B2.2 Commutative Algebra

Sheet 3 — HT25

Sections 1-10

Section A

1. Let R be a noetherian domain. Let $\mathfrak{m}$ be a maximal ideal in R. Let $r \in R\backslash\{0\}$ and suppose that (r) is a $\mathfrak{m}$-primary ideal. Show that $\text{height}((r)) = 1$.

   Solution: By assumption, the nilradical of (r) is $\mathfrak{m}$. Since the nilradical is the intersection of all the prime ideals containing (r), we see that every prime ideal containing (r) also contains $\mathfrak{m}$. On the other hand, a prime ideal containing $\mathfrak{m}$ must be equal to $\mathfrak{m}$. We conclude that $\mathfrak{m}$ is the only prime ideal containing (r). In particular, $\mathfrak{m}$ is minimal among the prime ideals containing (r) and thus $\text{height}((r)) = \text{height}(\mathfrak{m}) \leqslant 1$ by Krull's principal ideal theorem. On the other hand, $\text{height}(\mathfrak{m}) = 1$, since we have the chain $\mathfrak{m} \supset (0)$ (note that R is a domain).

2. Let R be a PID. Show that $\dim R \leqslant 1$, and that $\dim R = 0$ if and only if R is a field.

   Solution: We have the prime ideal (0), since R is a domain. If R is a field, then we have no other prime ideals, and $\dim R = 0$.

   If R is not a field, then it has at least one non-trivial proper prime ideal. Every such ideal is maximal (see Sheet 0), and hence $\dim R = 1$.

Section B

3. Let A, B be integral domains and suppose that $A \subseteq B$. Suppose that A is integrally closed and that B is integral over A. Let

$$\mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_n$$

be a descending chain of prime ideals in A. Let $k \in \{0, \dots, n-1\}$ and let

$$\mathfrak{q}_0 \supset \mathfrak{q}_1 \supset \cdots \supset \mathfrak{q}_k$$

be a descending chain of prime ideals in B, such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for all $i \in \{0, \dots, k\}$. Then there is a descending chain of prime ideals

$$\mathfrak{q}_k \supset \mathfrak{q}_{k+1} \supset \cdots \supset \mathfrak{q}_n$$

such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for all $i \in \{k+1, \dots, n\}$. This is the "Going-down Theorem", see AT, Th. 5.16, p. 64.

Let L (resp. K) be the fraction field of B (resp. A). Prove the Going-down Theorem when L is a (finite) Galois extension of K.

Solution: One immediately reduces the question to $n = 1$ and $k = 0$. Let $\bar{A}$ be the integral closure of A in L. Note that by assumption we have $B \subseteq \bar{A}$ and that $\bar{A}$ is integral over B (since it is integral over A). Let $\mathfrak{q}_0'$ be a prime ideal of $\bar{A}$ such that $\mathfrak{q}_0' \cap B = \mathfrak{q}_0$ (this exists by the Going-up Theorem). Let $\mathfrak{a}$ be a prime ideal of $\bar{A}$ such that $\mathfrak{a} \cap A = \mathfrak{p}_1$ (again this exists by the Going-up Theorem). According to Q6 of sheet 2, there is a prime ideal $\mathfrak{b}$ in $\bar{A}$ such that $\mathfrak{b} \supset \mathfrak{a}$ and such that $\mathfrak{b} \cap A = \mathfrak{p}_0$. According to Proposition 12.10, there is an element $\sigma \in \mathrm{Gal}(L|K)$ such that $\sigma(\mathfrak{b}) = \mathfrak{q}_0'$. We have $\sigma(\mathfrak{a}) \cap A = \mathfrak{p}_1$ and $\sigma(\mathfrak{a}) \subset \sigma(\mathfrak{b}) = \mathfrak{q}_0'$. Hence $\sigma(\mathfrak{a}) \cap B \subseteq \mathfrak{q}_0' \cap B = \mathfrak{q}_0$ and $(\sigma(\mathfrak{a}) \cap B) \cap A = \sigma(\mathfrak{a}) \cap A = \mathfrak{p}_1$. Furthermore, we have $\sigma(\mathfrak{a}) \cap B \subset \mathfrak{q}_0$ because $\sigma(\mathfrak{a}) \cap A = \mathfrak{p}_1 \subset \mathfrak{p}_0 = \mathfrak{q}_0 \cap A$. So we may set $\mathfrak{q}_1 = \sigma(\mathfrak{a}) \cap B$.

4. Let R be an integrally closed domain. Let $K = \mathrm{Frac}(R)$. Let $L|K$ be an algebraic field extension. Show that an element $e \in L$ is integral over R if and only if the minimal polynomial of e over K has coefficients in R.

   Solution: Let $m_e(x) \in K[x]$ be the minimal polynomial of e. If $m_e(x) \in R[x]$ then e is integral over R by the definition of integrality. On other hand, suppose that e is integral over R and let $Q(x) \in R[x]$ be a monic polynomial such that $Q(e) = 0$. Then $m_e(x)$ divides $Q(x)$ by the definition of the minimal polynomial and $m_e(x) \in R[x]$ by Q5 of sheet 2.

5. Let R be a PID. Suppose that $2 = 1 + 1$ is a unit in R. Let $c_1, \dots, c_t \in R$ be distinct irreducible elements with $t \geqslant 1$, and let $c = c_1 \cdots c_t$. Show that the ring $R[x]/(x^2 - c)$ is a Dedekind domain. Use this to show that $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$ is a Dedekind domain.

   Solution: Let $K = \mathrm{Frac}(R)$. Notice first that c is not a square in K.

   Indeed, suppose for contradiction that there is an element $e \in K$ such that $e^2 = c$. Write $e = f/g$, with $f, g \in R$ and f and g coprime. We then have $f^2/g^2 = c$ and hence $f^2 = g^2 c$. In particular, c_1 divides f and thus c_1^2 divides $g^2 c$. Since $(f, g) = 1$, we deduce that c_1^2 divides c, which contradicts our assumptions.

   We deduce that the polynomial $x^2 - c$ is irreducible over K, since it has no roots in K. Let $L = K[x]/(x^2 - c)$. Note that L is a field, since $x^2 - c$ is irreducible. Now let $\phi\colon R[x] \to L$ be the obvious homomorphism of R-algebras. We have $\phi(Q(x)) = 0$ if and only if $x^2 - c$ divides $Q(x)$ in $K[x]$. On the other hand, if $x^2 - c$ divides $Q(x)$ in $K[x]$, then $x^2 - c$ divides $Q(x)$ in $R[x]$ by the unicity statement in the Euclidean algorithm (see preamble). Hence $\ker(\phi) = (x^2 - c)$. We thus see that $R[x]/(x^2 - c)$ can be identified with the sub-R-algebra of L generated by x. Under this identification, the elements of $R[x]/(x^2 - c)$ correspond to the elements of the form $\lambda + \mu x$, with $\lambda, \mu \in R$, whereas the elements of K can all be written as $\lambda + \mu x$, with $\lambda, \mu \in K$.

   We claim that that L is the fraction field of $R[x]/(x^2 - c)$. Note first that the fraction field of $R[x]/(x^2 - c)$ naturally embeds in L, since L is field containing $R[x]/(x^2 - c)$. To prove the claim, we only have to show that every element of L can be written as a fraction in L of elements of $R[x]/(x^2 - c)$. This simply follows from the fact that if $f, g, h, j \in R$ and $f/g + (h/j)x \in L$, then

$$f/g + (h/j)x = \frac{fj + hgx}{gj}.$$

   Now to prove that $R[x]/(x^2 - c)$ is a Dedekind domain, we have to show that it is noetherian, that is has dimension 1 and that it is integrally closed.

Since R contains an irreducible element c_1, it cannot be a field.

The ring $R[x]/(x^2-c)$ is clearly noetherian (by the Hilbert basis theorem and stability of noetherianity under quotients). Also, the ring $R[x]/(x^2-c)$ is integral over R by construction and R has dimension one by Question 2. We deduce from Lemma 11.29 that $R[x]/(x^2-c)$ also has dimension 1.

To show that $R[x]/(x^2-c)$ is integrally closed, we have to show that the integral closure of $R[x]/(x^2-c)$ in L is $R[x]/(x^2-c)$. The integral closure of $R[x]/(x^2-c)$ in L is also the integral closure of R in L, since $R[x]/(x^2-c)$ consists of elements that are integral over R. Furthermore, by Question 4, an element $\lambda+\mu x\in L$ is integral over R if and only if its minimal polynomial $P(t)\in K[t]$ has coefficients in R. Thus we have to show that if $\lambda+\mu x\in L$ has a minimal polynomial $P(t)\in R[t]$ then $\lambda,\mu\in R$. We prove this statement.

If $\mu=0$ then $\lambda+\mu x\in K$ and thus the minimal polynomial of $\lambda+\mu x$ is $t-\lambda$. So the statement certainly holds in this situation.

If $\mu\neq 0$, we note that the polynomial

$$(t-(\lambda+\mu x))(t-(\lambda-\mu x))=t^2-2\lambda+\lambda^2-\mu^2x^2=t^2-2\lambda t+\lambda^2-c\mu^2$$

annihilates $\lambda+\mu y$ and has coefficients in K. It must thus coincide with the minimal polynomial $P(t)$ of $\lambda+\mu y$, since we know that $\deg(P(t))>1$.

Thus we have to show that if $-2\lambda\in R$ and $\lambda^2-c\mu^2\in R$, then $\lambda,\mu\in R$. So suppose that $-2\lambda\in R$ and $\lambda^2-c\mu^2\in R$. We have $\lambda\in R$, since -2 is a unit in R by assumption. Hence $c\mu^2\in R$. We claim that $\mu\in R$. Indeed, let $\mu=f/g$, where $f,g\in R$ and f and g are coprime. Then $cf^2=g^2r$ for some $r\in R$. Let $i\in\{1,\dots,t\}$ and suppose first that c_i divides g. Then c_i^2 divides rg^2 and since c_i appears with multiplicity one in c by assumption, we thus see that c_i divides f, which is a contradiction (because $(f,g)=1$). Hence c_i does not divide g and thus c_i divides r. Since all the c_i are distinct, we thus see that c divides r and thus $(f/g)^2=r/c=:d\in R$. Hence $f^2=g^2d$. Since f and g are coprime, we see that f^2 divides d and hence $d/f^2\in R$. Since $g^2(d/f^2)=1$, we conclude that g is a unit and hence $\mu=f/g\in R$.

To see that $\mathbb{R}[x,y]/(x^2+y^2-1)$ is a Dedekind domain, note that $\mathbb{R}[x,y]/(x^2+y^2-1)\simeq(\mathbb{R}[x])[y]/(y^2-(1-x^2))$ and apply the first statement of the question with $R=\mathbb{R}[x]$ and $c=1-x^2=(1-x)(1+x)$.

6. Let R be a PID. Let $c_1, c_2 \in R$ be two distinct irreducible elements and let $c = c_1 \cdot c_2$. Show that $(c) = (x, c_1)^2 \cdot (x, c_2)^2$ and that the ideals (x, c_i) are prime.

   Solution: Note first that (x, c_i) $(i = 1, 2)$ is indeed a prime ideal of $R[x]/(x^2 - c)$, because

$$(R[x]/(x^2 - c))/(x, c_i) = R[x]/(x^2 - c, x, c_i) = R/(-c, c_i) = R/(c_i),$$

   which is a domain, since c_i is irreducible.

   We only have to show that $(c_i) = (x, c_i)^2$.

   We first show that $(c_i) \subseteq (x, c_i)^2$. For this, note that $c_i^2 \in (x, c_i)^2$ by definition and

$$c_i(c_j - c_i) = c - c_i^2 = x^2 - c_i^2 \in (x, c_i)^2,$$

   where $\{i, j\} = \{1, 2\}$. But $\gcd_R(c_i^2, c_i(c_j - c_i)) = c_i$ (because $c_j - c_i$ is coprime to c_i in R, since c_j is irreducible and distinct from c_i), and in particular $c_i \in (x, c_i)^2$, so that $(c_i) \subseteq (x, c_i)^2$.

   The inclusion $(c_i) \supseteq (x, c_i)^2$ is clear, since $(x, c_i)^2$ is generated as an R-module by $x^2 = c, xc_i$, and ${c_i}^2$, and all these elements lie in (c_i).

7. Let R be a ring (not necessarily noetherian). Suppose that $\dim(R) < \infty$. Show that $\dim(R[x]) \leqslant 1 + 2\dim(R)$.

   Solution: Let

$$\mathfrak{q}_0 \supset \mathfrak{q}_1 \supset \mathfrak{q}_2 \supset \cdots \supset \mathfrak{q}_d$$

   be a descending chain of prime ideals in $R[x]$, where $d \geqslant 0$. By restriction, we obtain a descending chain of prime ideals

$$\mathfrak{q}_0 \cap R \supseteq \mathfrak{q}_1 \cap R \supseteq \mathfrak{q}_2 \cap R \supseteq \cdots \supseteq \mathfrak{q}_d \cap R \quad (*)$$

   (possibly with repetitions) in R. For each $i \in \{0, \dots, d\}$, let $\rho(i) \geqslant 0$ be the largest integer k such that $\mathfrak{q}_i \cap R = \mathfrak{q}_{i+1} \cap R = \cdots = \mathfrak{q}_{i+k} \cap R$. By Lemma 11.21, the remark before it, and Lemma 11.19 we have $\rho(i) \leqslant 1$ for all $i \in \{0, \dots, d\}$. Now let

$$\mathfrak{q}_{i_0} \cap R = \mathfrak{q}_0 \cap R \supset \mathfrak{q}_{i_1} \cap R \supset \cdots \supset \mathfrak{q}_{i_\delta} \cap R$$

   be an enumeration of all the prime ideals appearing in the chain $(*)$, in decreasing order of inclusion. We have

$$d + 1 = (1 + \rho(i_0)) + (1 + \rho(i_1)) + \cdots + (1 + \rho(i_\delta)) \leqslant 2(\delta + 1)$$

   and so that $d \leqslant 2\delta + 1$. Now we have $\delta \leqslant \dim(R)$ and the required inequality follows.

8. Let R be a Dedekind domain. Let I be a non zero ideal in R. Show that every ideal in R/I is principal. Deduce that every ideal in a Dedekind domain can be generated by two elements.

   Solution: We first prove the first statement. Since R is a Dedekind domain, we have a primary decomposition

$$I = \bigcap_{i=1}^{k} \mathfrak{p}_i^{m_i}$$

   for some prime ideals $\mathfrak{p}_i$. Using Lemma 12.2 and the Chinese remainder theorem, we see that we have

$$R/I \simeq \bigoplus_{i=1}^{k} R/\mathfrak{p}_i^{m_i}.$$

   Now an ideal J of $\bigoplus_{i=1}^{k} R/\mathfrak{p}_i^{m_i}$ is of the form $\bigoplus_{i=1}^{k} J_i$, where J_i is an ideal of $R/\mathfrak{p}_i^{m_i}$. This follows from the fact that if $e \in J$ and $e = \oplus_{i=1}^{k} e_i$ then $e_i = e \cdot (0, \dots, 1, \dots, 0) \in J$, where 1 appears in the i-th place in the expression $(0, \dots, 1, \dots, 0)$. Hence, if we can find generators $g_i \in J_i$ for J_i in $R/\mathfrak{p}_i^{m_i}$, then $(g_1, \dots, g_k)$ will be a generator of J. We proceed to show that any ideal in $R/\mathfrak{p}_i^{m_i}$ can be generated by one element.

Consider the exact sequence

$$0 \to \mathfrak{p}_i^{m_i} \to R \to R/\mathfrak{p}_i^{m_i} \to 0.$$

Localising this sequence at $R \smallsetminus \mathfrak{p}_i$, we get the exact sequence of $R_{\mathfrak{p}_i}$-modules

$$0 \to (\mathfrak{p}_i^{m_i})_{\mathfrak{p}_i} \to R_{\mathfrak{p}_i} \to (R/\mathfrak{p}_i^{m_i})_{\mathfrak{p}_i} \to 0.$$

Now the $R_{\mathfrak{p}_i}$-submodule $(\mathfrak{p}_i^{m_i})_{\mathfrak{p}_i}$ of $R_{\mathfrak{p}_i}$ is the ideal generated by the image of $\mathfrak{p}_i^{m_i}$ in $R_{\mathfrak{p}_i}$ (see the beginning of the proof of Lemma 5.6). If we let $\mathfrak{m}$ be the maximal ideal of $R_{\mathfrak{p}_i}$, this is also $\mathfrak{m}^{m_i}$. On the other hand, $\mathfrak{p}_i$ is contained in the nilradical of $\mathfrak{p}_i^{m_i}$ and since $\mathfrak{p}_i$ is maximal (by Lemma 12.1) it coincides with the radical of $\mathfrak{p}_i^{m_i}$. Hence $R/\mathfrak{p}_i^{m_i}$ has only one maximal ideal, namely $\mathfrak{p}_i \mod \mathfrak{p}_i^{m_i}$. Since the image of $R \smallsetminus \mathfrak{p}_i$ in $R/\mathfrak{p}_i^{m_i}$ lies outside $\mathfrak{p}_i \mod \mathfrak{p}_i^{m_i}$, we see that this image consists of units. Hence $(R/\mathfrak{p}_i^{m_i})_{\mathfrak{p}_i} \simeq R/\mathfrak{p}_i^{m_i}$. All in all, there is thus an isomorphism

$$R_{\mathfrak{p}_i}/\mathfrak{m}^{m_i} \simeq R/\mathfrak{p}_i^{m_i}.$$

Now by Proposition 12.4, every ideal in $R_{\mathfrak{p}_i}/\mathfrak{m}^{m_i}$ is principal, and so we have proven the first statement.

For the second one, let $e \in I$ be any non-zero element. Then the ideal $I \mod (e) \subseteq R/(e)$ is generated by one element, say g. Let $g' \in R$ be a preimage of g. Then $I = (e, g')$.

9. Let A (resp. B) be a noetherian local ring with maximal ideal $\mathfrak{m}_A$ (resp. $\mathfrak{m}_B$). Let $\phi : A \to B$ be a ring homomorphism and suppose that $\phi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$ (such a homomorphism is said to be 'local').

   Suppose that

   (a) B is finite over A via ϕ;

   (b) the map $\mathfrak{m}_A \to \mathfrak{m}_B/\mathfrak{m}_B^2$ induced by ϕ is surjective;

   (c) the map $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ induced by ϕ is bijective.

   Prove that ϕ is surjective. [Hint: use Nakayama's lemma twice].

Solution: By Corollary 3.6, (b) implies that the image of $\mathfrak{m}_A$ in $\mathfrak{m}_B$ generates $\mathfrak{m}_B$ as a B-module. In other words, $\phi(\mathfrak{m}_A)B = \mathfrak{m}_B$. On the other hand, since B is finitely generated as an A-module, the homomorphism ϕ is surjective if and only if the induced map $A/\mathfrak{m}_A \to B/\phi(\mathfrak{m}_A)B$ is surjective, again by Corollary 3.6. Now $B/\phi(\mathfrak{m}_A)B = B/\mathfrak{m}_B$ by the above, and by (c) the map $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ is surjective. The conclusion follows.

Section C

10. Let R be a Dedekind domain. Show that R is a PID if and only if it is a UFD.

Solution: Every PID is a UFD.

For the converse, first note that it is enough to prove that all prime ideals are principal, since every non-trivial proper ideal in a Dedekind domain is a product of prime ideals.

Let $\mathfrak{p}$ be a non-trivial prime ideal in R. Since R is a UFD, there is a prime element $p \in \mathfrak{p}$. Hence we have the inclusions

$$(0) \subset (p) \subseteq \mathfrak{p},$$

and since $\dim R = 1$ we must have $\mathfrak{p} = (p)$.