Problem Sheet 2 (with correction)

1. Section B. *An axisymmetric, spreading drop*

   A finite volume V of liquid (with density ρ and viscosity μ) is released on a horizontal rigid plate situated at $z = 0$. We neglect the effect of surface tension and consider the subsequent axisymmetric spreading of the drop under gravity. By considering the lubrication approximation in this geometry, show that the radial velocity $u(r,t)$ is given by

$$u = \frac{\rho g}{2\mu} z(z - 2h)\frac{\partial h}{\partial r}$$

   where $z = h(r,t)$ describes the free surface of the drop. Show that the conservation of mass and kinematic boundary condition in this axisymmetric geometry give that the drop profile evolves according to

$$\frac{\partial h}{\partial t} = \frac{\rho g}{3\mu}\frac{1}{r}\frac{\partial}{\partial r}\left(rh^3\frac{\partial h}{\partial r}\right). \tag{1}$$

   Explain why we must also have

$$V = 2\pi \int_0^{a(t)} rh(r,t)\ \mathrm{d}r \tag{2}$$

   where $a(t)$ is the droplet radius. Use scaling arguments to show that

$$a(t) \sim \left(\frac{\rho g V^3}{\mu}t\right)^{1/8}.$$

   By finding a similarity solution of the governing equation (1) subject to (2) show that in fact

$$a(t) = \left(\frac{2^{10}}{3^5\pi^3}\right)^{1/8}\left(\frac{\rho g V^3}{\mu}t\right)^{1/8}.$$

2. Section B. For the time-dependent Landau–Levich problem in which the wall moves with speed $u(t)U$, in which U gives the scale and u is dimensionless, the outer problem is still capillary static, where the rate of change of $u(t)$ is sufficiently slow. Show that in the "turnaround" region (which is beyond the intermediate region where the meniscus equation and thin film equation match) the leading order equation generates

$$u(t)\bar{h} + \frac{\bar{h}^3}{3}\bar{h}_{\bar{z}\bar{z}\bar{z}} = u(t)\bar{h}_0,$$

   with boundary conditions

$$\bar{h} \to \bar{h}_0 \quad \text{as } \bar{z} \to \infty,$$

$$\bar{h} \sim \frac{\bar{z}^2}{\sqrt{2}} \qquad \text{as } \bar{z} \to -\infty.$$

By rescaling $\bar{h}$ and $\bar{z}$ to eliminate $u(t)$ and $\bar{h}_0$ deduce that

$$\bar{h}_0 = 0.948\, u(t)^{2/3}.$$

3. Section C. Consider an infinite film of fluid above a horizontal plate; after a suitable non-dimensionalisation, the film thickness, h, is governed by

$$h_t + \left[\frac{h^3}{3\mathrm{Ca}}\left(h_{xxx} - \mathrm{Bo}\; h_x\right)\right]_x = 0.$$

If we perturb the film slightly from a uniform thickness h_0 by setting $h = h_0 + \delta h_1(x,t)$ where $\delta \ll 1$ show that, to leading order in δ,

$$h_{1,t} + \frac{h_0^3}{3\mathrm{Ca}}\left(h_{1,xxxx} - \mathrm{Bo}\; h_{1,xx}\right) = 0.$$

Show that seeking a solution in the form $h_1 = \Re(e^{\sigma t + ikx})$ gives

$$\sigma = -\frac{h_0^3}{3\mathrm{Ca}}\left(k^4 + \mathrm{Bo}\; k^2\right),$$

and deduce that the film is stable to all perturbations.

Now consider a film lying underneath a horizontal plate. Show that the corresponding analysis gives

$$\sigma = -\frac{h_0^3}{3\mathrm{Ca}}\left(k^4 - \mathrm{Bo}\; k^2\right),$$

so that the gravity term is now destabilising but the surface tension term is stabilising. Sketch σ as a function of k, identifying which wavenumbers are unstable. Find the most unstable wavenumber.

4. Section A.

The coffee stain problem: Pinning effects

Consider the quasi-steady evaporation of a two dimensional blob of fluid with pinned contact lines at $x = \pm s$ and a evaporation rate per unit length of E.

By choosing an appropriate velocity scale, determine the non-dimensionalization that renders the evaporation rate equal to unity in the kinematic boundary condition, which you should write clearly.

Neglecting the role of gravity in the liquid pressure, show that the evolution of the profile of the droplet $h(x,t)$ is governed by

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\frac{h^3}{3\mathrm{Ca}}\frac{\partial^3 h}{\partial x^3}\right) = -1$$

where you should give the definition of Ca in terms of physical parameters.

Assuming that $\mathrm{Ca} \ll 1$ show that the fluid thickness satisfies

$$h_0 = \frac{3A(t)}{4s^3}(s^2 - x^2)$$

at leading order where $A(t)$ is the cross-sectional area of the droplet. Hence show that

$$\dot{A} = -2s.$$

and that the mean fluid velocity is

$$\bar{u} = \frac{2sx}{3A(t)} + O\left(\mathrm{Ca}\right).$$

Explain why

$$s^2 \ll \frac{\gamma}{\rho g}$$

is sufficient to allow the neglect of gravity, where γ and ρ are the surface tension and density of the liquid and g is the gravitational acceleration.

5. Section B. Use lubrication theory to show that for flow on a horizontal plane with the slip law $u = \lambda u_z$ the average fluid velocity is

$$\bar{u} \quad = \quad \frac{\gamma}{\mu}\left(\frac{h^2}{3} + \lambda h\right)\left(h_{xxx} - \ell_c^{-2} h_x\right)$$

for a constant ℓ_c^2 that you should define. Thus write down the associated thin film equation.

6. Section C. A simple model for a blink of an eye involves the time dependent Landau-Levich problem described in Exercise 3, with $u(t) = e^{-t}$ for $t > 0$. By rescaling $\bar{z}$ show that a dominant balance for the smallest film thickness scale (which is beyond the intermediate region where the meniscus equation and thin film equation match) is given by

$$h_t + u(t)h_z = 0, \tag{3}$$

with boundary condition

$$h(0,t) = 0.948\, u(t)^{2/3}.$$

Use the method of characteristics to show that the general solution of (3) is

$$h = G(z + e^{-t}),$$

for an arbitrary function G. Find the film height and deduce that as $t \to \infty$ the final film thickness is

$$h = 0.948\, z^{2/3}, \quad 0 < z < 1.$$

This provides an initial condition for a slow evolution due to gravity and surface tension on a longer timescale (between blinks).

Comments and corrections to Eamonn Gaffney