B2.2 Commutative Algebra

Sheet 3 — HT25

Sections 1-10

Section A

1. Let R be a subring of a ring T. Suppose that T is integral over R. Let $\mathfrak{p}$ be a prime ideal of R and let $\mathfrak{q}_1, \mathfrak{q}_2$ be prime ideals of T such that $\mathfrak{q}_1 \cap R = \mathfrak{q}_2 \cap R = \mathfrak{p}$. Show that if $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ then $\mathfrak{q}_1 = \mathfrak{q}_2$.

   Solution: The ring $R/\mathfrak{p}$ is can be viewed as a subring of $T/\mathfrak{q}_1$ and by assumption we have $(\mathfrak{q}_2 \mod \mathfrak{q}_1) \cap R/\mathfrak{p} = (0)$. We may thus assume without loss of generality that R and T to be domains and that $\mathfrak{q}_1$ and $\mathfrak{p}$ are zero ideals.

   Now let $e \in \mathfrak{q}_2 \smallsetminus \{0\}$ and let $P(x) \in R[x]$ be a non-zero monic polynomial such that $P(e) = 0$. Since T is a domain, we may assume that the constant coefficient of $P(x)$ is non-zero (otherwise, replace $P(x)$ by $P(x)/x^k$ for a suitable $k \geqslant 1$). But then the constant term $P(0)$ is a linear combination of positive powers of e (since $P(e) = 0$), so $P(0) \in R \cap \mathfrak{q}_2 = (0)$. This is a contradiction.

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Section B

2. Let R be a ring. Show that the two following conditions are equivalent:

   (a) R is a Jacobson ring.

   (b) If $\mathfrak{p} \in \mathrm{Spec}(R)$ and $R/\mathfrak{p}$ contains an element b such that $(R/\mathfrak{p})[b^{-1}]$ is a field, then $R/\mathfrak{p}$ is a field.

   Here we write $(R/\mathfrak{p})[b^{-1}]$ for the localisation of $R/\mathfrak{p}$ at the multiplicative subset $1, b, b^2, \ldots$.

   Solution:

   (a) $\Rightarrow$ (b): If R is a Jacobson, then so is $R/\mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Spec}(R)$. Hence (ii) follows from Lemma 10.2.

   $\neg$(a) $\Rightarrow$ $\neg$(b): Note first that R is a Jacobson ring if and only if any prime ideal of R is the intersection of the maximal ideals containing it (this is straightforward). Now suppose that R is not Jacobson. Then there is a prime ideal $\mathfrak{p}$ of R and an element $e \notin \mathfrak{p}$ such that e is in the Jacobson radical of $\mathfrak{p}$. In other words, $e + \mathfrak{p} \neq \mathfrak{p}$ and $e + \mathfrak{p}$ lies in the Jacobson radical of $R/\mathfrak{p}$.

   Now let $\mathfrak{q}$ be an ideal maximal among the prime ideals of $R/\mathfrak{p}$, which do not contain $e + \mathfrak{p}$. The ideal $\mathfrak{q}$ exists, since it corresponds to a maximal ideal of $(R/\mathfrak{p})[(e + \mathfrak{p})^{-1}]$ by Lemma 5.6, and it is not maximal, since $e + \mathfrak{p}$ lies in the intersection of all the maximal ideals of $R/\mathfrak{p}$. The ring $(R/\mathfrak{p})/\mathfrak{q}$ has by construction the property that any of its non-zero prime ideals contains $(e + \mathfrak{p}) + \mathfrak{q}$. In particular, the ring

   $$((R/\mathfrak{p})/\mathfrak{q})[((e + \mathfrak{p}) + \mathfrak{q})^{-1}]$$

   is a field, because its only prime ideal is the zero ideal. On other hand, $((R/\mathfrak{p})/\mathfrak{q})$ is not a field, since $\mathfrak{q}$ is not maximal. Now if we let $q : R \to R/\mathfrak{p}$ be the quotient map, we have $((R/\mathfrak{p})/\mathfrak{q}) \simeq R/q^{-1}(\mathfrak{q})$ and thus this contradicts (b).

3. Let k be field and let R be a finitely generated algebra over k. Show that the two following conditions are equivalent:

   (a) $\mathrm{Spec}(R)$ is finite.

   (b) R is finite over k.

   Solution: (a) $\Rightarrow$ (b): Suppose that $\mathrm{Spec}(R)$ is finite. By Noether's normalisation lemma, there is an injection $k[x_1, \ldots, x_d] \to R$, which makes R into a finite $k[x_1, \ldots, x_d]$-algebra. Since the corresponding map of spectra $\mathrm{Spec}(R) \to \mathrm{Spec}(k[x_1, \ldots, x_d])$ is surjective by the Going-Up Theorem, this implies that $\mathrm{Spec}(k[x_1, \ldots, x_d])$ is finite. In

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particular, $k[x_1, \ldots, x_d]$ has only finitely many maximal ideals, say $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$. Since $k[x_1, \ldots, x_d]$ is a Jacobson ring, we have $\cap_i \mathfrak{m}_i = \mathfrak{r}((0)) = 0$ and so we may deduce from the Chinese remainder theorem that $k[x_1, \ldots, x_d] \simeq \oplus_i R/\mathfrak{m}_i$. Since $k[x_1, \ldots, x_d]$ is a domain, this implies that $t = 1$. In particular, $k[x_1, \ldots, x_d]$ is field, which is only possible if $d = 0$ (otherwise, x_1 is a non unit). Hence R is finite over k.

(ii) $\Rightarrow$ (i) : This follows from Proposition 8.12.

4. Let k be an algebraically closed field. Let $P_1, \ldots, P_d \in k[x_1, \ldots, x_d]$. Suppose that the set

$$\{(y_1, \ldots, y_d) \in k^d \mid P_i(y_1, \ldots, y_d) = 0 \, \forall i \in \{1, \ldots, d\}\}$$

is finite. Show that

$$\mathrm{Spec}(k[x_1, \ldots, x_d]/(P_1, \ldots, P_d))$$

is finite.

Solution: From Corollaries 9.5 and 9.3, we deduce that $\mathfrak{r}((P_1, \ldots, P_d))$ is the intersection of finitely many maximal ideals of $k[x_1, \ldots, x_d]$, say $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$. From the Chinese remainder theorem, we deduce that

$$k[x_1, \ldots, x_d]/\mathfrak{r}((P_1, \ldots, P_d)) \simeq \prod_i k[x_1, \ldots, x_d]/\mathfrak{m}_i \simeq \prod_i k.$$

In particular, $\mathrm{Spec}(k[x_1, \ldots, x_d]/\mathfrak{r}((P_1, \ldots, P_d)))$ is finite. Now we have

$$\mathrm{Spec}(k[x_1, \ldots, x_d]/\mathfrak{r}((P_1, \ldots, P_d))) \simeq \mathrm{Spec}(k[x_1, \ldots, x_d]/(P_1, \ldots, P_d))$$

so the conclusion follows.

5. Let R be a ring and let R_0 be the prime ring of R (see the preamble of the notes for the definition). Suppose that R is a finitely generated R_0-algebra. Suppose also that R is a field. Prove that R is a finite field.

Solution: Since R_0 is contained in a field, it is a domain and so R_0 is either a finite field or it is isomorphic to $\mathbb{Z}$. Suppose first that R_0 is a finite field. Then R is a finite field extension of a finite field by the weak Nullstellensatz and hence R is a finite field.

Now suppose that $R \simeq \mathbb{Z}$. A finitely generated $\mathbb{Z}$-algebra is simply a finitely generated abelian group; all subgroups of such groups are again finitely generated, and so such groups cannot contain $\mathbb{Q}$. But R contains the fraction field $\mathbb{Q}$ of $\mathbb{Z}$. This is a contradiction.

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6. Let k be a field and let $\mathfrak{m}$ be a maximal ideal of $k[x_1, \ldots, x_d]$. Show that there are polynomials $P_1(x_1), P_2(x_1, x_2), P_3(x_1, x_2, x_3), \ldots, P_d(x_1, \ldots, x_d)$ such that $\mathfrak{m} = (P_1, \ldots, P_d)$.

   Solution: By induction on $d \geqslant 1$. If $d = 1$ then this follows from the fact that $k[x_1]$ is a PID. We suppose that the statement holds for $d - 1$. Let $K = k[x_1, \ldots, x_d]/\mathfrak{m}$. By the weak Nullstellensatz, this is a finite field extension of k. Let $\phi\colon k[x_1, \ldots, x_d] \to K$ be the natural surjective homomorphism of k-algebras. Let $L = \phi(k[x_1, \ldots, x_{d-1}])$. This is a domain and by Lemma 8.9, L is a field, since it contains k and is contained inside an integral extension of k. Let $\psi\colon k[x_1, \ldots, x_{d-1}] \to L$ be the surjective homomorphism of k-algebras arising by restricting ϕ. The map ψ induces a surjective homomorphism of k-algebras

   $$\Psi\colon k[x_1, \ldots, x_d] \simeq (k[x_1, \ldots, x_{d-1}])[x_d] \to L[x_d]$$

   and there is a surjective homomorphism of L-algebras

   $$\Lambda\colon L[x_d] \to K$$

   that sends x_d to $\phi(x_d)$. By construction, we have $\phi = \Lambda \circ \Psi$. In particular, we have $\mathfrak{m} = \Psi^{-1}(\Lambda^{-1}(0))$. Since $L[x_d]$ is a PID and $\phi(x_d)$ is algebraic over k, we have $\Lambda^{-1}(0) = (P(x_d))$ for some non zero polynomial $P(x_d) \in L[x_d]$. Now let $P_d(x_1, \ldots, x_d) \in (k[x_1, \ldots, x_{d-1}])[x_d]$ be a preimage by Ψ of $P(x_d)$.

   We claim that $\mathfrak{m} = (\ker(\Psi), P_d)$. To see this, note that $\Psi((\ker(\Psi), P_d)) = (P(x_d))$ and so we have $(\ker(\Psi), P_d) \subseteq \mathfrak{m}$. On the other hand, if $e \in \mathfrak{m}$ then $\Psi(e) \in (P(x_d))$ and thus there is an element $e' \in (P_d)$ such that $\Psi(e) = \Psi(e')$ (since Ψ is surjective). In particular, we have $e - e' \in \ker(\Psi)$, so that $e \in (\ker(\Psi), P_d)$.

   Now by the inductive assumption, $\ker(\Psi)$ is generated by polynomials

   $$P_1(x_1), P_2(x_1, x_2), P_3(x_1, x_2, x_3), \ldots, P_{d-1}(x_1, \ldots, x_{d-1})$$

   and so $\mathfrak{m}$ is generated by $P_1(x_1), P_2(x_1, x_2), P_3(x_1, x_2, x_3), \ldots, P_d(x_1, \ldots, x_d)$.

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Section C

7. Let R be a domain. Show that $R[x]$ is integrally closed if R is integrally closed.

   Solution: Suppose that R is integrally closed in its fraction field K. The fraction field of $R[x]$ is $K(x) = (K[x])(K[x] \smallsetminus \{1\})^{-1}$. Let $Q(x) \in K(x)$ and suppose that $Q(x)$ is integral over $R[x]$. Suppose for a contradiction that $Q(x) \notin R[x]$, and take $Q(x)$ of smallest possible degree. Clearly $Q(x)$ is not the zero polynomial.

   Then $Q(x)$ is in particular integral over $K[x]$ and we saw in the solution of Question 4, sheet 2, that $K[x]$ is integrally closed, since it is a PID. So we deduce that $Q(x) \in K[x]$.

   Now let
   $$Q^n + P_{n-1}Q^{n-1} + \cdots + P_0 = 0$$
   be a non trivial integral equation for Q with $P_i \in R[x]$ for all n. Evaluating at $x = 0$ shows that the constant term of $Q(x)$ is integral over R, and hence lies in R. Since the integral closure of $R[x]$ is a ring, we may subtract the constant term, and assume that the constant term of Q is zero. But then we can also divide by a power of x, and decrease the degree of Q. Contradiction.

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