B2.2 Commutative Algebra

Sheet 1 — HT25

Sections 1-5

Section A

1. Let R be a ring. Show that the Jacobson radical of R coincides with the set

$$\{x \in R \mid 1 - xy \text{ is a unit for all } y \in R\}.$$

Solution: Suppose that x lies in the Jacobson radical of R. Suppose for contradiction that $1 - xy$ is not a unit for some $y \in R$. Let $\mathfrak{m}$ be a maximal ideal containing $1 - xy$. We know that $xy \in \mathfrak{m}$ since $x \in \mathfrak{m}$ and thus we conclude that $1 \in \mathfrak{m}$, a contradiction.

Suppose now that $x \in R$ and that there is a maximal ideal $\mathfrak{m}$ not containing x. Then $x + \mathfrak{m}$ is non-trivial in the field $R/\mathfrak{m}$ and hence it is a unit; thus, there is a $y \in R$ such that $xy + \mathfrak{m} = 1 + \mathfrak{m}$. In other words, $1 - xy \in \mathfrak{m}$ and so $1 - xy$ is not a unit.

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Section B

2. Let R be a ring.

   (a) Show that if $P(x) = a_0 + a_1x + \cdots + a_kx^k \in R[x]$ is a unit of $R[x]$ then a_0 is a unit of R and a_i is nilpotent for all $i \geqslant 1$.

   (b) Show that the Jacobson radical and the nilradical of $R[x]$ coincide.

   Solution:

   (a) Let $Q(x) = b_0 + \cdots + b_tx^t \in R[x]$ be an inverse of $P(x)$. Then $P(0)Q(0) = a_0b_0 = 1$, forcing a_0 and b_0 to be units.

   Let $\mathfrak{p}$ be a prime ideal. Let $j \geqslant 0$ be the largest integer such that $a_j \notin \mathfrak{p}$ and let $l \geqslant 0$ be the largest integer such that $b_l \notin \mathfrak{p}$. If $j > 0$ we have $a_jb_l \in \mathfrak{p}$ (since $P(x)Q(x) = 1$), which is not possible because $R/\mathfrak{p}$ is a domain. Hence $j = 0$ and in particular $a_i \in \mathfrak{p}$ for all $i > 0$. Since $\mathfrak{p}$ was arbitrary, we see that a_i lies in the nilradical of R for all $i > 0$.

   (b) We only have to show that every element of the Jacobson radical of $R[x]$ is nilpotent. So let $P(x) \in a_0 + a_1x + \cdots + a_kx^k \in R[x]$ be an element of the Jacobson radical. By Question 1, we know that for any $T(x) \in R[x]$, the element $1 - P(x)T(x)$ is a unit. In particular,

   $$1 + xP(x) = 1 + a_0x + a_1x^2 + \cdots + a_kx^{k+1}$$

   is a unit. By (a), the element a_i is thus nilpotent for all $i \geqslant 0$. In particular $a_0 + a_1x + \cdots + a_kx^k$ is nilpotent (since the radical of a ring is an ideal).

3. Let R be a ring and let $N \subseteq R$ be its nilradical. Show that the following are equivalent:

   (a) R has exactly one prime ideal.

   (b) Every element of R is either a unit or is nilpotent.

   (c) R/N is a field.

   Solution: (a)$\Rightarrow$(b): Let $\mathfrak{p}$ be the unique prime ideal. Suppose that $r \in R$ is not a unit. Then r is contained in a maximal ideal, which must coincide with $\mathfrak{p}$. Since $\mathfrak{p}$ is the only prime ideal, the ideal $\mathfrak{p}$ is the nilradical N of R and hence r is nilpotent.

   $\neg$(c)$\Rightarrow$ $\neg$(b): Suppose that R/N is not a field. Then either R/N is the zero ring or there is an element $x \in (R/N) \setminus \{0\}$, which is not a unit. If R/N is the zero ring, then every element of R is nilpotent (and in fact R is the zero ring). If there is an element

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$x \in (R/N) \smallsetminus \{0\}$, let $x_1 \in R$ be a preimage of x. Then x_1 is not a unit and is not nilpotent.

$\neg$(a)$\Rightarrow \neg$(c): If R has more than one prime ideal then R/N has a non-zero prime ideal (since every prime ideal contains N), and hence is not a field.

4. Let R be a ring and let $I \subseteq R$ be an ideal. Let $S = \{1 + r \mid r \in I\}$.
   (a) Show that S is a multiplicative set.
   (b) Show that the ideal generated by the image of I in R_S is contained in the Jacobson radical of R_S.
   (c) Prove the following generalisation of Nakayama's lemma:

   Lemma. *Let M be a finitely generated R-module and suppose that $IM = M$. Then there exists $r \in R$, such that $r - 1 \in I$ and $rM = 0$.*

Solution:

(a) This is clear.

(b) The ideal I_S generated by I in R_S consists of the elements a/b such that $a \in I$ and $b \in S$. By Q1, we thus only have to show that if a/b is such that $a \in I$ and $b \in S$, then $1 - (a/b)(c/d)$ is a unit for all $c \in R$ and $d \in S$. Now $1/b$ and $1/d$ are units of R_S, hence we only have to show that $bd - ac$ is a unit for a, b, c, d as in the previous sentence. Now $bd = (1 + b_1)(1 + d_1) = 1 + b_1 + d_1 + b_1 d_1$ for some $b_1, d_1 \in I$, and thus $bd - ac = 1 + b_1 + d_1 + b_1 d_1 - ac$. Since $b_1 + d_1 + b_1 d_1 - ac \in I$ we see that $bd - ac = 1 + b_1 + d_1 + b_1 d_1 - ac \in S$ and hence is a unit of R_S.

(c) If $IM = M$ we clearly have $I_S M_S = M_S$. Hence by (b) and the form of Nakayama's lemma proven in the course, we have $M_S = 0$. Now let $m_1, \ldots, m_k$ be generators of M. Since M is the kernel of the natural map $M \to M_S$ (since $M_S = 0$), there is an element $s_i \in S$ such that $s_i m_i = 0$ for all i. Let $s = \prod_i s_i$. Then s annihilates all the generators m_i and hence M. By construction, $s - 1 \in I$.

5. Let R be a ring and let M be a finitely generated R-module. Let $\phi\colon M \to M$ be a surjective homomorphism of R-modules. Prove that ϕ is injective, and is thus an automorphism. [Hint: use ϕ to construct a structure of $R[x]$-module on M and use the previous question.]

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Solution: View M as an $R[x]$-module by setting $P(x) \cdot m = P(\phi)(m)$. We have $(x)M = M$ by construction and hence by Q4 (iii), there is a polynomial $Q(x) \in R[x]$ such that $Q(x) - 1 \in (x)$ and $Q(x)M = 0$. Let $m_0 \in \ker(\phi)$. Then $Q(x)(m_0) = m_0$ and hence $m_0 = 0$. Thus ϕ is injective.

6. Let R be a ring. Let $\mathcal{S}$ be the subset of the set of ideals of R defined as follows: an ideal I is in $\mathcal{S}$ if and only if all the elements of I are zero-divisors. Show that $\mathcal{S}$ has maximal elements (for the relation of inclusion) and that every maximal element is a prime ideal. Show that the set of zero-divisors of R is a union of prime ideals.

Solution: If $\mathcal{T}$ is a totally ordered subset of $\mathcal{S}$, then the union of its elements is an ideal that consists solely of zero-divisors. So every totally ordered subset of $\mathcal{T}$ has an upper bound and thus, by Zorn's lemma, the ordered set $\mathcal{S}$ has maximal elements. Note that we may refine this reasoning as follows. Let $I \in \mathcal{S}$. Consider the subset $\mathcal{S}_I$ of $\mathcal{S}$ that consists of ideals containing I. By a completely similar reasoning, the subset $\mathcal{S}_I$ has maximal elements for the relation of inclusion. We contend that if $J \in \mathcal{S}_I$ is a maximal element, then it is also maximal in $\mathcal{S}$. Indeed, suppose that $J' \supseteq J$ for some ideal $J' \in \mathcal{S}$. Then $J' \in \mathcal{S}_I$ and hence $J' = J$. Now note that

$$\{ \text{ zero-divisors of } R\} = \bigcup_{r \in R,\ r \text{ is a zero-div.}} (r) \subseteq \bigcup_{r \in R,\ r \text{ is a zero-div.}} J(r)$$

where $J(r)$ is a maximal element of $\mathcal{S}$ containing the ideal (r). Since $J(r)$ also consists of zero-divisors, we conclude that

$$\{ \text{ zero-divisors of } R\} = \bigcup_{r \in R,\ r \text{ is a zero-div.}} J(r).$$

Hence we only have to prove that the maximal elements of $\mathcal{S}$ are prime ideals.

Let J be a maximal element of $\mathcal{S}$. Let $x, y \in R \smallsetminus J$ and suppose for contradiction that $xy \in J$. Then we have

$$((x) + J)((y) + J) \subseteq J.$$

By maximality of J, there are elements $a \in (x) + J$ and $b \in (y) + J$, which are not zero-divisors. But $ab \in J$ and so ab is a zero-divisor, which is contradiction (note that the set of non-zero-divisors is a multiplicative set). So we must have $x \in J$ or $y \in J$, so J is prime.

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Section C

7. Let R be a ring. Consider the inclusion relation on the set $\mathrm{Spec}(R)$. Show that there are minimal elements in $\mathrm{Spec}(R)$.

   Solution: Let $\mathcal{T}$ be a totally ordered subset of $\mathrm{Spec}(R)$ for the relation $\supseteq$. Note that the maximal elements for the relation $\supseteq$ are the minimal elements for the inclusion relation (which is $\subseteq$). Let $I = \bigcap_{\mathfrak{p}\in\mathcal{T}}\mathfrak{p}$. Then I is an ideal. We claim that I is prime.

   To see this, let $x, y \in R$ and suppose for contradiction that $x, y \in R\setminus I$ and that $xy \in I$. By assumption there are prime ideals $\mathfrak{p}_x, \mathfrak{p}_y \in \mathcal{T}$ such that $x \notin \mathfrak{p}_x$ and $y \notin \mathfrak{p}_y$. Suppose without restriction of generality that $\mathfrak{p}_x \supseteq \mathfrak{p}_y$ (recall that $\mathcal{T}$ is totally ordered). We have $xy \in \mathfrak{p}_y$ and thus either x or y lies in $\mathfrak{p}_y$. This contradicts the fact that $x, y \notin \mathfrak{p}_y$. The ideal I thus lies in $\mathrm{Spec}(R)$ and it is a lower bound for $\mathcal{T}$. We may thus apply Zorn's lemma to conclude that there are minimal elements in $\mathrm{Spec}(R)$.

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