PS1 Q1

(a) $\{a_n\}_{n\in\mathbb{N}_0}$ is an asymptotic sequence as $\varepsilon\to 0$ iff

$$\frac{a_{n+1}(\varepsilon)}{a_n(\varepsilon)} \to 0 \text{ or/ } a_{n+1}(\varepsilon) = o(a_n(\varepsilon)) \quad \forall n\in\mathbb{N}_0.$$

(b) $\sum_{n=0}^{\infty} a_n(\varepsilon)$ is an asymptotic expansion of a function $f(\varepsilon)$ as $\varepsilon\to 0$

iff $$\frac{f(\varepsilon)-\sum_{n=0}^{N} a_n(\varepsilon)}{a_N(\varepsilon)} \to 0 \text{ or/ } f(\varepsilon)-\sum_{n=0}^{N} a_n(\varepsilon) = o(a_N(\varepsilon)) \quad \forall n\in\mathbb{N}_0$$

(c) $$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} \text{ for } |x|<1$$

$$\Rightarrow \log(1-\log\varepsilon) = \log\left(1+\log\left(\tfrac{1}{\varepsilon}\right)\right)$$
$$= \log\left(\log\left(\tfrac{1}{\varepsilon}\right)\right) + \log\left(1+\frac{1}{\log(1/\varepsilon)}\right)$$
$$\sim \log\left(\log\left(\tfrac{1}{\varepsilon}\right)\right) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n\log(1/\varepsilon)^n} \quad \text{as } \varepsilon\to 0^+$$

$\therefore$ $a_0 = \log(\log(\frac{1}{\varepsilon}))$ and $a_n = \dfrac{(-1)^n}{n\log(1/\varepsilon)^n}$ for $n\in\mathbb{N}$.

(d) $$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \text{ for } |x|<1$$

$$\Rightarrow \exp\left(\frac{-1}{\varepsilon^2+\varepsilon^3}\right) = \exp\left(-\frac{1}{\varepsilon^2}\sum_{n=0}^{\infty}(-1)^n\varepsilon^n\right)$$
$$= \exp\left(-\frac{1}{\varepsilon^2}\left(1-\varepsilon+\varepsilon^2-\varepsilon^3+\cdots\right)\right)$$
$$= \exp\left(-\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}-1+\sum_{n=1}^{\infty}(-1)^{n+1}\varepsilon^n\right)$$
$$= \exp\left(-\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}-1\right)\exp\left(\sum_{n=1}^{\infty}(-1)^{n+1}\varepsilon^n\right)$$
$$= \exp\left(-\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}-1\right)\sum_{n=0}^{\infty}\frac{1}{m!}\left(\sum_{n=1}^{\infty}(-1)^{n+1}\varepsilon^n\right)^m \quad \text{for } |\varepsilon|<1$$

$\therefore$ $a_n = b_n\varepsilon^n\left(-\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}-1\right)$ for $n\in\mathbb{N}_0$ where $b_n = O(1)$ as $\varepsilon\to 0$.

(a) $x^3 + x - \varepsilon = 0$ as $\varepsilon \to 0$

Iterative method:

$$\varepsilon = 0 \Rightarrow \quad x^3 + x = 0 \quad \Rightarrow \quad x = 0, \pm i$$

For the root near $x = 0$: rewrite as $x = \varepsilon - x^3$ ie $g(x;\varepsilon) = \varepsilon - x^3$

so that $x_{n+1} = g(x_n;\varepsilon) = \varepsilon - x_n^3$, with $x_0 = 0$.

Then

$$x_1 = \varepsilon$$

$$x_2 = \varepsilon - \varepsilon^3$$

$$x_3 = \varepsilon - (\varepsilon - \varepsilon^3)^3 \sim \varepsilon - \varepsilon^3 + 3\varepsilon^5 + \dots$$

$$x_4 = \varepsilon - (\varepsilon - \varepsilon^3 + 3\varepsilon^5 + \dots)^3 \sim \varepsilon - \varepsilon^3 + 3\varepsilon^5 + \dots$$

no change

$$\therefore \quad x \sim \varepsilon - \varepsilon^3 + 3\varepsilon^5 + \dots \quad \text{as } \varepsilon \to 0$$

For the roots close to $x = \pm i$: rewrite as $x^2 = \frac{\varepsilon}{x} - 1 \Rightarrow x = \pm i\sqrt{1 - \frac{\varepsilon}{x}}$

ie $g(x;\varepsilon) = \pm i\left(1 - \frac{\varepsilon}{x}\right)^{\frac{1}{2}}$ so that $x_{n+1} = \pm i\left(1 - \frac{\varepsilon}{x_n}\right)^{\frac{1}{2}}$ with $x_0 = \pm i$.

Then

$$x_1 = \pm i\left(1 \mp \frac{\varepsilon}{i}\right)^{\frac{1}{2}} \sim \pm i\left(1 \mp \frac{\varepsilon}{2i} \pm \dots\right) \sim \pm i - \frac{\varepsilon}{2} + \dots$$

$$x_2 = \pm i\left(1 - \frac{\varepsilon}{\pm i \mp \frac{\varepsilon}{2} + \dots}\right)^{\frac{1}{2}} \sim \pm i - \frac{\varepsilon}{2} \pm \frac{3i\varepsilon^2}{8} + \dots$$

$$x_3 = \pm i\left(1 - \frac{\varepsilon}{\pm i - \frac{\varepsilon}{2} \pm \frac{3i\varepsilon^2}{8} + \dots}\right)^{\frac{1}{2}} \sim \pm i - \frac{\varepsilon}{2} + \frac{3i\varepsilon^2}{8} + \dots$$

$$\therefore \quad x \sim \pm i - \frac{\varepsilon}{2} \pm \frac{3i\varepsilon^2}{8} + \dots \quad \text{as } \varepsilon \to 0$$

Expansion method $x \sim x_0 + \varepsilon x_1 + \cdots$ as $\varepsilon \to 0$

Substitute into $x^3 + x - \varepsilon = 0$:

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots)^3 + (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots) - \varepsilon = 0$$

$O(\varepsilon^0)$: $\quad x_0^3 + x_0 = 0 \quad \Rightarrow \quad x_0 = 0, +i, -i$

$O(\varepsilon^1)$: $\quad 3x_0^2 x_1 + x_1 - 1 = 0 \quad \Rightarrow \quad x_1 = \dfrac{1}{3x_0^2+1} = 1, -\frac{1}{2}, -\frac{1}{2}$

$O(\varepsilon^2)$: $\quad 3x_0 x_1^2 + 3x_0^2 x_2 + x_2 = 0 \quad \Rightarrow \quad x_2 = \dfrac{-3x_0 x_1^2}{3x_0^2+1} = 0, \frac{3i}{8}, -\frac{3i}{8}$

Hence for the roots close to $x = \pm i$, we have

$$x \sim \pm i - \frac{\varepsilon}{2} \pm \frac{3i\varepsilon^2}{8} + \ldots \quad \text{as } \varepsilon \to 0.$$

For the root close to $x=0$ we need to go to higher order:

$O(\varepsilon^3)$: $\quad x_1^3 + x_3 = 0 \quad \Rightarrow \quad x_3 = -x_1^3 = -1$

$O(\varepsilon^4)$: $\quad x_4 = 0 \quad \Rightarrow \quad x_4 = 0$

$O(\varepsilon^5)$: $\quad 3x_1^2 x_3 + x_5 = 0 \Rightarrow x_5 = -3x_1^2 x_3 = 3$

Hence, $\quad x \sim \varepsilon - \varepsilon^3 + 3\varepsilon^5 + \ldots \quad \text{as } \varepsilon \to 0.$

(b) $\varepsilon^3 x^2 + \varepsilon x + 1 = 0$

Analytic solution $x = \dfrac{-\varepsilon \pm \sqrt{\varepsilon^2 - 4\varepsilon^3}}{2\varepsilon^3} = \dfrac{-1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon^2}$

Expand for $|\varepsilon| < \frac{1}{4}$ to give $x = -\frac{1}{2\varepsilon^3}\left[-1 \left(1 - 2\varepsilon - 2\varepsilon^2 - 4\varepsilon^3 + \dots\right)\right]$

$$\therefore \quad x \sim \begin{cases} -\frac{1}{\varepsilon} - 1 - 2\varepsilon + \dots \\ -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} + 1 + \dots \end{cases} \quad \text{as } \varepsilon \to 0.$$

(expansions converge for $|\varepsilon| < \frac{1}{4}$)

Check via rescaling and expanding: we take $x = \dfrac{X}{\varepsilon^3}$ so that

$X^2 + X + \varepsilon = 0.$

Expand using $X = X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \cdots$ as $\varepsilon \to 0$

$$(X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \dots)^2 + (X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \dots) + \varepsilon = 0$$

$O(\varepsilon^0)$: $X_0^2 + X_0 = 0 \Rightarrow X_0 = -1, 0$

$O(\varepsilon^1)$: $2X_0X_1 + X_1 + 1 = 0 \Rightarrow X_1 = 1, -1$

$O(\varepsilon^2)$: $2X_0X_2 + X_1^2 + X_2 = 0 \Rightarrow X_2 = 1, -1$

$O(\varepsilon^3)$: $2X_0X_3 + 2X_1X_2 + X_3 = 0 \Rightarrow X_3 = 2, -2$

gives the same expansions as above.

NB No other scaling can be used to regularise this problem.

(c) $\varepsilon^2 x^3 + x^2 + 2x + \varepsilon = 0$

Find roots by taking different scalings which give different terms in the dominant balance...)

Balance terms ① and ② : $x = \frac{1}{\varepsilon^2} X \Rightarrow X^3 + X^2 + 2\varepsilon^2 X + \varepsilon^5 = 0$

Let $X = X_0 + \varepsilon X_1 + \cdots$ and substitute:

$O(\varepsilon^0)$: $X_0^3 + X_0^2 = 0 \Rightarrow X_0 = 0, 0, -1$

$O(\varepsilon^1)$: $3X_0^2 X_1 + 2X_0 X_1 = 0 \Rightarrow X_1 = ?, ?, 0$

$O(\varepsilon^2)$: $3X_0^2 X_2 + 3X_0 X_1^2 + 2X_0 X_2 + 2X_0 = 0 \Rightarrow X_2 = ?, ?, 2$

Hence $\exists$ a root of the form $x \sim -\frac{1}{\varepsilon^2} + 2 + \cdots$ as $\varepsilon \to 0$.

To find the other roots we need a different rescaling/dominant balance.

Balance terms ② and ③ : let $x = x_0 + \varepsilon x_1 + \cdots$ and substitute

$O(\varepsilon^0)$: $x_0^2 + 2x_0 = 0 \Rightarrow x_0 = 0, -2$

$O(\varepsilon^1)$: $2x_0 x_1 + 2x_1 = -1 \Rightarrow x_1 = -\frac{1}{2}, +\frac{1}{2}$

$O(\varepsilon^2)$: $x_0^3 + 2x_0 x_2 + x_1^2 + 2x_2 = 0 \Rightarrow x_2 = -\frac{1}{8}, -\frac{31}{8}$

Hence two further roots are $x \sim -2 + \frac{1}{2}\varepsilon + \cdots$

$x \sim -\frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \cdots$ as $\varepsilon \to 0$

Finally, balance terms ③ and ④ : let $x = \varepsilon X \Rightarrow \varepsilon^4 X^4 + \varepsilon X^2 + 2X + 1 = 0$

Let $X = X_0 + \varepsilon X_1 + \cdots$ and substitute

$O(\varepsilon^0)$: $2X_0 + 1 = 0 \Rightarrow X_0 = -\frac{1}{2}$

$O(\varepsilon^1)$: $X_0^2 + 2X_1 = 0 \Rightarrow X_1 = \frac{1}{8}$

Hence the final root is of the form $x \sim -\frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + \cdots$ as $\varepsilon \to 0$.

(a) $I(x) = \int_0^x e^{t^3}\,dt$ as $x \to \infty$

Try IBPs as is: let $I(x) = \int_0^x \underbrace{3t^2 e^{t^3}}_{\frac{dv}{dt}} \cdot \underbrace{\frac{1}{3t^2}}_{u}\,dt$

$\frac{dv}{dt} \Rightarrow v = e^{t^3}$ $\qquad u \Rightarrow \frac{du}{dt} = -\frac{2}{3}t^{-3}$

Then, $I(x) = \left[\frac{1}{3t^2} e^{t^3}\right]_0^x + \int_0^x \frac{2}{3} t^{-3} e^{t^3}\,dt = \infty$ ie this method fails.

So, re-write as $I(x) = \int_0^a e^{t^3}\,dt + \int_a^x e^{t^3}\,dt$ for some $a>0$

change variables: let $s = t^3 \Rightarrow I(x) = \frac{1}{3}\int_0^{x^3} s^{-\frac{2}{3}} e^s\,ds$

$\frac{ds}{dt} = 3t^2 = 3s^{\frac{2}{3}}$

Let $J_n(x) := \int_1^{x^3} \underbrace{s^{-n}}_{u} \underbrace{e^s}_{\frac{dv}{ds}}\,ds = \left[s^{-n}e^s\right]_1^{x^3} + \int_1^{x^3} n s^{-(n+1)} e^s\,ds = \frac{e^{x^3}}{x^{3n}} - e + nJ_{n+1}(x)$

$$\therefore J_{\frac{2}{3}}(x) = \frac{e^{x^3}}{x^2} - e + \frac{2}{3}J_{\frac{5}{3}}(x)$$

$$= \frac{e^{x^3}}{x^2} - e + \frac{2}{3}\left[\frac{e^{x^5}}{x^5} - e + \frac{5}{3}J_{\frac{8}{3}}(x)\right]$$

$$= \frac{e^{x^3}}{x^2} + \frac{2e^{x^3}}{3x^5} - \frac{5}{3}e + \frac{10}{9}\left[J_{\frac{8}{3}}(x) - e + \frac{8}{3}J_{\frac{11}{3}}(x)\right]$$

$$= \frac{e^{x^3}}{x^2} + \frac{2e^{x^3}}{3x^5} + \frac{10e^{x^3}}{9x^8} - \frac{25}{9}e + \frac{80}{27}J_{\frac{11}{3}}(x)$$

$$|J_{\frac{11}{3}}(x)| = \left|\int_1^{x^3} \frac{e^s}{s^{11/3}}\,ds\right| < \frac{e^s}{s^{11/3}}\Bigg|_{s=x^3} \int_1^{x^3} ds = \frac{e^{x^3}}{x^8} \text{ as } x \to \infty$$

Hence, $I(x) = \frac{1}{3}\int_0^1 s^{-\frac{2}{3}} e^s\,ds + \frac{1}{3}J_{\frac{2}{3}}(x)$

$$\Rightarrow I(x) = \frac{e^{x^3}}{3x^2} + \frac{2e^{x^3}}{9x^5} + O\left(\frac{e^{x^3}}{x^8}\right) \text{ as } x \to \infty$$

(b) $I(x) = \int_0^\infty \underbrace{te^{-t^2}}_{u} \underbrace{\cos(xt)}_{\frac{dv}{dt}}\,dt$ as $x \to \infty$.

$\frac{dv}{dt} = \quad v = \frac{1}{x}\sin(xt)$

$\Rightarrow \frac{du}{dt} = (1-2t^2)e^{-t^2}$

$$= \left[te^{-t^2} \cdot \frac{1}{x}\sin(xt) \right]_0^\infty - \int_0^\infty (1-2t^2)e^{-t^2} \cdot \frac{1}{x}\sin(xt)\,dt$$

$$= -\frac{1}{x}\int_0^\infty \underbrace{(1-2t^2)e^{-t^2}}_{u} \underbrace{\sin(xt)}_{\frac{dv}{dt}}\,dt$$

$\Rightarrow v = -\frac{1}{x}\cos(xt)$

$\frac{du}{dt} = -4t - 2t(1-2t^2))e^{-t^2} = (-6t + 4t^3)e^{-t^2}$

$$= -\frac{1}{x}\left\{ \left[(1-2t^2)e^{-t^2} \cdot -\frac{1}{x}\cos(xt) \right]_0^\infty + \int_0^\infty (6t-4t^3)e^{-t^2} \cdot \frac{1}{x}\cos(xt)\,dt \right\}$$

$$= -\frac{1}{x^2} + \frac{1}{x^2}\int_0^\infty \underbrace{(6t-4t^3)e^{-t^2}}_{u} \cos(xt)\,dt$$

$u \Rightarrow \frac{du}{dt} = (6-12t^2 - 2t(6t-4t^3))e^{-t^2}$

$\frac{dv}{dt} = \cos(xt) \Rightarrow v = \frac{1}{x}\sin(xt)$

$$= -\frac{1}{x^2} + \frac{1}{x^2}\left\{ \left[(6t-4t^3)e^{-t^2} \cdot \frac{1}{x}\sin(xt) \right]_0^\infty - \underbrace{\int_0^\infty \frac{1}{x}\sin(xt) \cdot \frac{du}{dt}\,dt}_{x^2 R(x)} \right\}$$

$$= -\frac{1}{x^2} + R(x)$$

$$|R(x)| = \frac{1}{x^3}\left| \int_0^\infty (6-24t^2+8t^4)e^{-t^2}\sin(xt)\,dt \right| \leq \frac{c}{x^3}$$

with $\quad c = \int_0^\infty |(6-24t^2+8t^4)e^{-t^2}|\,dt$

$\therefore \quad I(x) = -\frac{1}{x^2} + O\left(\frac{1}{x^3}\right)$ as $x \to \infty$.