Problem Sheet 1

1. Section A. By taking the scalar product with an arbitrary constant vector $\mathbf{k}$, use Stokes' Theorem to show that

$$\left[\int_C \mathbf{f} \wedge \mathbf{t}\, \mathrm{d}s\right]_i = \int_S \left[n_i(\nabla \cdot \mathbf{f}) - n_j \partial_i f_j\right]\, \mathrm{d}S,$$

   for any vector function $\mathbf{f}$, where $\mathbf{t}$ is the unit tangent to a curve C (which bounds the surface S), the vector $\mathbf{n}$ is the unit normal to S and we use the summation convention.

   Let $\boldsymbol{\nu} = \mathbf{t} \wedge \mathbf{n}$ and choose $\mathbf{f} = \gamma \mathbf{n}$, defined off the surface by the extension that is independent of position in the normal direction $\mathbf{n}$. Show that

$$\int_C \gamma \boldsymbol{\nu}\, \mathrm{d}s = \int_S \left[\nabla \gamma - \gamma(\nabla \cdot \mathbf{n})\mathbf{n}\right]\, \mathrm{d}S.$$

   [You may need the standard identities

$$(\mathbf{f} \wedge \mathbf{k}) \cdot \mathbf{t} = -\mathbf{k} \cdot (\mathbf{f} \wedge \mathbf{t}),$$

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}.$$

   ]

2. Section B. Recall the Laplace–Young equation governing the shape $z = h(x)$ of a meniscus near a vertical, planar wall

$$\ell_c^2 \frac{h_{xx}}{(1 + h_x^2)^{3/2}} = h.$$

   Integrating the full Laplace–Young equation once (without making an assumption of shallow slopes), show that the rise height of the meniscus on the wall, h_0, is given in terms of the contact angle of the liquid θ by

$$h_0 = \pm \ell_c \left[2(1 - \sin\theta)\right]^{1/2}$$

   and discuss when each of the $\pm$ branches of the result are appropriate.

   Show that the total area displaced by the meniscus is given by

$$A = \int_0^\infty h\, \mathrm{d}x = \ell_c^2 \cos\theta.$$

   Determine the weight of liquid displaced (per unit depth length in the y direction). How is this related to the vertical force exerted by surface tension at the contact line?

3. Section B. With r, ϕ, z denoting cylindrical polar coordinates, fluid occupies the domain

$$\left\{ (r, \phi, z) \ \middle| \ 0 < r < \infty, \quad \phi \in [-\alpha, \alpha], \quad z \in [0, h(r, \phi)] \right\}$$

where $h(r, \phi) > 0$ denotes the height of the free surface above the (x, y) plane.

The boundary at $z = 0$ is rigid as are the boundaries at $\phi = \pm \alpha$, while the boundary at $z = h(r, \phi)$ is a static free surface, with contact lines at $z = h(r, \pm\alpha)$.

Sketch the domain and show that the linearised Laplace–Young equation enforces the equation

$$h = \ell_c^2 \, \nabla^2 h,$$

for the region

$$\left\{ (r, \phi) \ \middle| \ 0 < r < \infty, \quad \phi \in [-\alpha, \alpha] \right\},$$

where the constant ℓ_c is to be determined.

Show that the boundary conditions are

$$\frac{1}{r} h_\phi(\alpha) = \cot \theta, \qquad \frac{1}{r} h_\phi(-\alpha) = - \cot \theta,$$

where θ is the contact angle between the air-fluid interface and the rigid boundary at $\phi = \pm\alpha$.

Explain why one must assume that $|\pi/2 - \theta| \ll 1$ for self consistency.

When $\alpha = \pi/4$ show that the unique solution is

$$h = \ell_c \cot \theta \left[e^{-r \sin(\pi/4 - \phi)/\ell_c} + e^{-r \sin(\phi + \pi/4)/\ell_c} \right].$$

4. Section B.

Consider a spherical bubble of constant radius a rising in fluid of viscosity μ at speed U in the laboratory reference frame under the influence of gravity, with $\boldsymbol{g} = -g\boldsymbol{e}_z$. The bubble is sufficiently small that the Reynolds number is negligible.

With z the coordinate in the bubble-fixed frame with origin at the centre of the bubble, the surface tension at the bubble-water interface is given by

$$\gamma = \gamma_0 + \gamma' z = \gamma_0 + \gamma' a \cos\theta,$$

where θ is the polar angle (so that $z = a$ on the bubble corresponds to $\theta = 0$ and $z = -a$ corresponds to $\theta = \pi$).

The resulting fluid flow, relative to the bubble-fixed frame, is given via the streamfunction

$$\psi = \left[-\frac{U}{2} r^2 + Cr - \frac{\gamma' a^4}{6\mu} \frac{1}{r} \right] \sin^2\theta, \quad C = \frac{1}{2} U a - \frac{\gamma' a^2}{6\mu},$$

where r is the distance from the bubble centre, as derived in the lecture notes.

You are given that the force exerted on the bubble by the fluid is given by

$$F_i = \int_B T_{ij} n_j \mathrm{d}S,$$

where B is the bubble surface, $\boldsymbol{T}$ is the Cauchy stress tenor and $\boldsymbol{n}$ is the outward normal for the bubble.

(i) Without calculation, explain why the force on the bubble only has a non-trivial component in the $\boldsymbol{e}_z$ direction.

(ii) Show the flow due to the term scaling with r^2 in the stream function is given by $\boldsymbol{u} = U\boldsymbol{e}_z$. Hence determine the force due to the term scaling with r^2 term in the streamfunction.

(iii) Show that the term scaling with $1/r$ in the streamfunction does not generate a force (*Hint: explicit calculation is not expected.*)

(iv) Determine the value of γ' for which the bubble is stationary, that is $U = 0$.

   [Hint. You are given that the flow associated with a sphere moving at speed $V\boldsymbol{e}_z$ has streamfunction

   $$\frac{3}{4} V a \left(r + \frac{\alpha}{r} \right) \sin^2\theta,$$

   where α is constant, with the flow exerting a force $-6\pi\mu V a\boldsymbol{e}_z$ on the sphere.]

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5. Section C. Consider a static blob of fluid lying underneath a horizontal plate, $z = 0$, and occupying the region

$$\left\{(x, z)\Big| - x_0 \leq x \leq x_0, \quad h(x) \leq z \leq 0\right\}.$$

The liquid has a contact angle $\theta \ll 1$. Assuming that $|h_x| \ll 1$, show that h satisfies

$$\ell_c^2 h_{xxx} + h_x = 0.$$

where ℓ_c is a constant, to be determined. Show that

$$h(x) = \theta \ell_c \left[\cot(x_0/\ell_c) - \frac{\cos(x/\ell_c)}{\sin(x_0/\ell_c)}\right],$$

and confirm that the assumption $|h_x| \ll 1$ is justified under certain circumstances, which you should state. Show that the cross sectional area of the drop, A, is given by

$$\frac{A}{2} = \theta \ell_c^2 \left(1 - \frac{x_0}{\ell_c} \cot(x_0/\ell_c)\right). \tag{1}$$

Is there a maximum cross sectional area of fluid which can hang beneath the plate in this way? Does this accord with your physical intuition? If not, why not?