B5.6 Nonlinear Dynamics, Bifurcations and Chaos

Sheet 4 — HT 2025

Solutions to all problems in Sections A and C

Section A: Problems 1, 2 and 3

1. Consider the ODE system

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= \mu\, x_1\,(1+x_1) \,-\, x_2 \,-\, x_1\,(x_1^2+x_2^2-1)^2 \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_1 \,+\mu\, x_2\,(1+x_1) \,-\, x_2\,(x_1^2+x_2^2-1)^2\end{aligned}$$

where $\mu \in \mathbb{R}$ is a parameter. Consider the Poincaré section

$$\Sigma = \left\{ [x_1, 0] \in \mathbb{R}^2 \,|\, x_1 > 0 \right\}$$

and the Poincaré map $P_\mu : \Sigma \to \Sigma$, where $P_\mu(x_1) = P(x_1; \mu)$ is defined such that the positive semi-orbit of $[x_1, 0]$ intersects Σ for the first time at $[P_\mu(x_1), 0]$.

(a) Sketch the Poincaré map $P(x_1; \mu) = P_\mu(x_1)$ for

$$\mu = -\frac{1}{10}, \qquad \mu = 0 \qquad \text{and} \qquad \mu = \frac{1}{10}.$$

(b) Calculate the local expansion of the Poincaré map $P(x_1; \mu) = P_\mu(x_1)$ close to the bifurcation point $\mu = 0$.

Solution:

(a) We first transform the ODE system to polar coordinates, *i.e.* we introduce new variables $r(t)$ and $\theta(t)$, where

$$x_1(t) = r(t)\cos\theta(t) \qquad \text{and} \qquad x_2(t) = r(t)\sin\theta(t).$$

We obtain

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \mu\, r\,(1 + r\,\cos\theta) \,-\, r\,(r^2-1)^2 \tag{1}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1\,. \tag{2}$$

Consequently, the first return time of all trajectories is $t = 2\pi$. To get the Poincaré map, we solve ODEs (1)–(2) in the time interval $[0, 2\pi]$ with initial condition $[x_1, 0]$. The Poincaré maps are visualized for $\mu = -1/10$, $\mu = -1/20$, $\mu = 0$, $\mu = 1/20$ and $\mu = 1/10$ in the figure on the next page:

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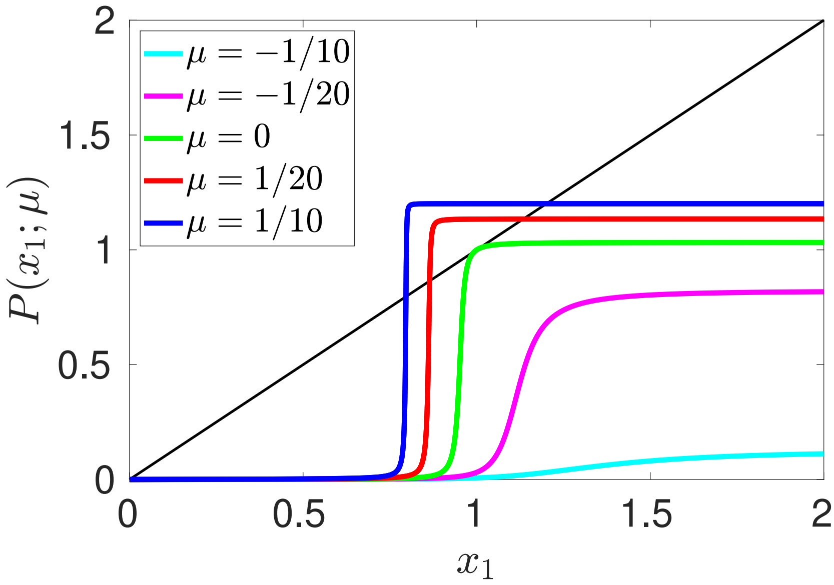


We observe that the Poincaré map has two fixed points for $\mu = 1/10$ corresponding to two limit cycles: one is stable and one is unstable. As μ decreases through the bifurcation point $\mu = 0$, the stable and unstable limit cycle collide at $\mu = 0$ (where we have only one half-stable cycle) and disappear for $\mu < 0$, i.e. there is a saddle-node bifurcation of cycles at $\mu = 0$.

(b) At the bifurcation point $\mu = 0$, the Poincaré map $P(x_1; 0) = P_0(x_1)$ has the fixed point at $x_1 = 1$. This fixed point corresponds to a periodic orbit. We expand $P(x_1; \mu)$ to get

$$P(1+\varepsilon;\mu) \;=\; P(1;0) \,+\, \varepsilon\, P_{x_1}(1;0) \,+\, \frac{\varepsilon^2}{2}\, P_{x_1 x_1}(1;0) \,+\, \mu\, P_\mu(1;0) \,+\, \dots \tag{3}$$

for $|\varepsilon| \ll 1$ and $|\mu| \ll 1$. We have $P(1; 0) = 1$, since $x_1 = 1$ is the fixed point for $\mu = 0$. To calculate the remaining terms in (3), we use ODEs (1)–(2). Substituting $r(t) = 1 + \xi(t)$ and $\theta = t$ in (1) and expanding, we get

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} \;=\; -4\,\xi^2 \,+\, \mu\,(1+\cos t) \,+\, \dots . \tag{4}$$

Using initial condition $\xi(0) = \varepsilon$, the Poincaré map is given by $P(1+\varepsilon; \mu) = \xi(2\pi)$. Consider that ξ depends not only on time t, but also on the initial condition ε and the parameter μ, *i.e.* we write

$$r(t) \;=\; 1 \,+\, \xi(t) \;\equiv\; 1 \,+\, \xi(t;\varepsilon,\mu).$$

Then, differentiating equation (4) with respect of ε and μ and evaluating at $\varepsilon = 0$ and $\mu = 0$, we get

$$\frac{\mathrm{d}\xi_\mu}{\mathrm{d}t} \;=\; (1+\cos t)\,, \qquad \frac{\mathrm{d}\xi_\varepsilon}{\mathrm{d}t} \;=\; 0 \qquad \text{and} \qquad \frac{\mathrm{d}\xi_{\varepsilon\varepsilon}}{\mathrm{d}t} \;=\; -8\,\xi_\varepsilon^2\,, \tag{5}$$

where we have used $\xi \equiv \xi(t;0,0) \equiv 0$ to simplify the right hand sides. The initial condition $\xi(0) = \varepsilon$ gives initial conditions $\xi_\mu(0) = 0$, $\xi_\varepsilon(0) = 1$ and $\xi_{\varepsilon\varepsilon}(0) = 0$. Solving (5) with these initial conditions, we obtain:

$$\begin{aligned}
P_\mu(1;0) &= \xi_\mu(2\pi;0,0) = \int_0^{2\pi} (1+\cos t)\,\mathrm{d}t = 2\pi \\
P_{x_1}(1;0) &= \xi_\varepsilon(2\pi;0,0) = 1 \\
P_{x_1x_1}(1;0) &= \xi_{\varepsilon\varepsilon}(2\pi;0,0) = -\int_0^{2\pi} 8\,\mathrm{d}t = -16\,\pi
\end{aligned}$$

Substituting into equation (3), we get the local expansion of the Poincaré map $P(x_1;\mu) = P_\mu(x_1)$ close to the bifurcation point $\mu = 0$ as

$$P(1+\varepsilon;\mu) = 1 + \varepsilon - 8\pi\,\varepsilon^2 + 2\pi\,\mu + \ldots$$

2. Find functions $f_1(x_1,x_2;\mu)$ and $f_2(x_1,x_2;\mu)$ such that

   (a) $f_1(x_1,x_2;\mu)$ and $f_2(x_1,x_2;\mu)$ are quadratic polynomials in variables x_1 and x_2 for any value of the parameter $\mu \in \mathbb{R}$; and

   (b) the planar ODE system

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= f_1(x_1,x_2;\mu) \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= f_2(x_1,x_2;\mu)
\end{aligned}$$

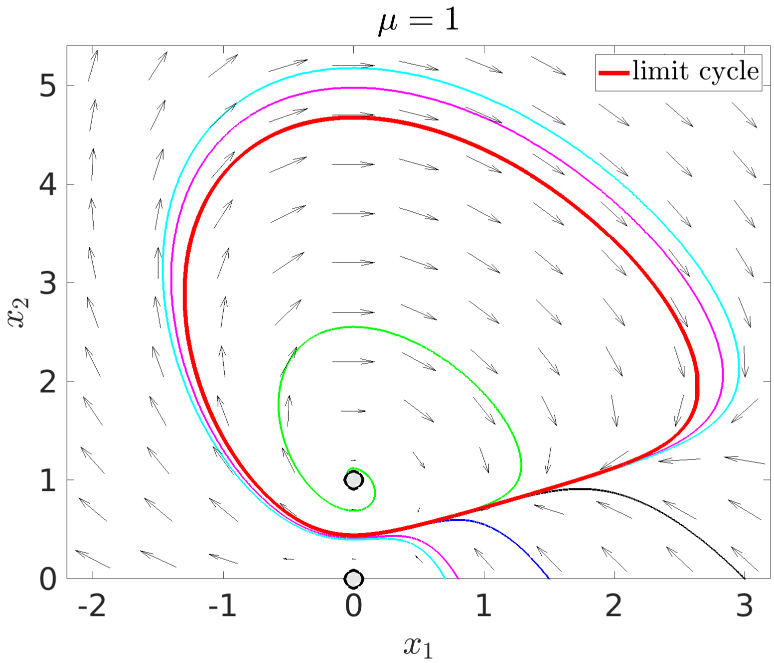
   is undergoing an infinite period (SNIC, SNIPER) bifurcation as the parameter μ passes through the bifurcation point $\mu = \mu_c$, which you should determine.

Solution: The infinite period (SNIC, SNIPER) bifurcation is one of the possible routes to destruction or creation of limit cycles. To find an ODE system satisfying (a) and (b), we start with a quadratic system which has a limit cycle. For example, consider the following ODE system

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -x_2 - x_1^2 + x_1\,x_2 + x_2^2 \tag{6}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1 + x_1^2 - 3\,x_1\,x_2 \tag{7}$$

which has two (unstable) critical points $[0,0]$ and $[0,1]$ and a stable limit cycle. We plot the x_1-x_2 phase plane in the following figure, where the limit cycle is visualized as the red (thick) line together with additional illustrative trajectories, visualized as thin lines and converging to the limit cycle:



Next, we introduce parameter $\mu \in \mathbb{R}$ to our system by defining

$$\begin{aligned} f_1(x_1, x_2; \mu) &= -\mu x_2 - x_1^2 + x_1 x_2 + x_2^2 \\ f_2(x_1, x_2; \mu) &= x_1 + x_1^2 - 3 x_1 x_2 \end{aligned}$$

where $\mu = 1$ corresponds to the ODE system (6)–(7). Then $f_1(x_1, x_2; \mu)$ and $f_2(x_1, x_2; \mu)$ are quadratic polynomials in variables x_1 and x_2 for any value of the parameter $\mu \in \mathbb{R}$. The steady states of the ODE system

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = f_1(x_1, x_2; \mu) = -\mu x_2 - x_1^2 + x_1 x_2 + x_2^2 \tag{8}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = f_2(x_1, x_2; \mu) = x_1 + x_1^2 - 3 x_1 x_2 \tag{9}$$

can be found by solving

$$f_1(x_1, x_2; \mu) = 0 \qquad \text{and} \qquad f_2(x_1, x_2; \mu) = 0.$$

Solving $f_2(x_1, x_2; \mu) = 0$, we obtain that

$$x_1 = 0 \qquad \text{or} \qquad x_2 = \frac{x_1 + 1}{3}.$$

Substituting $x_1 = 0$ into $f_1(x_1, x_2; \mu) = 0$, we get two critical points which exist for all values of parameter $\mu \in \mathbb{R}$:

$$\mathbf{x}_{c1} = [0, 0] \qquad \text{and} \qquad \mathbf{x}_{c2} = [0, \mu]\,.$$

Substituting $x_2 = (x_1+1)/3$ into $f_1(x_1, x_2; \mu) = 0$, we get the following quadratic equation:

$$5\,x_1^2 + (3\,\mu - 5)\,x_1 + 3\,\mu - 1 = 0\,.$$

This equation has real solutions if

$$\mu \le 5 - 2\sqrt{5} \qquad \text{or} \qquad \mu \ge 5 + 2\sqrt{5}$$

giving us additional two critical points

$$\mathbf{x}_{c3} = \left[\frac{5 - 3\,\mu - 3\sqrt{\mu^2 - 10\mu + 5}}{10}, \frac{5 - \mu - \sqrt{\mu^2 - 10\mu + 5}}{10} \right]$$

and

$$\mathbf{x}_{c4} = \left[\frac{5 - 3\,\mu + 3\sqrt{\mu^2 - 10\mu + 5}}{10}, \frac{5 - \mu + \sqrt{\mu^2 - 10\mu + 5}}{10} \right].$$

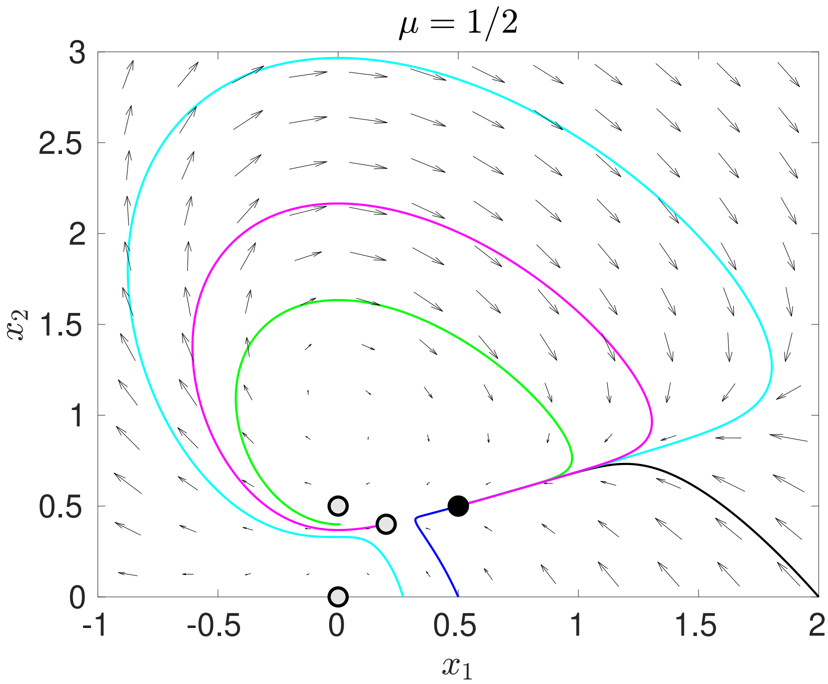
We choose

$$\mu_c = 5 - 2\sqrt{5} = 0.52786\ldots$$

Let $\mu = 1/2$, then $\mu < \mu_c$ and we have four critical points

$$\mathbf{x}_{c1} = [0,0]\,, \qquad \mathbf{x}_{c2} = [0,1/2]\,, \qquad \mathbf{x}_{c3} = [1/5,2/5] \qquad \text{and} \qquad \mathbf{x}_{c4} = [1/2,1/2]\,,$$

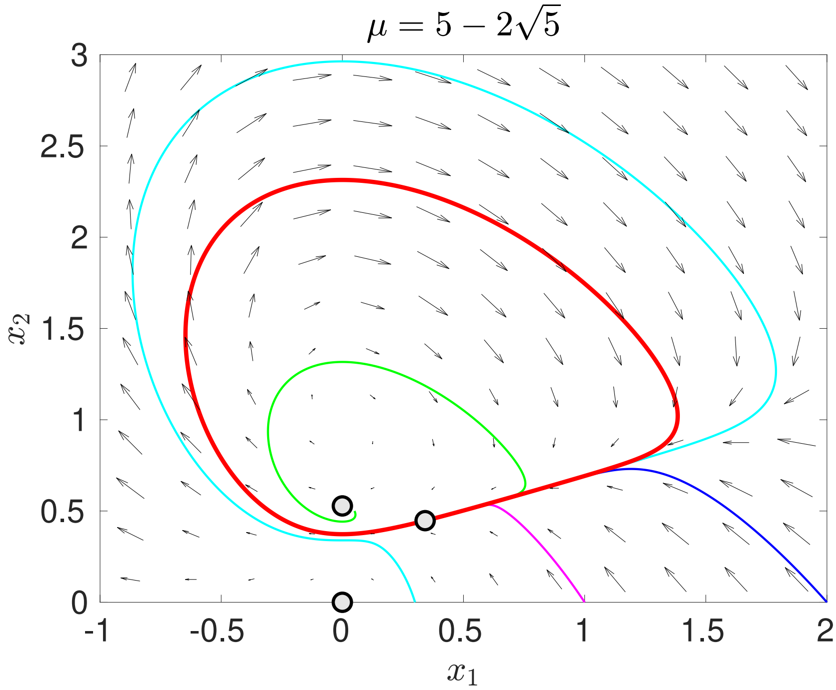
where $\mathbf{x}_{c3}$ is a saddle and $\mathbf{x}_{c4}$ is a stable node. The phase plane for $\mu = 1/2$ is shown below, where all illustrative trajectories converge to the stable node $\mathbf{x}_{c4}$:



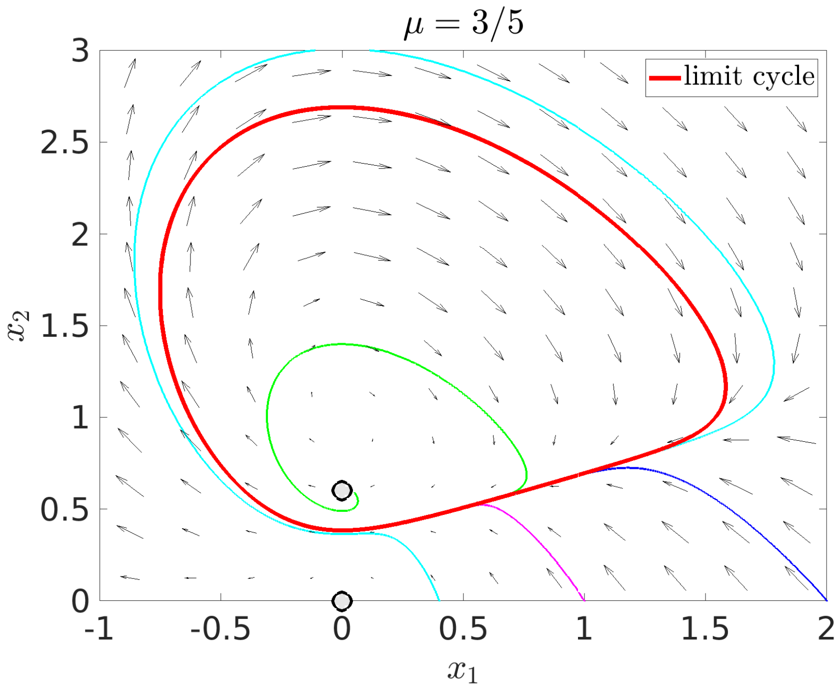
Increasing μ towards μ_c, the saddle $\mathbf{x}_{c3}$ and the stable node $\mathbf{x}_{c4}$ approach each other. At the bifurcation point $\mu = \mu_c$, we have

$$\mathbf{x}_{c3} = \mathbf{x}_{c4} = [3\sqrt{5}/5 - 1, \sqrt{5}/5] = [0.34164\ldots, 0.44721\ldots].$$

We have a saddle-node bifurcation at $\mu = \mu_c$ with the phase plane shown below:



Moreover, a limit cycle exists for $\mu > \mu_c$, which has an infinite period at $\mu = \mu_c$, corresponding to the homoclinic loop in the figure above (red thick line). If $\mu > \mu_c$, the saddle $\mathbf{x}_{c3}$ and the stable node $\mathbf{x}_{c4}$ no longer exist and there is a limit cycle visualized as the red (thick) line below for $\mu = 3/5$:



The limit cycle also exists for larger values of μ satisfying $\mu > \mu_c$, as we have already shown for the parameter value $\mu = 1$ in the ODE system (6)–(7).

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Consequently, the ODE system (8)–(9) provides an example of a planar quadratic system which undergoes an infinite period (SNIC, SNIPER) bifurcation as the parameter μ passes through the bifurcation point

$$\mu \;=\; \mu_c \;=\; 5 - 2\sqrt{5} \;=\; 0.52786\ldots.$$

3. Let $x_0 \in [0,1)$ and $F : [0,1) \to [0,1)$.
Define sequence $x_k \in [0,1)$, $k = 0,1,2,\ldots$, iteratively by

$$x_{k+1} = F(x_k)\,,$$

where

$$F(x) = \begin{cases} 2x & \text{for } x \in [0,1/2)\,; \\ 2x-1 & \text{for } x \in [1/2,1)\,. \end{cases}$$

(a) Consider $x_0 \in [0,1)$ which is not a dyadic rational, *i.e.* it does not have a terminating binary expansion. Show that if x_0 has the binary expansion

$$x_0 \;=\; 0.a_1a_2a_3\ldots \;=\; \sum_{i=1}^{\infty} \frac{a_i}{2^i}$$

with $a_i \in \{0,\,1\}$, then

$$x_k = 0.a_{k+1}a_{k+2}a_{k+3}\ldots.$$

(b) Prove the existence of a countable infinity of periodic orbits.

(c) Prove the existence of an uncountable infinity of nonperiodic orbits.

(d) Show that there is a dense orbit.

(e) Show that this dynamical system has sensitive dependence on initial conditions.

Solution:

(a) If $x_0 \in [0,1/2)$, then $a_1 = 0$ and

$$F(x_0) \;=\; 2x_0 \;=\; \sum_{i=1}^{\infty} \frac{a_{i+1}}{2^i} \;=\; 0.a_2a_3a_4\ldots$$

If $x_0 \in (1/2,1)$, then $a_1 = 1$ and

$$F(x_0) \;=\; 2x_0 - 1 \;=\; \sum_{i=1}^{\infty} \frac{a_{i+1}}{2^i} \;=\; 0.a_2a_3a_4\ldots$$

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If $x_0 = 1/2$, then x_0 is a dyadic rational, which is not considered in part (a). In fact, if $x_0 = 1/2$, then there are two ways to represent x_0: either as $x_0 = 0.100000\ldots$ or as $x_0 = 0.011111\ldots$. Avoiding this ambiguity, we conclude that

$$x_1 \;=\; F(x_0) \;=\; 0.a_2a_3a_4\ldots \qquad \text{for} \quad x_0 \neq 1/2,$$

and, by induction,

$$x_k = 0.a_{k+1}a_{k+2}a_{k+3}\ldots$$

for any $x_0 \in [0,1)$ which is not a dyadic rational.

(b) Using (a), we observe that any periodic string of 0's and 1's gives rise to a periodic orbit. For example, the orbit of $0.01\,01\,01\,01\,01\,01\,01\,\ldots$ is a cycle of period 2 and the orbit of $0.001\,001\,001\,001\,001\,001\,001\,\ldots$ is a cycle of period 3. Thus, we obtain a countable infinity of periodic orbits by considering orbits of points x_0 with binary expansions obtained by repeating blocks $0\ldots01$, *i.e* by considering the orbits of points x_0 in the sequence:

$$\begin{aligned}
x_0 &= 0.01\;01\;01\;01\;01\;01\;01\;01\;01\;01\;01\;01\;01\;01\;\ldots\\
x_0 &= 0.001\;001\;001\;001\;001\;001\;001\;001\;001\;001\;\ldots\\
x_0 &= 0.0001\;0001\;0001\;0001\;0001\;0001\;0001\;0001\;\ldots\\
x_0 &= 0.00001\;00001\;00001\;00001\;00001\;00001\;00001\;\ldots\\
\vdots\;\; & \quad \vdots\\
x_0 &= 0.0\ldots01\;0\ldots01\;0\ldots01\;0\ldots01\;0\ldots01\;0\ldots01\;0\ldots01\;\ldots
\end{aligned}$$

(c) Suppose (for a contradiction) that there were only countably many nonperiodic orbits. Then since a countable number of countable sets is countable, the set of all numbers the nonperiodic orbits visit would be countable. In particular, there must exist an irrational number which none of the nonperiodic orbits visit. However, this irrational number generates a nonperiodic orbit which is not in our list of nonperiodic orbits, so we have a contradiction. This proves that there are an uncountable infinity of nonperiodic orbits.

(d) We construct a point $x_0 \in (0,1)$ which has a dense orbit under F by defining its binary expansion in the following way: starting with all possible '1-blocks' (*i.e.* 0 followed by 1), and continuing with all possible '2-blocks' (*i.e.* 00 01 10 11), all possible '3-blocks' in the lexicographical order, all possible '4-blocks', and so on, to get

$$x_0 = 0.0\,1\,00\,01\,10\,11\,000\,001\,010\,011\,100\,101\,110\,111\,0000\,0001\,0010\,0011\,0100\,\ldots$$

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To show that the orbit of x_0 is dense in $[0,1)$, consider an arbitrary point $b \in [0,1)$ which has a binary expansion

$$b \;=\; 0.b_1 b_2 b_3 \ldots$$

Let $\varepsilon > 0$ and choose $j \in \mathbb{N}$ large enough so that $\varepsilon > 2^{-j}$. Since the string $b_1 b_2 \ldots b_j$ appears in the binary expansion of x_0, there exists $\ell \in \mathbb{N}$ such that

$$F^{(\ell)}(x_0) = 0.b_1 b_2 \ldots b_j c_{j+1} c_{j+2} \ldots$$

for some $c_{j+1}, c_{j+2}, \dots \in \{0,1\}$. Then

$$\left| F^{(\ell)}(x_0) - b \right| \le \sum_{i=j+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^j} < \varepsilon\,.$$

Therefore, any point $b \in [0,1)$ is arbitrarily close to the orbit of x_0 under F, i.e. the orbit of x_0 is dense in [0,1).

(e) Consider $\delta < 1/2$ in our definition of sensitive dependence on initial conditions. Consider $x_0 = 0.a_1 a_2 a_3 \ldots$ which is not a dyadic rational. Let $\varepsilon > 0$ be arbitrary and choose $j \in \mathbb{N}$ sufficiently large such that

$$\frac{1}{2^{j+1}} < \varepsilon\,.$$

Define

$$b = \begin{cases} x_0 + \dfrac{1}{2^{j+1}} & \text{if} \quad a_{j+1} = 0\,, \\ x_0 - \dfrac{1}{2^{j+1}} & \text{if} \quad a_{j+1} = 1\,. \end{cases}$$

Then

$$\left| x_0 - b \right| = \frac{1}{2^{j+1}} < \varepsilon\,,$$

i.e. we have found point b which is close to x_0 (with their distance less than ε), but images of the j-th iteration of the map F applied to x_0 and b are the distance of $1/2$ apart. We have

$$\left| F^{(j)}(x_0) - F^{(j)}(b) \right| = \frac{1}{2} > \delta\,.$$

This shows the sensitive dependence on initial conditions.

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Section C: Problem 7

4. Find functions $f_1(x_1, x_2; \mu)$ and $f_2(x_1, x_2; \mu)$ such that

   (a) $f_1(x_1, x_2; \mu)$ and $f_2(x_1, x_2; \mu)$ are quadratic polynomials in variables x_1 and x_2 for any value of the parameter $\mu \in \mathbb{R}$; and

   (b) the planar ODE system

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= f_1(x_1, x_2; \mu) \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= f_2(x_1, x_2; \mu) \end{aligned}$$

   has two critical points $\mathbf{x}_{c1}$ and $\mathbf{x}_{c2}$ for all values of parameter $\mu \in \mathbb{R}$ such that

   (i) the ODE system undergoes a Hopf bifurcation at each critical point, $\mathbf{x}_{c1}$ and $\mathbf{x}_{c2}$, as the parameter μ passes through the same bifurcation point $\mu = \mu_c$, which you should determine;

   (ii) there exist two limit cycles in the phase plane for some parameter values.

Solution: Consider the ODE system

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{1}{2}\left(x_2^2 - 1\right) \tag{10}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu - x_1 x_2 + (1-\mu)\, x_2^2 \tag{11}$$

where $\mu \in \mathbb{R}$ is a parameter. Then the critical points are obtained by solving:

$$\begin{aligned} 0 &= \frac{x_2^2 - 1}{2} \\ 0 &= \mu - x_1 x_2 + (1-\mu)\, x_2^2 \end{aligned}$$

The first equation implies $x_2 = \pm 1$. Substituting into the second equation, we get two critical points

$$\mathbf{x}_{c1} = [1, 1] \qquad \text{and} \qquad \mathbf{x}_{c2} = [-1, -1]$$

for all values of parameter $\mu \in \mathbb{R}$. The Jacobian matrix is

$$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} 0 & x_2 \\ -x_2 & 2(1-\mu)x_2 - x_1 \end{pmatrix},$$

giving

$$D\mathbf{f}(\mathbf{x}_{c1}) = -D\mathbf{f}(\mathbf{x}_{c2}) = \begin{pmatrix} 0 & 1 \\ -1 & 1-2\mu \end{pmatrix}.$$

The eigenvalues of $D\mathbf{f}(\mathbf{x}_{c1})$ are

$$\lambda_\pm(\mu) = \frac{1}{2} - \mu \pm \frac{1}{2}\sqrt{4\mu(\mu-1) - 3}\,.$$

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Choosing

$$\mu_c = \frac{1}{2},$$

we have

$$\lambda_\pm(\mu_c) = \pm i,$$

while the eigenvalues of $D\mathbf{f}(\mathbf{x}_{c2})$ are $-\lambda_\pm(\mu)$. To classify the bifurcations, we transform the ODE system (10)–(11) using variables

$$\begin{aligned}
\overline{x}_1 &= x_1 - 1\,, \qquad \overline{x}_2 = x_2 - 1\,, \qquad \overline{\mu} = \mu - \frac{1}{2}\,, \qquad\qquad \text{at point } \mathbf{x}_{c1}\,,\\
\overline{x}_1 &= x_1 + 1\,, \qquad \overline{x}_2 = x_2 + 1\,, \qquad \overline{\mu} = \mu - \frac{1}{2}\,, \qquad\qquad \text{at point } \mathbf{x}_{c2}\,.
\end{aligned}$$

We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = M(\overline{\mu})\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\overline{x}_2^2\\ (1-2\overline{\mu})\,\overline{x}_2^2 - 2\,\overline{x}_1\,\overline{x}_2\end{pmatrix}, \qquad \text{at point } \mathbf{x}_{c1}\,, \tag{12}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = -\,M(\overline{\mu})\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\overline{x}_2^2\\ (1-2\overline{\mu})\,\overline{x}_2^2 - 2\,\overline{x}_1\,\overline{x}_2\end{pmatrix}, \qquad \text{at point } \mathbf{x}_{c2}\,, \tag{13}$$

where matrix $M(\overline{\mu})$ is

$$M(\overline{\mu}) = \begin{pmatrix}0 & 1\\ -1 & -2\overline{\mu}\end{pmatrix}.$$

Considering the linear part of equation (12), matrix $M(\overline{\mu})$ has its eigenvalues (close to the bifurcation point $\overline{\mu} = 0$) given by

$$\lambda_\pm(\overline{\mu}) = \alpha(\overline{\mu}) \,\pm\, i\,\omega(\overline{\mu})$$

where

$$\alpha(\overline{\mu}) = -\overline{\mu} \qquad \text{and} \qquad \omega(\overline{\mu}) = -\sqrt{1-\overline{\mu}^2}$$

which implies

$$\alpha(0) = 0\,, \qquad \omega(0) = -1\,, \qquad \alpha'(0) = -1\,, \qquad \text{and} \qquad \omega'(0) = 0\,.$$

Consequently, at the bifurcation point $\overline{\mu} = 0$, equation (12) can be rewritten in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = \begin{pmatrix}0 & -\omega(0)\\ \omega(0) & 0\end{pmatrix}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + \begin{pmatrix}h_1(\overline{x}_1,\overline{x}_2)\\ h_2(\overline{x}_1,\overline{x}_2)\end{pmatrix}, \tag{14}$$

where

$$h_1(\overline{x}_1,\overline{x}_2) = \frac{\overline{x}_2^2}{2} \quad \text{and} \quad h_2(\overline{x}_1,\overline{x}_2) = \frac{\overline{x}_2^2}{2} - \overline{x}_1\,\overline{x}_2\,. \tag{15}$$

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Evaluating the partial derivatives at the origin $\mathbf{0} = [0, 0]$, we get

$$\begin{aligned} a(0) &= \frac{1}{16}\left(\frac{\partial^3 h_1}{\partial \overline{x}_1^3} + \frac{\partial^3 h_1}{\partial \overline{x}_1 \partial \overline{x}_2^2} + \frac{\partial^3 h_2}{\partial \overline{x}_1^2 \partial \overline{x}_2} + \frac{\partial^3 h_2}{\partial \overline{x}_2^3}\right) + \frac{1}{16\,\omega(0)}\left[\frac{\partial^2 h_1}{\partial \overline{x}_1 \partial \overline{x}_2}\left(\frac{\partial^2 h_1}{\partial \overline{x}_1^2} + \frac{\partial^2 h_1}{\partial \overline{x}_2^2}\right)\right. \\ &\qquad \left. - \frac{\partial^2 h_2}{\partial \overline{x}_1 \partial \overline{x}_2}\left(\frac{\partial^2 h_2}{\partial \overline{x}_1^2} + \frac{\partial^2 h_2}{\partial \overline{x}_2^2}\right) - \frac{\partial^2 h_1}{\partial \overline{x}_1^2}\frac{\partial^2 h_2}{\partial \overline{x}_1^2} + \frac{\partial^2 h_1}{\partial \overline{x}_2^2}\frac{\partial^2 h_2}{\partial \overline{x}_2^2}\right] \\ &= \frac{1}{16}(0 + 0 + 0 + 0) + \frac{1}{16\,\omega(0)}[0(0+1) - (-1)(0+1) - 0(0) + 1(1)] \\ &= \frac{1}{8\omega(0)} = -\frac{1}{8}\,. \end{aligned}$$

Since $a(0) < 0$, we have a supercritical Hopf bifurcation at $\overline{\mu} = 0$, *i.e.* the original system has a supercritical Hopf bifurcation at $\mu = \mu_c = 1/2$ at the critical point $\mathbf{x}_{c1}$. The normal form is

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}t} &= -\,\overline{\mu}\,r - \frac{1}{8}r^3 + \dots \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= -\,1 + \dots \end{aligned}$$

Considering the bifurcation at the point $\mathbf{x}_{c2}$, we observe that the linear part of equation (13) is equal to $-M(\mu)$ and the nonlinear part is the same as in equation (12). Consequently, at the bifurcation point $\overline{\mu} = 0$, equation (13) can be rewritten in the form (14) with $\omega(0) = 1$ and the nonlinear part is given by (15). In particular, we get

$$\begin{aligned} a(0) &= \frac{1}{16}\left(\frac{\partial^3 h_1}{\partial \overline{x}_1^3} + \frac{\partial^3 h_1}{\partial \overline{x}_1 \partial \overline{x}_2^2} + \frac{\partial^3 h_2}{\partial \overline{x}_1^2 \partial \overline{x}_2} + \frac{\partial^3 h_2}{\partial \overline{x}_2^3}\right) + \frac{1}{16\,\omega(0)}\left[\frac{\partial^2 h_1}{\partial \overline{x}_1 \partial \overline{x}_2}\left(\frac{\partial^2 h_1}{\partial \overline{x}_1^2} + \frac{\partial^2 h_1}{\partial \overline{x}_2^2}\right)\right. \\ &\qquad \left. - \frac{\partial^2 h_2}{\partial \overline{x}_1 \partial \overline{x}_2}\left(\frac{\partial^2 h_2}{\partial \overline{x}_1^2} + \frac{\partial^2 h_2}{\partial \overline{x}_2^2}\right) - \frac{\partial^2 h_1}{\partial \overline{x}_1^2}\frac{\partial^2 h_2}{\partial \overline{x}_1^2} + \frac{\partial^2 h_1}{\partial \overline{x}_2^2}\frac{\partial^2 h_2}{\partial \overline{x}_2^2}\right] \\ &= \frac{1}{16}(0 + 0 + 0 + 0) + \frac{1}{16\,\omega(0)}[0(0+1) - (-1)(0+1) - 0(0) + 1(1)] \\ &= \frac{1}{8\omega(0)} = \frac{1}{8}\,. \end{aligned}$$

Since $a(0) > 0$, we have a subcritical Hopf bifurcation at $\overline{\mu} = 0$, *i.e.* the original system has a subcritical Hopf bifurcation at $\mu = \mu_c = 1/2$ at the critical point $\mathbf{x}_{c2}$. The normal form is

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}t} &= \overline{\mu}\,r + \frac{1}{8}r^3 + \dots \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= 1 + \dots \end{aligned}$$

Consequently, the ODE system (10)–(11) undergoes a supercritical Hopf bifurcation at $\mathbf{x}_{c1}$ and a subcritical Hopf bifurcation at $\mathbf{x}_{c2}$ at the bifurcation point $\mu = \mu_c = 1/2$. If $\mu \geq \mu_c$, then the critical point at $\mathbf{x}_{c1}$ is stable, the critical point at $\mathbf{x}_{c2}$ is unstable and there are no limit cycles. First, we visualize the phase plane for the parameter value

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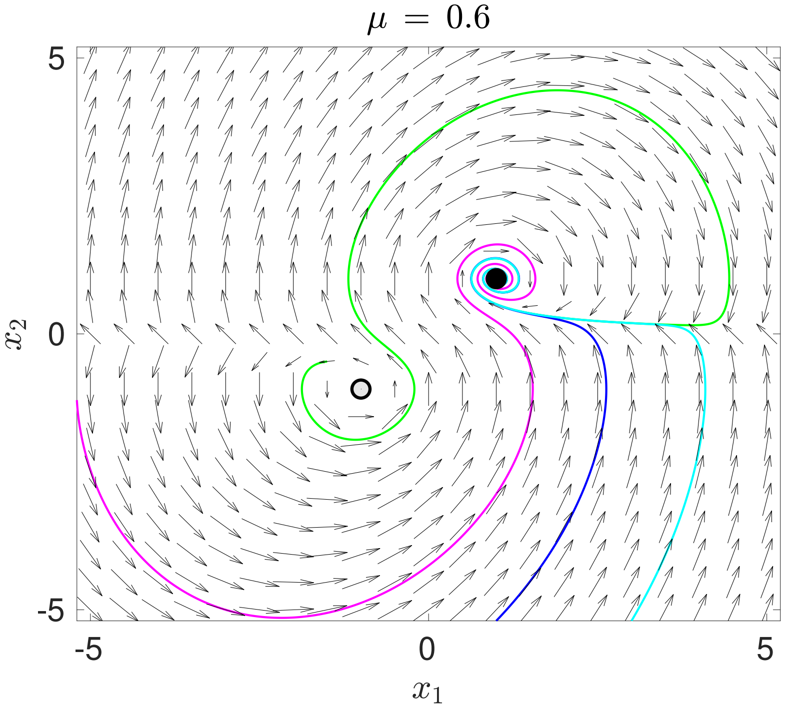
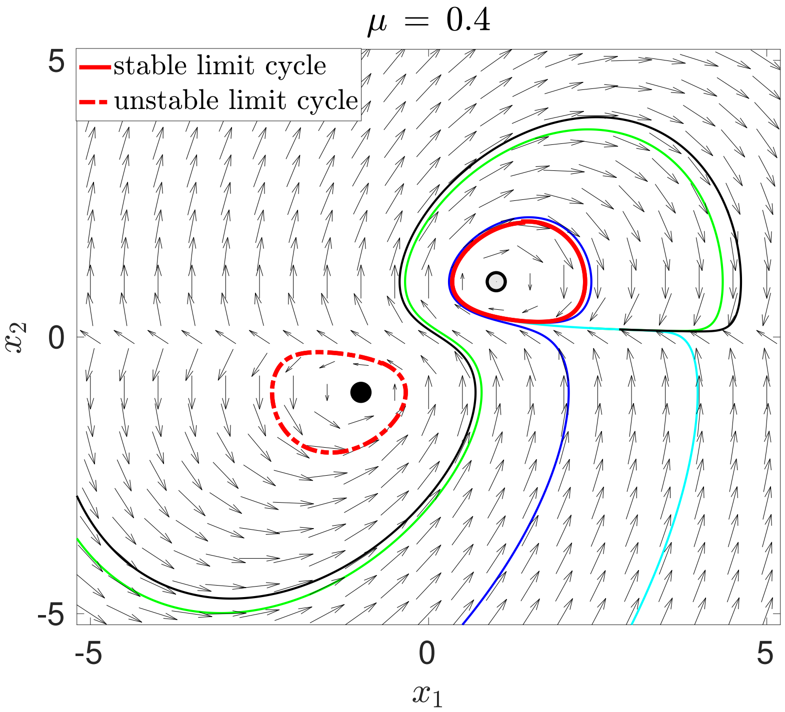
$\mu = 0.6 > \mu_c$. Four illustrative trajectories are plotted using the blue, green, magenta and cyan lines. They all converge to the critical point at $\mathbf{x}_{c1} = [1, 1]$:



Two limit cycles are created by two Hopf bifurcations for $\mu < 1/2$ as illustrated in the next phase plane for $\mu = 0.4 < \mu_c$. The stable limit cycle is visualized as the red solid line and the unstable limit cycle is visualized as the red dashed line. Four illustrative trajectories are plotted using the blue, green, black and cyan lines.



The unstable critical point $\mathbf{x}_{c1} = [1, 1]$ is in the interior of the stable limit cycle, while the stable critical point $\mathbf{x}_{c2} = [-1, -1]$ is in the interior of the unstable limit cycle. The two limit cycles co-exist in the phase plane for $0 < \mu < 1/2$. As μ is decreasing, the size of limit cycles increases. We illustrate this by plotting the phase plane below for the parameter value $\mu = 1/4 < \mu_c$: