\overline{\mu})\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} -\frac{2}{\sqrt{3}}\, y_1\, y_2 - \left(4\,\overline{\mu} + \frac{1}{3}\right) y_1^2 - y_1^3 \\ y_1\, y_2 \end{pmatrix}, \tag{1}$$

where matrix $M(\overline{\mu})$ is

$$M(\overline{\mu}) = \begin{pmatrix} -\overline{\mu} - 3\,\overline{\mu}^2 & -\frac{6\,\overline{\mu}+2}{3\sqrt{3}} \\ \frac{2+3\,\overline{\mu}-9\,\overline{\mu}^2}{3\sqrt{3}} & 0 \end{pmatrix}$$

Close to the bifurcation point $\overline{\mu} = 0$, matrix $M(\overline{\mu})$ has eigenvalues

$$\lambda_\pm(\overline{\mu}) = \alpha(\overline{\mu}) \pm i\,\omega(\overline{\mu})$$

where

$$\alpha(\overline{\mu}) = -\,\frac{\overline{\mu} + 3\,\overline{\mu}^2}{2}$$

and

$$\omega(\overline{\mu}) = \frac{2}{3\sqrt{3}}\sqrt{1 + \frac{27\,\overline{\mu}}{6} - \frac{27\,\overline{\mu}^2}{16} - \frac{378\,\overline{\mu}^3}{16} - \frac{243\,\overline{\mu}^4}{16}}$$

which implies

$$\alpha(0) = 0\,, \qquad \omega(0) = \frac{2}{3\sqrt{3}}\,, \qquad \alpha'(0) = -\frac{1}{2}\,, \qquad \text{and} \qquad \omega'(0) = \frac{\sqrt{3}}{2}.$$

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Substituting $\overline{\mu} = 0$ into equation (1), the system reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & -\omega(0) \\ \omega(0) & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} h_1(y_1, y_2) \\ h_2(y_1, y_2) \end{pmatrix}$$

where

$$h_1(y_1, y_2) \;=\; -\frac{2}{\sqrt{3}}\, y_1\, y_2 \,-\, \frac{1}{3}\, y_1^2 \,-\, y_1^3$$

and

$$h_2(y_1, y_2) \;=\; y_1\, y_2\,.$$

Evaluating the partial derivatives at the origin $\mathbf{0} = [0, 0]$, we get

$$\begin{aligned} a(0) \;&=\; \frac{1}{16}\left(\frac{\partial^3 h_1}{\partial y_1^3} + \frac{\partial^3 h_1}{\partial y_1 \partial y_2^2} + \frac{\partial^3 h_2}{\partial y_1^2 \partial y_2} + \frac{\partial^3 h_2}{\partial y_2^3}\right) + \frac{1}{16\,\omega(0)}\left[\frac{\partial^2 h_1}{\partial y_1 \partial y_2}\left(\frac{\partial^2 h_1}{\partial y_1^2} + \frac{\partial^2 h_1}{\partial y_2^2}\right)\right. \\ &\qquad \left. -\frac{\partial^2 h_2}{\partial y_1 \partial y_2}\left(\frac{\partial^2 h_2}{\partial y_1^2} + \frac{\partial^2 h_2}{\partial y_2^2}\right) - \frac{\partial^2 h_1}{\partial y_1^2}\frac{\partial^2 h_2}{\partial y_1^2} + \frac{\partial^2 h_1}{\partial y_2^2}\frac{\partial^2 h_2}{\partial y_2^2}\right] \\ &=\; \frac{1}{16}(-\,6 \,+\, 0 \,+\, 0 \,+\, 0) + \frac{1}{16\,\omega(0)}\left[-\frac{2}{\sqrt{3}}\left(-\frac{2}{3} + 0\right) - 1\,(0+0) - 0 + 0\right] \\ &=\; -\frac{1}{4}\,. \end{aligned}$$

Since $a(0) < 0$, we have a supercritical Hopf bifurcation at $\overline{\mu} = 0$, *i.e.* the original system has a supercritical Hopf bifurcation at $\mu = 1/3$. The normal form is

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{1}{2}\,\overline{\mu}\, r \,-\, \frac{1}{4}\, r^3 \,+\, \ldots$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{2}{3\sqrt{3}} \,+\, \frac{\sqrt{3}}{2}\,\overline{\mu} \,+\, \ldots$$

Origin $\mathbf{0}$ is stable for $\overline{\mu} > 0$ (*i.e.* the critical point $\mathbf{x}_{c1}$ is stable for $\mu > 1/3$) and unstable for $\overline{\mu} < 0$ (*i.e.* the critical point $\mathbf{x}_{c1}$ is unstable for $\mu < 1/3$). A stable limit cycle is born with amplitude

$$\sqrt{2\left(\frac{1}{3} - \mu\right)}$$

and period $3\,\pi\,\sqrt{3}$ at the bifurcation point for $\mu < 1/3$. The limit cycle can be approximated by

$$y_1^2 + y_2^2 = 2\left(\frac{1}{3} - \mu\right),$$

which corresponds to an ellipse in x_1 and x_2 variables

$$\big(x_1 - \mu\big)^2 + 3\big(x_2 - \mu + \mu^2\big)^2 \;=\; 2\left(\frac{1}{3} - \mu\right).$$

The bifurcation diagram can be drawn in the μ–x_1 plane as follows:

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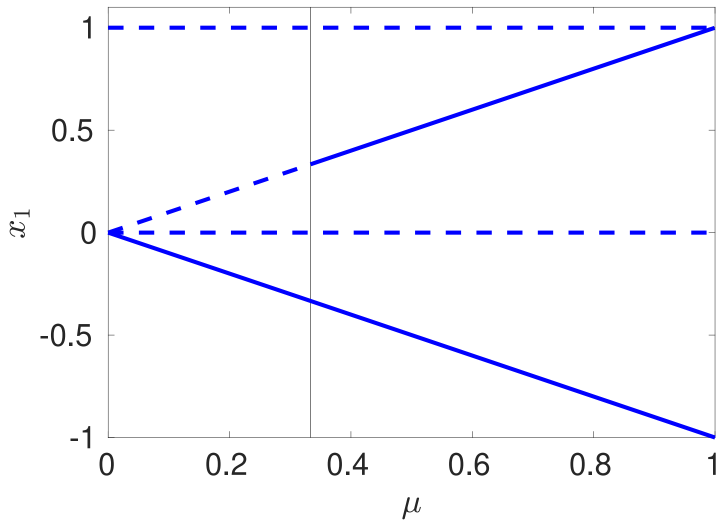
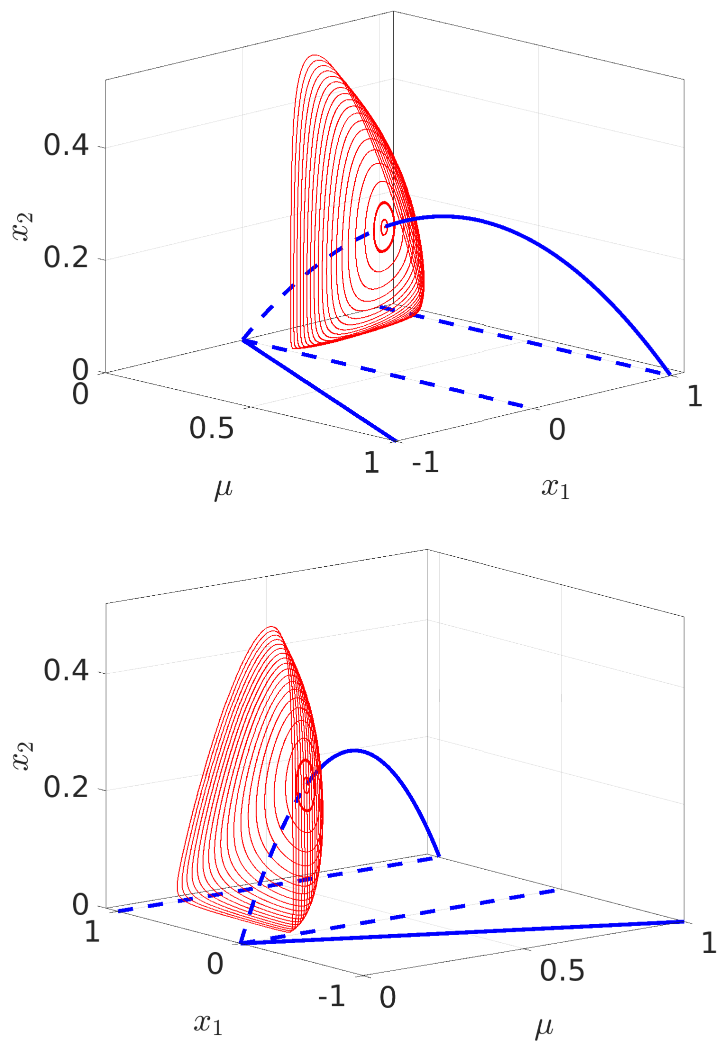
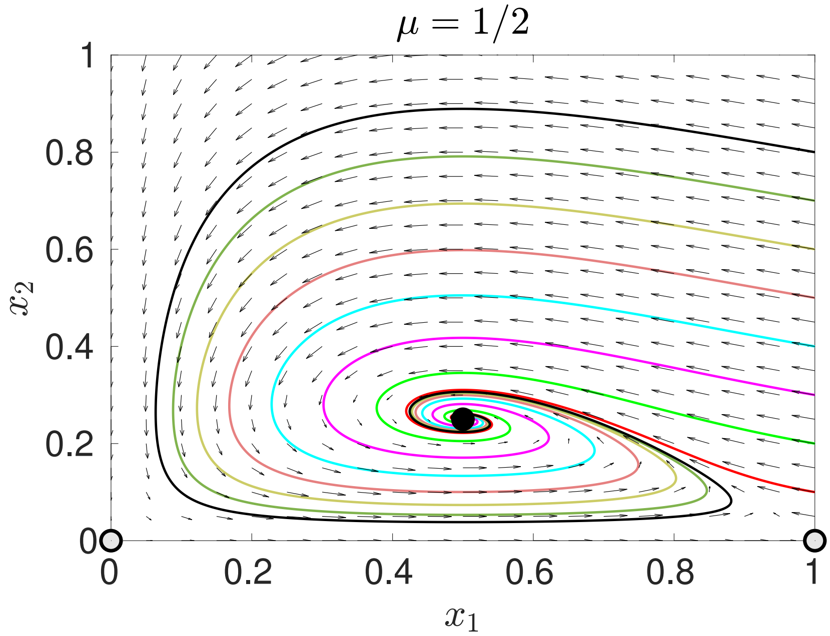


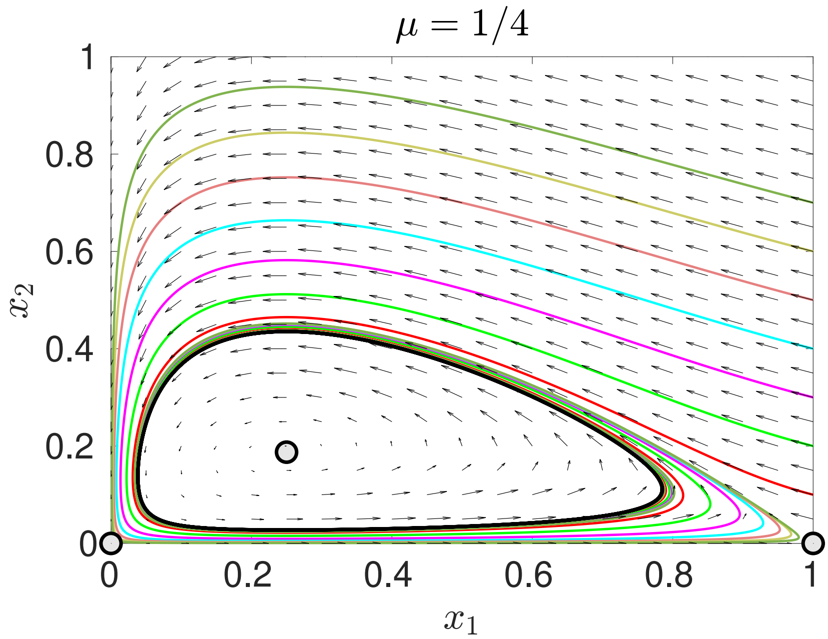
We can also draw the bifurcation diagram in the μ–x_1–x_2 space, when we can also add the (stable) limit cycles for $\mu < 1/3$. This plot (from two different viewing angles) is visualized below:



(b) If $\mu = 1/2$, then the fixed point $\mathbf{x}_{c1} = [\mu, \mu - \mu^2] = [1/2, 1/4]$ is a stable spiral and trajectories approach this stable critical point as shown below, where we plot eight trajectories starting at the right boundary of the square $[0,1] \times [0,1]$:



If $\mu = 1/4$, then the fixed point $\mathbf{x}_{c1} = [\mu, \mu - \mu^2] = [1/4, 3/16]$ is unstable and the system has a limit cycle as illustrated below, where we observe that trajectories (starting at the right boundary of the square $[0,1] \times [0,1]$) approach the limit cycle which is plotted using the black solid line:



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2. Consider the system of $n = 2$ chemical species X_1 and X_2 which are subject to the following $\ell = 4$ chemical reactions:

$$X_1 \xrightarrow{k_1} X_2 \qquad \emptyset \xrightarrow{k_2} X_1 \qquad X_1 \xrightarrow{k_3} \emptyset \qquad 2X_1 + X_2 \xrightarrow{k_4} 3X_1$$

Let $x_1(t)$ and $x_2(t)$ be the concentrations of the chemical species X_1 and X_2, respectively.

(a) Assuming mass action kinetics, write a system of ODEs (reaction rate equations) describing the time evolution of $x_1(t)$ and $x_2(t)$.

(b) Assume the problem has already been non-dimensionalized and choose the values of dimensionless rate constants as

$$k_1 = \mu \qquad \text{and} \qquad k_2 = k_3 = k_4 = 1\,,$$

where $\mu > 0$ is a single parameter that we will vary.

Show that a supercritical Hopf bifurcation occurs at some parameter value $\mu = \mu_c$, where you should determine the value of $\mu = \mu_c$ at the bifurcation point.

(c) Find an approximation of the amplitude and the period of the limit cycle close to the bifurcation value $\mu = \mu_c$.

(d) Sketch the phase plane for μ close to $\mu = \mu_c$.

Solution:

(a) Using the definition of mass action kinetics (covered in Lecture 1), we have:

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= k_2 - (k_1 + k_3)\,x_1 + k_4\,x_1^2\,x_2 \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= k_1\,x_1 - k_4\,x_1^2\,x_2 \end{aligned}$$

(b) Using our values of parameters $k_1 = \mu$, $k_2 = k_3 = k_4 = 1$, we have

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= 1 - (\mu + 1)\,x_1 + x_1^2\,x_2 \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= \mu\,x_1 - x_1^2\,x_2 \end{aligned}$$

This system only has one critical point

$$\mathbf{x}_c = [1, \mu].$$

The Jacobian matrix is

$$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} 2x_1x_2 - \mu - 1 & x_1^2 \\ \mu - 2x_1x_2 & -x_1^2 \end{pmatrix},$$

giving

$$D\mathbf{f}(\mathbf{x}_c) = \begin{pmatrix} \mu - 1 & 1 \\ -\mu & -1 \end{pmatrix}.$$

The eigenvalues solve the quadratic equation

$$\lambda^2 + (2-\mu)\,\lambda + 1 = 0,$$

which implies

$$\lambda_\pm = \frac{\mu - 2 \pm \sqrt{\mu(\mu-4)}}{2}.$$

If $\mu \in (0,4)$, then we have two complex conjugate eigenvalues

$$\lambda_\pm = \frac{\mu-2}{2} \pm i\,\frac{\sqrt{\mu(4-\mu)}}{2}.$$

The real part is positive for $\mu > 2$ and negative for $\mu < 2$. We have a pair of purely imaginary eigenvalues at the bifurcation point $\mu = 2$, when

$$\lambda_\pm = \pm\, i$$

and the stability of the critical point $\mathbf{x}_c$ changes at the bifurcation point $\mu = 2$. Introducing new variables

$$\overline{x}_1 = x_1 - 1, \qquad \overline{x}_2 = x_2 - \mu\,, \qquad \overline{\mu} = \frac{\mu-2}{2}\,, \tag{2}$$

the ODE system transforms to

$$\begin{aligned} \frac{\mathrm{d}\overline{x}_1}{\mathrm{d}t} &= (2\,\overline{\mu}+1)\,\overline{x_1} + \overline{x_2} + 2\,\overline{x}_1\,\overline{x}_2 + 2\,(\,\overline{\mu}+1)\,\overline{x}_1^2 + \overline{x}_1^2\,\overline{x}_2 \\ \frac{\mathrm{d}\overline{x}_2}{\mathrm{d}t} &= -\,2\,(\overline{\mu}+1)\,\overline{x}_1 - \overline{x}_2 - 2\,\overline{x}_1\,\overline{x}_2 - 2\,(\overline{\mu}+1)\,\overline{x}_1^2 - \overline{x}_1^2\,\overline{x}_2 \end{aligned}$$

which can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = M(\overline{\mu})\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + \left(2\,\overline{x}_1\,\overline{x}_2 + 2\,(\overline{\mu}+1)\,\overline{x}_1^2 + \overline{x}_1^2\,\overline{x}_2\right)\begin{pmatrix}1\\ -1\end{pmatrix}, \tag{3}$$

where matrix $M(\overline{\mu})$ is

$$M(\overline{\mu}) = \begin{pmatrix} 2\,\overline{\mu} + 1 & 1 \\ -\,2\,(\overline{\mu}+1) & -1 \end{pmatrix}$$

Close to the bifurcation point $\overline{\mu} = 0$, matrix $M(\overline{\mu})$ has eigenvalues

$$\lambda_\pm(\overline{\mu}) = \alpha(\overline{\mu}) \pm i\,\omega(\overline{\mu})$$

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where

$$\alpha(\overline{\mu}) = \overline{\mu} \qquad \text{and} \qquad \omega(\overline{\mu}) = \sqrt{1-\overline{\mu}^2}$$

which implies

$$\alpha(0) = 0\,, \qquad \omega(0) = 1\,, \qquad \alpha'(0) = 1\,, \qquad \text{and} \qquad \omega'(0) = 0\,.$$

Matrix $M(0)$ has eigenvalues $\lambda_\pm = \pm i$ with eigenvectors

$$\mathbf{v}_\pm = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \mp i \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We introduce the change of variables

$$\begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{with inverse} \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix}. \tag{4}$$

Using (3) at the bifurcation point $\overline{\mu} = 0$, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} M(0) \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &\quad + \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \left(2\,\overline{x}_1\,\overline{x}_2 + 2\,\overline{x}_1^2 + \overline{x}_1^2\,\overline{x}_2\right) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \left(y_2^2 - y_1^2 + y_1^3 - 2\,y_1^2\,y_2 + y_1\,y_2^2\right) \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \end{aligned}$$

which is in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & -\omega(0) \\ \omega(0) & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} h_1(y_1, y_2) \\ h_2(y_1, y_2) \end{pmatrix}$$

where

$$h_2(y_1, y_2) = -h_1(y_1, y_2) = y_2^2 - y_1^2 + y_1^3 - 2\,y_1^2\,y_2 + y_1\,y_2^2\,.$$

Evaluating the partial derivatives at the origin $\mathbf{0} = [0, 0]$, we get

$$\begin{aligned} a(0) &= \frac{1}{16}\left(\frac{\partial^3 h_1}{\partial y_1^3} + \frac{\partial^3 h_1}{\partial y_1 \partial y_2^2} + \frac{\partial^3 h_2}{\partial y_1^2 \partial y_2} + \frac{\partial^3 h_2}{\partial y_2^3}\right) + \frac{1}{16\,\omega(0)}\left[\frac{\partial^2 h_1}{\partial y_1 \partial y_2}\left(\frac{\partial^2 h_1}{\partial y_1^2} + \frac{\partial^2 h_1}{\partial y_2^2}\right)\right. \\ &\qquad \left. - \frac{\partial^2 h_2}{\partial y_1 \partial y_2}\left(\frac{\partial^2 h_2}{\partial y_1^2} + \frac{\partial^2 h_2}{\partial y_2^2}\right) - \frac{\partial^2 h_1}{\partial y_1^2}\frac{\partial^2 h_2}{\partial y_1^2} + \frac{\partial^2 h_1}{\partial y_2^2}\frac{\partial^2 h_2}{\partial y_2^2}\right] \\ &= \frac{1}{16}(-6 - 2 - 4 + 0) + \frac{1}{16\,\omega(0)}[0(2-2) - 0(-2+2) - 2(-2) - 2(2)] \\ &= -\frac{3}{4}\,. \end{aligned}$$

Since $a(0) < 0$, we have a supercritical Hopf bifurcation at $\overline{\mu} = 0$, *i.e.* the original system has a supercritical Hopf bifurcation at $\mu = 2$.

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(c) The normal form is

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \overline{\mu}\, r - \frac{3}{4} r^3 + \dots$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1 + \dots$$

Origin $\mathbf{0}$ is stable for $\overline{\mu} < 0$ (*i.e.* the critical point $\mathbf{x}_c$ is stable for $\mu < 2$) and unstable for $\overline{\mu} > 0$ (*i.e.* the critical point $\mathbf{x}_c$ is unstable for $\mu > 2$). A stable limit cycle is born with amplitude

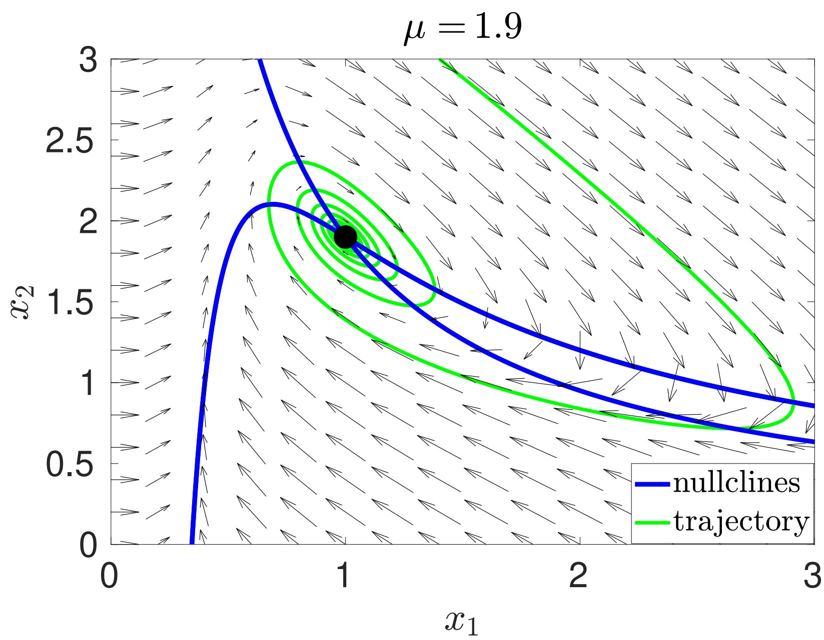
$$\sqrt{\frac{2(\mu - 2)}{3}}$$

and period 2π at the bifurcation point for $\mu > 2$. The limit cycle can be approximated by

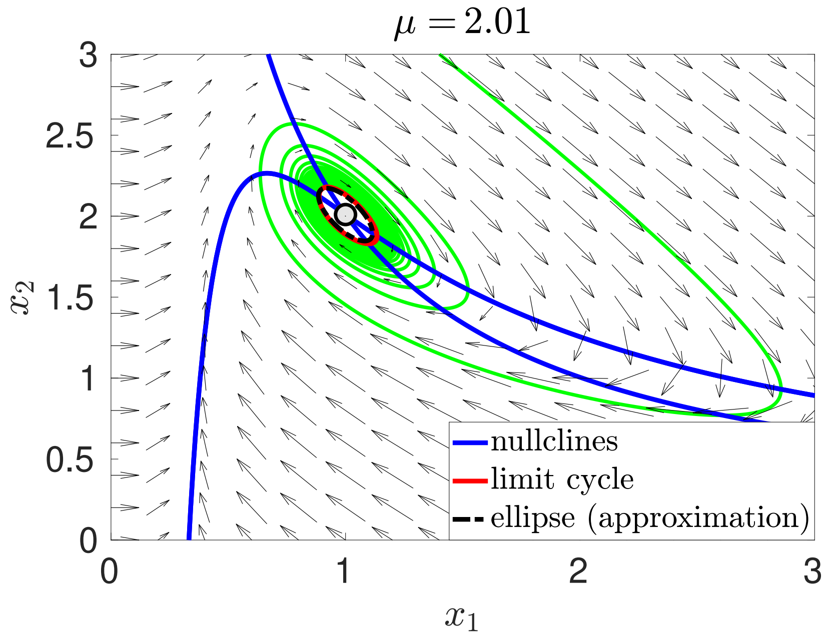
$$y_1^2 + y_2^2 = \frac{2(\mu - 2)}{3} \tag{5}$$

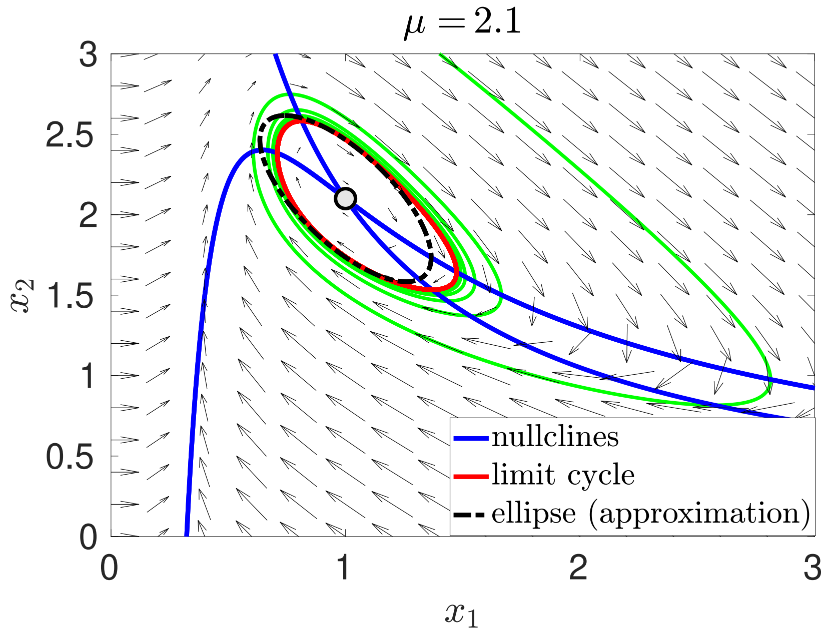
which, using transformation of variables (2) and (4), corresponds to an ellipse in x_1 and x_2 variables, visualized as the black dot-dashed line on the next page.

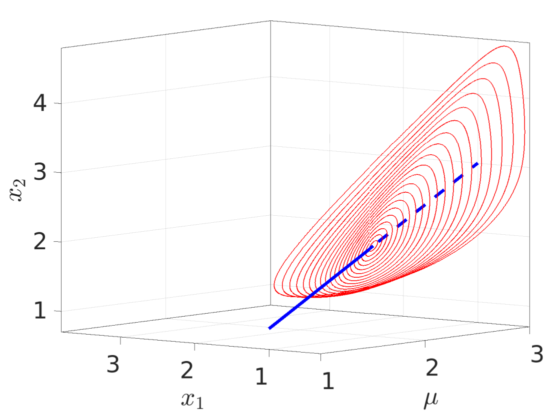
(d) If $\mu < 2$, then the fixed point $\mathbf{x}_c = [1, \mu]$ is a stable spiral and trajectories approach this stable critical point as shown below for $\mu = 1.9$, where we plot a trajectory starting at $[1.4, 3]$ as the green line. Nullclines are visualized as blue lines:

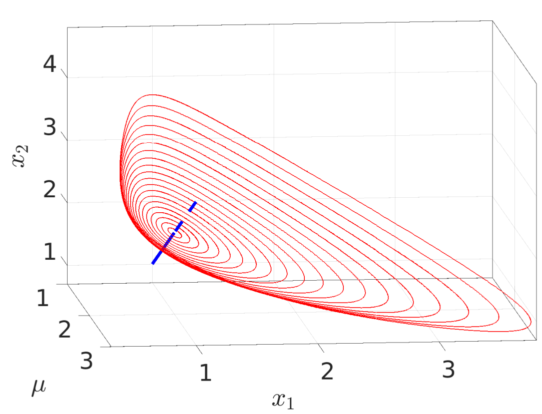


If $\mu > 2$, then the fixed point $\mathbf{x}_c = [1, \mu]$ is unstable and the system has a limit cycle as shown on the next page, where we observe that the green trajectory approaches the red limit cycle. The approximating ellipse (5) is visualized as the black dot-dashed line. The bifurcation diagram can also be visualized in the μ–x_1–x_2 space, as shown on the next page, with (stable) limit cycles included for $\mu > 2$.


$\mu = 2.01$
x_2
x_1
nullclines
limit cycle
ellipse (approximation)


$\mu = 2.1$
x_2
x_1
nullclines
limit cycle
ellipse (approximation)


x_2
x_1
μ


x_2
μ
x_1

3. Consider the second-order ODE describing an 'asymmetric spring' in the form

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\,x \,+\, \varepsilon\, x^2$$

(a) Rewrite the ODE as a planar system of autonomous ODEs.

(b) Find and classify all critical points.

(c) Consider the periodic orbit satisfying

$$x(0) = A, \qquad \frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0.$$

Use the Poincaré-Lindstedt method to find the expansion of the frequency of this orbit up to [and including] terms of $\mathcal{O}(\varepsilon^2)$.

Solution:

(a) Denoting

$$y_1 = x \qquad \text{and} \qquad y_2 = \frac{\mathrm{d}x}{\mathrm{d}t}\,,$$

we can rewrite this second order equation as the following planar system of autonomous ODEs

$$\begin{aligned} \frac{\mathrm{d}y_1}{\mathrm{d}t} &= y_2 \\ \frac{\mathrm{d}y_2}{\mathrm{d}t} &= -\,y_1 \,+\, \varepsilon\, y_1^2 \end{aligned}$$

(b) The critical points are obtained by solving $0 = -\,y_1 \,+\, \varepsilon\, y_1^2$ and $y_2 = 0$. We get $\mathbf{y}_{c1} = [0,0]$ which exists for all $\varepsilon \in \mathbb{R}$ and $\mathbf{y}_{c2} = [\varepsilon^{-1}, 0]$ which exists for $\varepsilon \neq 0$. The Jacobian matrix is

$$D\mathbf{f}(\mathbf{y}) = \begin{pmatrix} 0 & 1 \\ -1 \,+\, 2\,\varepsilon\, y_1 & 0 \end{pmatrix},$$

giving

$$D\mathbf{f}(\mathbf{y}_{c1}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad D\mathbf{f}(\mathbf{y}_{c2}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues are $\lambda_\pm = \pm\, i$ at $\mathbf{y}_{c1}$ and $\lambda_\pm = \pm 1$ at $\mathbf{y}_{c2}$. Consequently, $\mathbf{y}_{c1}$ is a center and $\mathbf{y}_{c2}$ is a saddle whenever it exists.

(c) We transform the time variable as $\tau = \omega(\varepsilon)\, t$ where $2\pi/\omega(\varepsilon)$ is the period of the periodic solution. We obtain

$$\omega^2(\varepsilon)\, \frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} = -\,x \,+\, \varepsilon\, x^2 \tag{6}$$

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where we denote the solution by $x(\tau;\varepsilon)$. We have $x(\tau+2\pi;\varepsilon) = x(\tau;\varepsilon)$ and, by translating time if necessary, we also have

$$\frac{\mathrm{d}x}{\mathrm{d}\tau}(0;\varepsilon) = 0 \qquad \text{and} \qquad x(0;\varepsilon) = A\,,$$

where A is the amplitude. We expand

$$x(\tau;\varepsilon) = x_0(\tau) + \varepsilon\, x_1(\tau) + \varepsilon^2\, x_2(\tau) + \dots \quad \text{and} \quad \omega(\varepsilon) = \omega_0 + \varepsilon\,\omega_1 + \varepsilon^2\,\omega_2 + \dots$$

Substituting into (6) and equating coefficients of ε^0 gives

$$\omega_0^2\,\frac{\mathrm{d}^2 x_0}{\mathrm{d}\tau^2} = -\,x_0 \qquad \text{with} \qquad x_0(\tau+2\pi) = x_0(\tau)\,.$$

Thus $\omega_0 = 1$ and

$$x_0(\tau) = A\,\cos(\tau)\,.$$

Equating coefficients of ε^1 gives

$$\omega_0^2\,\frac{\mathrm{d}^2 x_1}{\mathrm{d}\tau^2} + 2\,\omega_0\,\omega_1\,\frac{\mathrm{d}^2 x_0}{\mathrm{d}\tau^2} = -\,x_1 + x_0^2 \qquad \text{with} \qquad x_1(\tau+2\pi) = x_1(\tau)\,.$$

Substituting $\omega_0 = 1$ and $x_0(\tau) = A\,\cos(\tau)$, we get

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}\tau^2} + x_1 = 2\,\omega_1\,A\cos(\tau) + A^2\cos^2(\tau) = 2\,\omega_1\,A\cos(\tau) + \frac{A^2}{2}\,(1+\cos(2\tau))\,.$$

Eliminating the secular term gives $\omega_1 = 0$. Solving the resulting equation, we get

$$x_1(\tau) = c_1\cos(\tau) + c_2\sin(\tau) + \frac{A^2}{2} - \frac{A^2}{6}\cos(2\tau)\,. \tag{7}$$

Using the initial conditions, we get $c_1 = -A^2/3$ and $c_2 = 0$. Consequently, we have

$$x(\tau;\varepsilon) = A\,\cos(\tau) + \varepsilon\left(\frac{A^2}{2} - \frac{A^2}{3}\cos(\tau) - \frac{A^2}{6}\cos(2\tau)\right) + \mathcal{O}(\varepsilon^2).$$

Equating coefficients of ε^2 gives

$$\omega_0^2\,\frac{\mathrm{d}^2 x_2}{\mathrm{d}\tau^2} + 2\,\omega_0\,\omega_1\,\frac{\mathrm{d}^2 x_1}{\mathrm{d}\tau^2} + (2\,\omega_0\,\omega_2 + \omega_1^2)\,\frac{\mathrm{d}^2 x_0}{\mathrm{d}\tau^2} = -\,x_2 + 2\,x_0\,x_1\,.$$

Substituting $\omega_0 = 1$, $\omega_1 = 0$, $x_0(\tau) = A\,\cos(\tau)$ and equation (7) for $x_1(\tau)$, we get

$$\begin{aligned}\frac{\mathrm{d}^2 x_2}{\mathrm{d}\tau^2} + x_2 &= 2\,\omega_2\,A\,\cos(\tau) + 2\,A\,\cos(\tau)\left(\frac{A^2}{2} - \frac{A^2}{3}\cos(\tau) - \frac{A^2}{6}\cos(2\tau)\right)\\ &= \left(2\,\omega_2\,A + \frac{5A^3}{6}\right)\cos(\tau) - \frac{A^3}{6}\,(2 + 2\cos(2\tau) + \cos(3\tau))\,.\end{aligned}$$

Eliminating the secular term gives

$$2\,\omega_2\,A + \frac{5A^3}{6} = 0$$

which implies

$$\omega_2 = -\frac{5A^2}{12}.$$

Thus, we conclude that

$$\omega = 1 - \frac{5\,\varepsilon^2 A^2}{12} + \mathcal{O}\big(\varepsilon^3\big)\,.$$

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Section C: Problem 7

7. In our lectures, we defined Sharkovsky's ordering:

$$3 \rhd 5 \rhd 7 \rhd \ldots \rhd 2 \times 3 \rhd 2 \times 5 \rhd \ldots \rhd 2^2 \times 3 \rhd 2^2 \times 5 \rhd \ldots \rhd 2^3 \times 3 \rhd 2^3 \times 5 \rhd \ldots$$

$$\ldots \rhd 2^n \times 3 \rhd 2^n \times 5 \rhd \ldots \rhd 2^n \rhd 2^{n-1} \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1$$

and we stated Sharkovsky's Theorem:

Let $\Omega = [a, b] \subset \mathbb{R}$ be an interval and $F : \Omega \to \Omega$ be continuous.

If F has a point of period n, then it has points of period k for all $k \in \mathbb{N}$ with $n \rhd k$.

We did not prove this theorem in our lectures. In this question, you are asked to prove some special cases of Sharkovsky's Theorem and also some special cases of its converse, which states that for every $n \in \mathbb{N}$, there exists a continuous map $F : \Omega \to \Omega$ that has a point of period n but no cycles of period k for any k appearing before n in Sharkovsky's ordering.

(a) Give an example of interval $\Omega = [a, b] \subset \mathbb{R}$ and a continuous function $F : \Omega \to \Omega$ which has a point of period 5, but no points of period 3.

(b) Give an example of interval $\Omega = [a, b] \subset \mathbb{R}$ and a continuous function $F : \Omega \to \Omega$ which has a point of period 7, but no points of period 5.

(c) Assume that the continuous function $F : \Omega \to \Omega$ has a point of period 3. Show that the map F has a fixed point.

(d) Assume that the continuous function $F : \Omega \to \Omega$ has a point of period 3. Show that the map F has a point of period 2.

(e) Assume that the continuous function $F : \Omega \to \Omega$ has a point of period 3. Show that the map F has a point of period 4.

Solution:

(a) Consider $\Omega = [0, 4]$ and function $F : \Omega \to \Omega$ defined by

$$F(x) = \begin{cases} 2 + 2x & \text{for} \quad x \in [0, 1]\,; \\ 5 - x & \text{for} \quad x \in [1, 2]\,; \\ 7 - 2x & \text{for} \quad x \in [2, 3]\,; \\ 4 - x & \text{for} \quad x \in [3, 4]\,. \end{cases}$$

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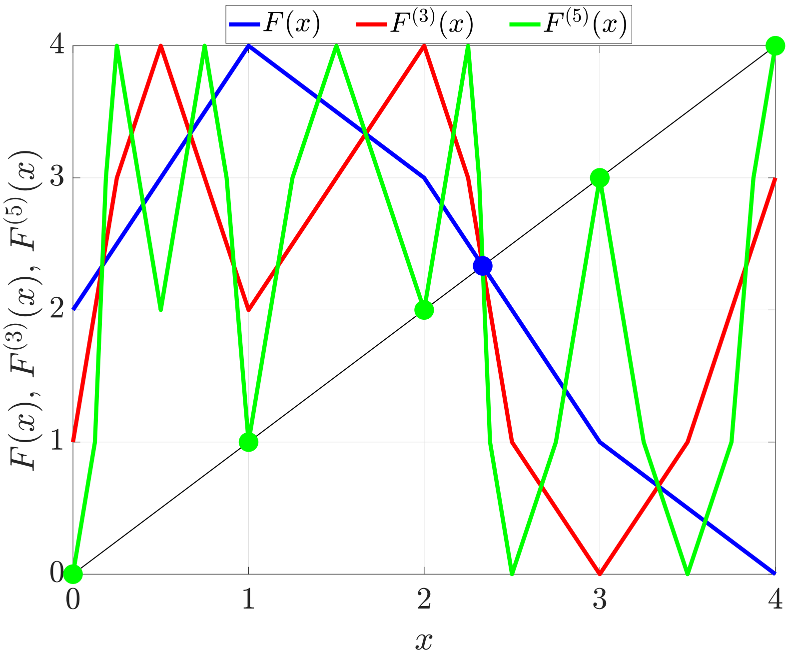
Then $F : [0, 4] \to [0, 4]$ is a continuous function, which has five points of period 5 which form the 5-cycle

$$\{0,\ 1,\ 2,\ 3,\ 4\}.$$

The following figure includes plots of $F(x)$, $F^{(3)}(x)$ and $F^{(5)}(x)$ illustrating that the only fixed point of $F(x)$ and $F^{(3)}(x)$ is $x_c = 7/3$, *i.e.* F has no points of period 3, but it has points of period 5.



To prove that $x_c = 7/3$ is the only fixed point of $F^{(3)}(x)$, we could first exclude intervals $[0, 1]$, $[1, 2]$ and $[3, 4]$ by showing that $F^{(3)}[0, 1] = [1, 4]$, $F^{(3)}[1, 2] = [2, 4]$ and $F^{(3)}[3, 4] = [1, 3]$, *i.e* any fixed point of $F^{(3)}(x)$ must be in interval $[2, 3]$ and $F^{(3)}(x)$ is decreasing in this interval, so there is only one fixed point $x_c = 7/3$.

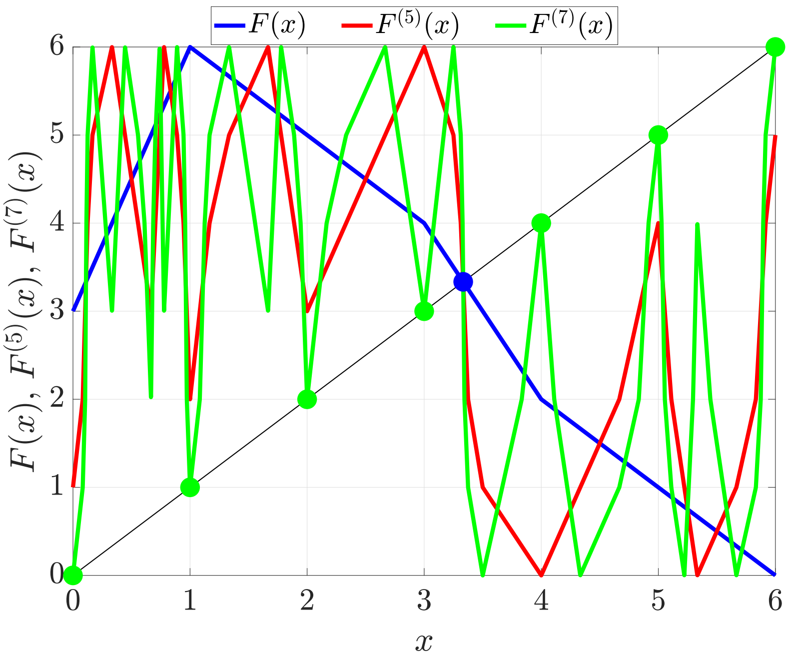
(b) Consider $\Omega = [0, 6]$ and function $F : \Omega \to \Omega$ defined by

$$F(x) = \begin{cases} 3 + 3x & \text{for} \quad x \in [0, 1]\,; \\ 7 - x & \text{for} \quad x \in [1, 3]\,; \\ 10 - 2x & \text{for} \quad x \in [3, 4]\,; \\ 6 - x & \text{for} \quad x \in [4, 6]\,. \end{cases}$$

Then $F : [0, 6] \to [0, 6]$ is a continuous function, which has seven points of period 7 which form the 7-cycle

$$\{0,\ 1,\ 2,\ 3,\ 4, 5, 6\}.$$

The following figure includes plots of $F(x)$, $F^{(5)}(x)$ and $F^{(7)}(x)$ illustrating that the only fixed point of $F(x)$ and $F^{(5)}(x)$ is $x_c = 10/3$, *i.e.* F has no points of period 5, but it has points of period 7.



(c) Consider that the 3-cycle is $\{a, b, c\}$ with $F(a) = b$, $F(b) = c$ and $F(c) = a$. Then it is sufficient to investigate the case $a < b < c$ (and other cases follow by symmetry). Define intervals $\Omega_1 = [a, b]$ and $\Omega_2 = [b, c]$. Then we have

$$\Omega_2 \subset F(\Omega_1) \qquad \text{and} \qquad \Omega_1 \cup \Omega_2 \subset F(\Omega_2)\,. \tag{8}$$

In particular, $\Omega_2 \subset F(\Omega_2)$ and F has a fixed point in subinterval Ω_2.

(d) Using (8), we get $\Omega_1 \subset F(\Omega_2)$. Therefore, there exists interval $\Omega_0 \subset \Omega_2$ such that $\Omega_1 = F(\Omega_0)$. Using (8), we have $\Omega_0 \subset \Omega_2 \subset F(\Omega_1) = F(F(\Omega_0)) = F^{(2)}(\Omega_0)$. Therefore, $F^{(2)}$ has a fixed point in Ω_0, which is a point of period 2 of map F.

(e) Using (8), we get the existence of interval $\Omega_0 \subset \Omega_2$ such that $\Omega_1 = F(\Omega_0)$ as in part (d). Moreover, there exists interval $\Omega_3 \subset \Omega_2$ such that $\Omega_0 = F(\Omega_3)$ and there exists interval $\Omega_4 \subset \Omega_2$ such that $\Omega_3 = F(\Omega_4)$. Then

$$\Omega_1 = F(\Omega_0) = F(F(\Omega_3)) = F(F(F(\Omega_4))) = F^{(3)}(\Omega_4).$$

Using (8), we have $\Omega_4 \subset \Omega_2 \subset F(\Omega_1) = F^{(4)}(\Omega_4)$. Therefore there exists a fixed point of $F^{(4)}$ in Ω_4, which is a point of period 4 of map F.