PS2 Q1

As $x \to \infty$

$\int_0^{\pi/2} e^{ix\cos t}\,dt$ – method of stationary phase for first term (then method of steepest descents for more..)

$\int_0^1 \ln t\, e^{ixt}\,dt$ – method of steepest descents (note that using IBPs and the method of stationary phase does not work...)

$\int_0^x t^{-\frac{1}{2}} e^{-t}\,dt$ – write as $\int_0^\infty t^{-\frac{1}{2}}e^{-t}\,dt - \int_x^\infty t^{-\frac{1}{2}}e^{-t}\,dt$ and then use IBPs for the second integral

$\int_0^{\pi/2} e^{-x\sin^2 t}\,dt$ – Laplace's method

$\int_0^1 e^{ixe^{-1/t}}\,dt$ – method of steepest descents with $s = e^{-1/t}$

As $x \to 0^+$

$\int_0^{10} \frac{e^{-xt}}{1+t}\,dt$ – Taylor expand the integrand and integrate term-by-term

$\int_0^{\pi/2} \frac{1}{\sqrt{\cos^2 t + x\sin^2 t}}\,dt$ – write as $\int_0^{\pi/2-\delta} + \int_{\pi/2-\delta}^{\pi/2}$ where $x \ll \delta \ll 1$

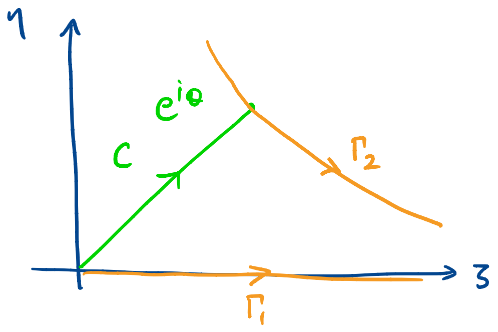
$\int_0^1 \frac{\sin(xt)}{t}\,dt$ – Taylor expand and integrate term-by-term.

$\int_x^\infty t^{a-1}e^{-t}\,dt$ – write as $\int_0^\infty - \int_0^x$ and then Taylor expand and integrate term-by-term for the second integral when $\mathrm{Re}(a) > 0$. (NB v. tricky o/w!)

$\int_0^1 \frac{\ln t}{x+t}\,dt$ – write as $\int_0^\delta + \int_\delta^1$ where $x \ll \delta \ll 1$.

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-z^2}\,dz = \frac{2r}{\sqrt{\pi}}\int_0^{e^{i\theta}} e^{-r^2t^2}\,dt \quad \text{with } z = re^{i\theta} \text{ and } s = rt$$

$$\varphi(t) = -t^2 = -(\xi + i\eta)^2 = \underbrace{\eta^2 - \xi^2}_{u(\xi,\eta)} - \underbrace{2\xi\eta i}_{v(\xi,\eta)}$$



Contour of steepest descent through $(0,0)$ is $\eta = 0$.

Contour of steepest descent through $t = e^{i\theta}$ is $2\xi\eta = 2\cos\theta\sin\theta = \sin 2\theta$

$(\theta \in (0, \pi/2) \Rightarrow \xi, \eta > 0)$

Then, by the deformation theorem,

$$\operatorname{erf}(z) = \Big(\int_{\Gamma_1} - \int_{\Gamma_2}\Big)\frac{2r}{\sqrt{\pi}} e^{r^2\varphi(t)}\,dt$$

$\Gamma_1 = \Gamma_1(r)$ $\quad \Gamma_2 = \Gamma_2(r,\theta)$

$$I_1(r) = \frac{2r}{\sqrt{\pi}}\int_0^\infty e^{-r^2\xi^2}\,d\xi = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2}\,du = 1$$

with $u = r\xi$

For I_2, we have

$$\Gamma_2 = \Big\{ \overbrace{\xi + i\,\underbrace{\frac{\sin 2\theta}{2\xi}}}^{t},\ \xi > \cos\theta \Big\}$$

$2\xi\eta = \sin 2\theta$
$\Rightarrow \eta = \dfrac{\sin 2\theta}{2\xi}$

and

$$\varphi(t) = \overbrace{\eta^2 - \xi^2}^{u} \overset{+}{-} \overbrace{2\xi\eta i}^{iv} = \eta^2 - \xi^2 - i\sin 2\theta$$

which gives

$$I_2(r,\theta) = \frac{2r}{\sqrt{\pi}}\int_{\cos\theta}^\infty e^{r^2(\eta^2-\xi^2-i\sin 2\theta)}\cdot(1+\eta'(\xi)i)\,d\xi$$

$$\eta'(\xi) = -\frac{\sin 2\theta}{2\xi^2}$$

$$= \frac{2r}{\sqrt{\pi}} e^{-r^2 i\sin 2\theta}\int_{\cos\theta}^\infty F(\xi)e^{r^2\Phi(\xi)}\,d\xi$$

$$F(\xi) = 1 - \frac{i\sin 2\theta}{2\xi^2}$$

$$\Phi(\xi) = \frac{\sin^2 2\theta}{4\xi^2}$$

Since Γ_2 is a contour of steepest descent then $\Phi(\xi)$ is a decreasing function of ξ on Γ_2

$\Rightarrow$ Apply Laplace's method to give

$$I_2(r,\theta) = \frac{-2r}{\sqrt{\pi}} e^{-r^2 i\sin 2\theta} \frac{F(\cos\theta)\, e^{r^2 \Phi(\cos\theta)}}{r^2\, \Phi'(\cos\theta)} \quad \text{as } r\to\infty.$$

$$F(\cos\theta) = \frac{e^{-i\theta}}{\cos\theta}, \quad \Phi(\cos\theta) = -\cos 2\theta, \quad \Phi'(\cos\theta) = \frac{-2}{\cos\theta}$$

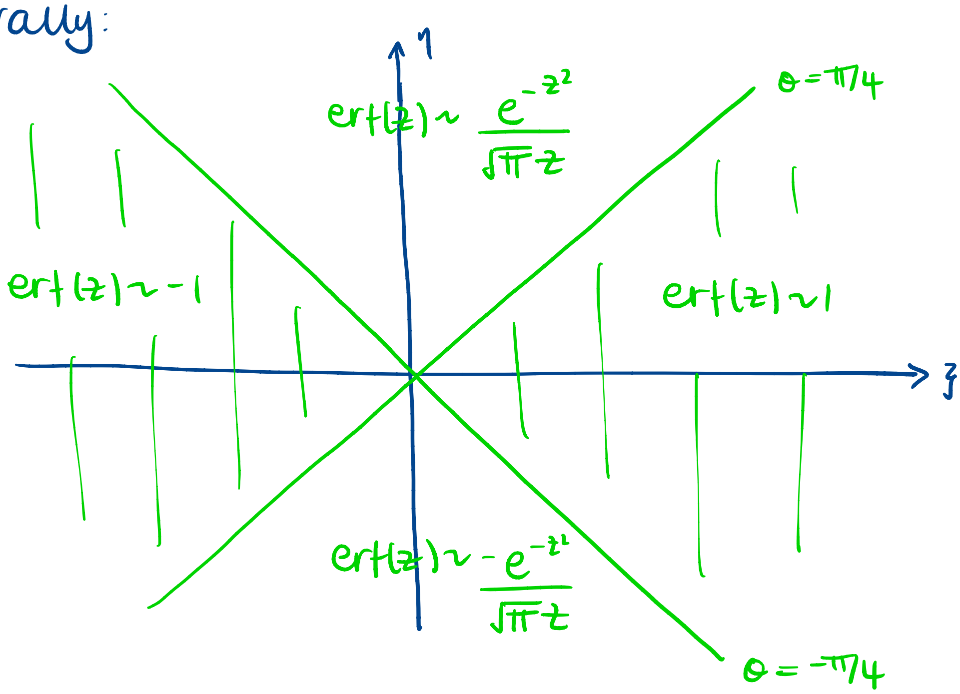
$$\Rightarrow \quad I_2(r,\theta) \sim \frac{1}{\sqrt{\pi}\, r e^{i\theta}}\, e^{-r^2 e^{2i\theta}} \quad \text{as } r\to\infty.$$

Hence $I_1(r) \sim 1$ and $I_2(r,\theta) \sim \frac{1}{\sqrt{\pi} z} e^{-z^2}$ as $r = |z| \to \infty$ for $0 < \theta = \arg(z) < \frac{\pi}{2}$

$$|I_2(r)| \sim \frac{1}{r} e^{-r^2\cos 2\theta} = \begin{cases} \ll 1 & \text{for } 0<\theta\leq\frac{\pi}{4} \\ \gg 1 & \text{for } \frac{\pi}{4}<\theta<\frac{\pi}{2} \end{cases} \quad \text{as } z\to\infty$$

$$\therefore \ \operatorname{erf}(z) = \begin{cases} 1 & \text{for } 0<\theta\leq\frac{\pi}{4} \\ -\frac{1}{\sqrt{\pi} z} e^{-z^2} & \text{for } \frac{\pi}{4}<\theta<\frac{\pi}{2} \end{cases} \quad \text{as } |z|\to\infty.$$

More generally:



- Different asymptotic expansions in different regions → Stokes Phenomena
- While e^{-z^2} is active, it has an essential singularity at ∞ ↑
- $\theta = \pm\frac{\pi}{2}$ – Stokes' lines (across which topology of SD contour changes).
- $|\theta| = \frac{\pi}{4}, \frac{5\pi}{4}$ – anti Stokes' lines (across which dominance of end point and saddle point changes).

PS2 Q6

$$I(\varepsilon) = \int_0^1 \frac{f(x)}{x+\varepsilon}\,dx \quad \text{as } \varepsilon \to 0^+ \text{ with } f \text{ smooth.}$$

$$= \underbrace{\int_0^\delta \frac{f(x)}{x+\varepsilon}\,dx}_{I_1} + \underbrace{\int_\delta^1 \frac{f(x)}{x+\varepsilon}\,dx}_{I_2} \quad \text{where } 0<\varepsilon \ll \delta \ll 1$$

$$I_1(\varepsilon) = \int_0^{\delta/\varepsilon} \frac{f(\varepsilon y)}{y+1}\cdot dy \quad (\text{Letting } x = \varepsilon y)$$

$$= \int_0^{\delta/\varepsilon} \frac{1}{y+1}\left[f(0) + \varepsilon y f'(0) + O(\varepsilon^2)\right] dy$$

ok since $\varepsilon y \ll \delta \ll 1$.

$$= \left[f(0)\ln(y+1)\right]_0^{\delta/\varepsilon} + O(\delta)$$

$O(\varepsilon \cdot \frac{\delta}{\varepsilon})$

$$= f(0)\ln\left(1+\frac{\delta}{\varepsilon}\right) + O(\delta)$$

$$= f(0)\ln\left(\frac{\delta}{\varepsilon}\right) + f(0)\ln\left(1+\frac{\varepsilon}{\delta}\right) + O(\delta)$$

$$= -f(0)\ln\varepsilon + f(0)\ln\delta + O\left(\delta, \frac{\varepsilon}{\delta}\right)$$

$$I_2(\varepsilon) = \int_\delta^1 \frac{f(x)}{x+\varepsilon}\,dx$$

$$= \int_\delta^1 \frac{f(x)}{x(1+\varepsilon/x)}\,dx$$

$$= \int_\delta^1 \frac{f(x)}{x}\left(1-\frac{\varepsilon}{x} + O(\varepsilon^2)\right)dx$$

ok since $\frac{\varepsilon}{x} < \frac{\varepsilon}{\delta} \ll 1$

$$= \int_\delta^1 \frac{f(x)-f(0)}{x}\,dx + \int_\delta^1 \frac{f(0)}{x}\,dx + \dots$$

$$= \int_\delta^1 \frac{f(x)-f(0)}{x}\,dx - f(0)\ln\delta + \dots$$

$$\therefore\ I(\varepsilon) \sim -f(0)\ln\varepsilon + f(0)\ln\delta + \int_\delta^1 \frac{f(x)-f(0)}{x}\,dx - f(0)\ln\delta + \dots$$

$$\sim -f(0)\ln\varepsilon + \int_0^1 \frac{f(x)-f(0)}{x}\,dx + \dots \quad \text{as } \varepsilon \to 0^+.$$