Part B: Lie Algebras

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1 Course Details

1.1 Outline Lie Algebras are mathematical objects which, besides being of interest in their own right, elucidate problems in several areas in mathematics. The classification of the finite-dimensional complex Lie algebras is a beautiful piece of applied linear algebra. The aims of this course are to introduce Lie algebras, develop some of the techniques for studying them, and outline the techniques and structures that go into this classification theorem, which shows that semisimple Lie algebras are encoded in finite sets of highly symmetric vectors in a Euclidean vector space known as root systems, which in turn are classified by a kind of graph known as a Dynkin diagram.

1.2 Learning Outcomes By the end of the course, students will be able to identify the basic classes of Lie algebras - nilpotent, solvable and semisimple, give examples of each, and appreciate the role they play in understanding the structure of Lie algebras. They should be able to use basic notions such as ideals and representations to analyse the structure of Lie algebras, and employ the Cartan criteria. They should also be able to analyse concrete examples of semisimple Lie algebras and identify the associated Dynkin diagram.

Although not all of the key theorems are proved in this course, should they need to, a student who has internalised the techniques used in these lectures should not have too much difficulty filling in these gaps using any of the standard textbooks on the subject.

1.3 Course Synopsis

- Definition of Lie algebras, small-dimensional examples, some classical groups and their Lie algebras (treated informally). Ideals, subalgebras, homomorphisms, isomorphism theorems.
- Basics of representation theory: $\mathfrak{g}$-modules (or equivalently $\mathfrak{g}$-representations). Irreducible and indecomposable representations, semisimplicity. Composition series and the Jordan-Hölder theorem for modules. Operations on representations: subrepresentations, quotients, Homs and tensor products.
- Composition series for Lie algebras. Short exact sequences and the notion of an extension. Split sequences and semi-direct products. Definition of solvable and nilpotent Lie algebras. A representation in which every element of the Lie algebra acts nilpotently has the trivial representation as its only composition factor. Engel's theorem.
- Representations of solvable Lie algebras over an algebraically closed field of characteristic zero including Lie's theorem. Decomposition of representations of nilpotent Lie algebras into generalised weight spaces.
- Cartan subalgebras and the Cartan decomposition. Trace forms and Cartan's criterion for solvability. The solvable radical and Cartan's criterion for semisimplicity. Semisimple and simple Lie algebras. The Jordan decomposition. The representations of $\mathfrak{sl}_2$ (*to be done through problem set questions – only the classification of irreducibles is needed for use elsewhere in the course*).
- The Cartan decomposition of a semisimple Lie algebra and the structure of the root system using the representation theory of $\mathfrak{sl}_2$. The abstract root system attached to a semisimple Lie algebra and the reduction of the classification of abstract root systems to the classification of Cartan matrices/Dynkin diagrams. Statement of the classification theorem and informal discussion of the proof of the classification of semisimple Lie algebras.

1.4 Reading: There are a number of good reference texts for the material of this course, and the Moodle site for the Part C course includes a reading list with some of these. Of those texts, the book by Erdmann & Wildon is one of the most approachable, though its approach differs from the one adopted in this course in a number of significant ways. The book by Roger Carter "*Lie Algebras of Finite and Affine type*" is closer in its approach to that of this course, and is carefully, though perhaps less engagingly, written. Complete lecture notes will be made available on the course Moodle site at the start of the term.

1.5 Recommended reading

(i) *Introduction to Lie algebras*, K. Erdmann, M. Wildon, Springer Undergraduate Mathematics Series. (Available online through the Bodleain.)

(ii) *Introduction to Lie Groups and Lie algebras*, A. Kirillov, Jr. Cambridge Studies in Advanced Mathematics, C.U.P.

(iii) *Lie algebras: Theory and algorithms*, Willem A. de Graff, North-Holland Mathematical Library.

(iv) *Lie algebras of finite and affine type*, R. Carter, Cambridge Studies in Advanced Mathematics, C.U.P.

(v) *Lie Groups, Lie Algebras, and Representations*, Brian C. Hall, Graduate Texts in Mathematics, Springer.

(vi) *Representation theory: A First Course*, W. Fulton, J. Harris, Graduate Texts in Mathematics, Springer.

References (i) (ii) and (v) are written at the level of the lecture course, while (iii) and (iv) are perhaps pitched at a slightly more advanced level. The book (vi) by Fulton and Harris discusses both the representation theory of finite groups and the structure of Lie groups and Lie algebras, along with their representation theory. It has many detailed examples.