PS3 Q1



(a) Van Dyke's matching rule $(m t i)(n t o) = (n t o)(m t i)$

↳ n terms of the outer solution, written in the inner variable and then expanded to m terms, is the same as m terms of the inner solution, written in terms of the outer variable and then expanded to n terms.

(b) $f(x;\varepsilon) = [1+(x+\varepsilon)^{1/2}]^{1/2}$

$\varepsilon \to 0^+$ with $x = O(1) \Rightarrow f(x;\varepsilon) = [1 + x^{\frac{1}{2}}(1+\varepsilon/x)^{1/2}]^{1/2}$

$$\sim [1 + x^{\frac{1}{2}}(1 + \tfrac{\varepsilon}{2x} + \cdots)]^{\frac{1}{2}}$$

$$= [(1+x^{\frac{1}{2}}) + \tfrac{\varepsilon}{2x^{1/2}} + \cdots]^{\frac{1}{2}}$$

$$= (1+x^{\frac{1}{2}})^{\frac{1}{2}}\left[1 + \frac{\varepsilon}{2x^{1/2}(1+x^{1/2})} + \cdots\right]^{\frac{1}{2}}$$

$$\sim (1+x^{\frac{1}{2}})^{\frac{1}{2}}\left[1 + \frac{\varepsilon}{4x^{1/2}(1+x^{1/2})} + \cdots\right]$$

$$= (1+x^{1/2})^{1/2} + \frac{\varepsilon}{4x^{1/2}(1+x^{1/2})^{1/2}}$$

$\therefore\ (1 t o) = (1+x^{1/2})^{1/2}$

$(2 t o) = (1+x^{1/2})^{1/2} + \dfrac{\varepsilon}{4x^{1/2}(1+x^{1/2})^{1/2}}$

$\varepsilon \to 0^+$ with $X = \frac{x}{\varepsilon}$ and $X = O(1) \Rightarrow f(\varepsilon X;\varepsilon) = [1 + (\varepsilon X + \varepsilon)^{1/2}]^{1/2}$

$$= [1 + \varepsilon^{1/2}(X+1)^{1/2}]^{1/2}$$

$$\sim 1 + \tfrac{1}{2}\varepsilon^{1/2}(X+1)^{1/2} + \cdots$$

$\therefore\ (1 t i) = 1$

$(2 t i) = 1 + \varepsilon^{1/2}(X+1)^{1/2}$

$(m,n) = (1,1)$

$(1 t o) = (1+x^{1/2})^{1/2}$

$$= (1 + (\varepsilon X)^{1/2})^{1/2}$$

$$\sim 1 + \tfrac{1}{2}\varepsilon^{1/2}X^{1/2} + \cdots$$

$(m,n) = (1,1)$

$$(1t_o) = (1+x^{1/2})^{1/2}$$
$$= (1+(\varepsilon X)^{1/2})^{1/2}$$
$$\sim 1+\tfrac{1}{2}\varepsilon^{1/2}X^{1/2}+\ldots$$
$$(1t_i)(1t_o) = \underline{1}$$

$$(1t_i) = 1$$
$$(1t_o)(1t_i) = \underline{1}$$

hence $(1t_o)(1t_i) = (1t_i)(1t_o)$ ✓✓

$(m,n) = (1,2)$

$$(2t_o) = (1+x^{1/2})^{1/2} + \frac{1}{4x^{1/2}(1+x^{1/2})^{1/2}}$$
$$= (1+(\varepsilon X)^{1/2})^{1/2} + \frac{1}{4(\varepsilon X)^{1/2}(1+(\varepsilon X)^{1/2})^{1/2}}$$

expand

$$\sim 1+\varepsilon^{1/2}X^{1/2} + \frac{\varepsilon^{1/2}}{4X^{1/2}}$$
$$(1t_i)(2t_o) = 1$$

$$(1t_i) = 1$$
$$\Rightarrow (2t_o)(1t_i) = 1$$

hence, $(1t_i)(2t_o) = (2t_o)(1t_i)$ ✓✓

$(m,n) = (2,1)$

$$(1t_o) = (1+x^{1/2})^{1/2}$$
$$= (1+(\varepsilon X)^{1/2})^{1/2}$$
$$\sim 1+\tfrac{1}{2}\varepsilon^{1/2}X^{1/2}+\ldots$$
$$(2t_i)(1t_o) = 1+\tfrac{1}{2}\varepsilon^{1/2}X^{1/2}$$

$$(2t_i) = 1+\tfrac{1}{2}\varepsilon^{1/2}(X+1)^{1/2}$$
$$= 1+\tfrac{1}{2}\varepsilon^{1/2}(x/\varepsilon+1)^{1/2}$$
$$= 1+\tfrac{1}{2}x^{1/2}(1+\varepsilon/x)^{1/2}$$
$$\sim 1+\tfrac{1}{2}x^{1/2}+\ldots$$
$$(1t_o)(2t_i) = 1+\tfrac{1}{2}x^{1/2}$$

Hence $(2t_i)(1t_o) = (1t_o)(2t_i)$ ✓✓

$(m,n) = (2,2)$

$$(2t_o) = (1+x^{1/2})^{1/2} + \frac{1}{4x^{1/2}(1+x^{1/2})^{1/2}}$$
$$= (1+(\varepsilon X)^{1/2})^{1/2} + \frac{1}{4(\varepsilon X)^{1/2}(1+(\varepsilon X)^{1/2})^{1/2}}$$
$$\sim 1+\tfrac{1}{2}\varepsilon^{1/2}X^{1/2} + \frac{\varepsilon^{1/2}}{4X^{1/2}}+\ldots$$
$$= 1+\varepsilon^{1/2}\left(\tfrac{1}{2}X^{1/2}+\frac{1}{4X^{1/2}}\right)+\ldots$$

$$(2t_i)(2t_o)$$
$$= 1+\varepsilon^{1/2}\left(\tfrac{1}{2}X^{1/2}+\frac{1}{4X^{1/4}}\right)$$

$$(2ti) = 1 + \tfrac{1}{2}\varepsilon^{1/2}(X+1)^{1/2}$$
$$= 1 + \tfrac{1}{2}\varepsilon^{1/2}(x/\varepsilon + 1)^{1/2}$$
$$\sim 1 + \tfrac{1}{2}x^{1/2} + \frac{\varepsilon}{4x^{1/2}} + \cdots$$

Hence $(2ti)(2to) = (2to)(2ti)$ ✓✓.

$$(2to)(2ti) = 1 + \tfrac{1}{2}x^{1/2} + \frac{\varepsilon}{4x^{1/2}}$$

(c) $g(x) = 1 + \dfrac{\log x}{\log \varepsilon}$ with $\varepsilon \to 0^+$, $x = O(1)$ and $X = \frac{x}{\varepsilon}$ with $X \sim O(1)$.

$$g(x;\varepsilon) \sim \begin{cases} 1 + \dfrac{\log x}{\log \varepsilon} & \text{as } \varepsilon \to 0^+ \text{ with } x = O(1) \\ 2 + \dfrac{\log X}{\log \varepsilon} & \text{as } \varepsilon \to 0^+ \text{ with } X = O(1) \text{ and } X = \frac{x}{\varepsilon} \end{cases}$$

Then, $(1to) = 1$ and $(1ti) = 2$ $\Rightarrow$ $(1ti)(1to) = 1 \neq 2 = (1to)(1ti)$.

We can resolve the situation by treating $\log\varepsilon$ as $O(1)$ in the matching procedure:

$$(1to) = 1 + \frac{\log x}{\log\varepsilon} = 1 + \frac{\log(\varepsilon X)}{\log\varepsilon} = 2 + \frac{\log X}{\log \varepsilon} = (1ti)(1to)$$

$$(1ti) = 2 + \frac{\log X}{\log\varepsilon} = 2 + \frac{\log(x/\varepsilon)}{\log\varepsilon} = 1 + \frac{\log x}{\log\varepsilon} = (1to)(1ti)$$

These are now equal ☺

$y'' + \varepsilon y' = 0$ as $\varepsilon \to 0^+$ with $0 < x < L$ and $y(0) = 0$, $y(L) = 1$.

(a) Suppose $L = O(1)$ as $\varepsilon \to 0^+$. Let $y = y_0 + \varepsilon y_1 + \dots$ as $\varepsilon \to 0^+$

$O(\varepsilon^0)$: $y_0'' = 0$ with $y_0(0) = 0$, $y_0(L) = 1 \Rightarrow y_0 = \frac{x}{L}$.

$O(\varepsilon^1)$: $y_1'' + y_0' = 0$ for $0 < x < 1$ with $y_1(0) = 0$, $y_1(L) = 0$

$\Rightarrow y_1'' = -\frac{1}{L} \quad \therefore \quad y_1 = \frac{1}{2L} x(L - x)$

$\therefore y(x) \sim \frac{x}{L} + \varepsilon \cdot \frac{1}{2L} x(L-x) + \dots$ as $\varepsilon \to 0^+$ with $L = O(1)$.

(b) Note that the expansion is not valid for $L \gg \frac{1}{\varepsilon}$ (and hence in the large $L$ ($L \to \infty$) limit).

Differentiating gives $y'(x) \sim \frac{1}{L} + \frac{\varepsilon}{2L}(L - 2x) + \dots$ as $\varepsilon \to 0^+$ with $L = O(1)$

$\Rightarrow y'(0) \sim \frac{1}{L} + \frac{\varepsilon}{2} + \dots$ as $\varepsilon \to 0^+$

So this expansion is not valid when $\frac{\varepsilon}{L} = o(1)$ as $\varepsilon \to 0^+$.
This corresponds to a distinguished limit in which we have $L = \frac{\ell}{\varepsilon}$ with $\ell = O(1)$ as $\varepsilon \to 0^+$.

Scaling $x = \frac{X}{\varepsilon}$ and $y = Y(X) \Rightarrow Y'' + Y' = 0$ for $0 < X < \ell$

with $Y(0) = 0$, $Y(\ell) = 1$. Hence $Y(X) = \frac{1 - e^{-X}}{1 - e^{-L}}$.

In this case (we have scaled properly) we have

$Y'(0) = \frac{1}{1 - e^{-\ell}} \to 1$ as $\ell \to \infty$. This agrees with what is

obtained from the exact soln $y = \frac{1 - e^{-\varepsilon x}}{1 - e^{-\varepsilon L}} \Rightarrow y'(0) = \varepsilon$ as $L \to \infty$.

(a) $\varepsilon\nabla^2 u = u$ in $r^2 = x^2+y^2 < 1$ with $u=1$ on $r=1$ as $\varepsilon \to 0^+$

OUTER: $u \sim u_0 + \varepsilon u_1 + \dots$ as $\varepsilon \to 0^+$ with $1-r \sim O(1)$

$O(\varepsilon^0)$: $u_0 = 0$

$O(\varepsilon^1)$: $u_1 = \nabla^2 u_0 \Rightarrow u_1 = 0$

$O(\varepsilon^2)$: $u_2 = \nabla^2 u_1 \Rightarrow u_2 = 0$

$\Big\}$ $u = o(\varepsilon^n)\ \forall n \in \mathbb{N}$ as $\varepsilon \to 0^+$.

INNER: $u(r,\theta) = U(R,\theta)$ with $r = 1-\delta(\varepsilon)R$ with $\delta(\varepsilon) \to 0$ and $R = O(1)$ as $\varepsilon \to 0^+$.

$$\Rightarrow \underbrace{\frac{\varepsilon}{\delta^2} U_{RR}}_{①} - \underbrace{\frac{\varepsilon}{\delta(1-\delta R)} U_R}_{②} + \underbrace{\frac{\varepsilon}{(1-\delta R)^2} U_{\theta\theta}}_{③} - \underbrace{U}_{④} = 0$$

① – ④ balance by setting $\delta = \varepsilon^{\frac{1}{2}}$

$$\Rightarrow U_{RR} - \frac{\varepsilon^{\frac{1}{2}}}{(1-\varepsilon^{\frac{1}{2}}R)} U_R + \frac{\varepsilon}{(1-\varepsilon^{1/2}R)^2} U_{\theta\theta} - U = 0$$

Expand: $U \sim U_0(R,\theta) + \varepsilon^{\frac{1}{2}} U_1(R,\theta) + \dots$ as $\varepsilon \to 0^+$ with $R = O(1)$.

$O(\varepsilon^0)$: $U_{0_{RR}} - U_0 = 0$ in $R > 0$ with $U_0 = 1$ on $R = 0$

$\Rightarrow U_0 = Ae^{R} + (1-A)e^{-R}$ $(A \in \mathbb{R})$

Matching: $(1 t o) = 0 \Rightarrow (1 t i)(1 t o) = 0$

$\Rightarrow (1 t o)(1 t i) = 0$ (by VDMR)

$\Rightarrow U_0 \to 0$ as $R \to \infty$

$= A = 0$

$\therefore u = e^{-R} + O(\varepsilon^{1/2})$ as $\varepsilon \to 0^+$ with $\varepsilon^{1/2}(1-r) = R = O(1)$.

Exact solution: $u = \dfrac{I_0(r/\sqrt{\varepsilon})}{I_0(1/\sqrt{\varepsilon})}$

$$I_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(ix\sin\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{\pi} \left(e^{-i(ix\sin\theta)} + e^{+i(ix\sin\theta)}\right)d\theta$$

$$= \frac{1}{2\pi}\int_0^{\pi} \left(e^{x\sin\theta} + e^{-x\sin\theta}\right)d\theta$$

$$\sim \frac{1}{2\pi}\int_0^{\pi} e^{x\sin\theta}\,d\theta \quad \text{as } x\to\infty$$ (1st term dominates because $\sin\theta > 0$ on $(0,\pi)$)

$$\sim \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{x[1-\frac{1}{2}(\theta-\pi/2)^2+\dots]}\,d\theta$$ (use Laplace's method because $\varphi(\theta)=\sin\theta$ has a maximum at $\theta=\frac{\pi}{2}$)

$$\sim \frac{e^x}{2\pi}\int_{-\infty}^{\infty} e^{-xs^2/2}\,ds$$ $\leftarrow$ $(\theta-\pi/2=s)$

$$= \frac{e^x}{\sqrt{2\pi}}\sqrt{\frac{2}{x}}\underbrace{\int_{-\infty}^{\infty} e^{-t^2}\,dt}_{=\sqrt{\pi}}$$ $\leftarrow$ $\left(s=\sqrt{\frac{2}{x}}\,t\right)$

$$\therefore\ I_0(x) \sim \frac{e^x}{\sqrt{2\pi x}} \quad \text{as } x\to\infty.$$

Hence $u \sim \dfrac{1}{\sqrt{r}}e^{-(1-r)/\sqrt{\varepsilon}}$ as $\varepsilon\to 0^+$ with $r=O(1)$, $1-r=O(1)$

$u \sim \dfrac{\sqrt{2\pi}\,e^{-1/\sqrt{\varepsilon}}}{\varepsilon^{1/4}}\,I_0(\rho)$ as $\varepsilon\to 0^+$ with $\rho=\varepsilon^{-\frac{1}{2}}r=O(1)$

$r = 1-\varepsilon^{1/2}R \Rightarrow 1-r=\varepsilon^{1/2}R$

$u \sim \dfrac{1}{\sqrt{1-\varepsilon^{1/2}R}}\,e^{-R} = e^{-R}+O(\varepsilon^{\frac{1}{2}})$ as $\varepsilon\to 0^+$ with $R=\varepsilon^{-\frac{1}{2}}(1-r) = O(1)$

$\Rightarrow$ consistent with the result from BL expansion.

(b) $\varepsilon \nabla^2 u = u_x$ in $y>0$ with $u=1$ on $y=0,\ x>0$

$u_y = 0$ on $y=0,\ x<0$

$u \to 0$ as $x^2+y^2 \to \infty,\ y>0$

OUTER: $u \sim u_0 + \varepsilon u_1 + \dots$ as $\varepsilon \to 0^+$ with $x, y = O(1)$.

$O(\varepsilon^0)$: $u_{0x} = 0$ with $u_0 = 0$ at $\infty$ $\Rightarrow$ $u_0 \equiv 0$.

$O(\varepsilon^1)$: $u_{1x} = 0$ with $u_1 = 0$ at $\infty$ $\Rightarrow$ $u_1 \equiv 0$

$\Big\}$ $u = o(\varepsilon^n)\ \forall n \in \mathbb{N}$ as $\varepsilon \to 0^+$ with $x, y = O(1)$.

INNER: $u(x,y) = U(x,Y)$ with $y = \delta(\varepsilon) Y$ and $\delta \to 0$, $Y = O(1)$ as $\varepsilon \to 0^+$

$$\Rightarrow \varepsilon U_{xx} + \underbrace{\frac{\varepsilon}{\delta^2} U_{YY} - U_x}_{\text{Balance}} = 0$$

Balance $\Rightarrow$ $\delta = \varepsilon^{\frac{1}{2}}$ $\Rightarrow$ $\varepsilon U_{xx} + U_{YY} - U_x = 0$.

$U \sim U_0 + \varepsilon U_1 + \dots$ as $\varepsilon \to 0^+$ with $Y = O(1)$.

$O(\varepsilon^0)$: $U_{0YY} - U_{0x} = 0$ in $Y>0,\ x>0$ with $U_0(x,0) = 1$ for $x>0$

Matching: $(1t0) = 0 \Rightarrow (1ti)(1t0) = 0$

$\Rightarrow (1t0)(1ti) = 0$ $\quad$ $\downarrow$ VDMR

$\Rightarrow U_0 \to 0$ as $Y \to \infty$ for $x>0$

Seek a similarity solution $U_0 = f(\eta)$ with $\eta = Y/\sqrt{x}$.

Substituting: $\eta_x = -\dfrac{\eta}{2x}$, $\eta_Y = \dfrac{1}{x^{1/2}}$

$$\therefore\ U_{0x} = f'(\eta)\eta_x = -\frac{\eta f'(\eta)}{2x}$$

$$U_{0YY} = f''(\eta)\eta_Y^2 = \frac{f''(\eta)}{x}$$

$\Big\}$ $\Rightarrow$ $f'' + \frac{1}{2}\eta f' = 0 \quad (\eta > 0)$

BCs $U_0 = 1$ on $Y=0,\ x>0$ $\Rightarrow f(0) = 1$

$U_0 \to 0$ as $Y \to \infty,\ x>0$ $\Rightarrow f(\infty) = 0$

$$\therefore \quad \frac{f''(\eta)}{f'(\eta)} = -\tfrac{1}{2}\eta \quad \Rightarrow \quad \ln|f'(\eta)| = c_1 - \tfrac{1}{4}\eta^2 \qquad (c_1 \in \mathbb{R})$$

$$\therefore \quad f'(\eta) = e^{c_1 - \frac{1}{4}\eta^2}$$

$$f(\eta) = c_2 - c_1 \int_\eta^\infty e^{-\frac{1}{4}s^2}\, ds$$

$\downarrow$ $s = 2t$

$$= c_2 - 2c_1 \int_{\eta/2}^\infty e^{-t^2}\, dt$$

$$= c_2 - 2c_1 \operatorname{erfc}(\eta/2)$$

$$\left.\begin{aligned} f(\infty) = 0 &\Rightarrow c_2 = 0 \\ f(0) = 1 &\Rightarrow 1 = -2c_1 \operatorname{erfc}(0) = -2c_2 \end{aligned}\right\} \Rightarrow f(\eta) = \operatorname{erfc}(\eta/2)$$

$$\therefore \quad u = \operatorname{erfc}\left(\frac{y}{2\sqrt{x}}\right) + o(\varepsilon) \quad \text{as } \varepsilon \to 0^+ \quad \text{with } Y = \varepsilon^{-\frac{1}{2}} y = O(1)$$

and $x = O(1)$

Neither approximation holds for $X = \frac{x}{\varepsilon} = O(1)$, $Y = \frac{y}{\varepsilon} = O(1)$

$$\Rightarrow \quad u_{XX} + u_{YY} = u_X \quad \text{in } Y > 0.$$