1a) $\ddot{x} + \varepsilon \dot{x} + x = 0$ as $\varepsilon \to 0^+$

let $x = x(t, T)$ with $T = \varepsilon t$ $\Rightarrow$ $\frac{d}{dt} \mapsto \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T}$

substituting: $x_{tt} + 2\varepsilon x_{tT} + \varepsilon^2 x_{TT} + \varepsilon(x_t + \varepsilon x_T) + x = 0$

Expand: $x \sim x_0(t, \tau) + \varepsilon x_1(t, \tau) + \cdots$ and collect terms

$O(\varepsilon^0)$: $x_{0tt} + x_0 = 0$ $\Rightarrow$ $x_0 = \frac{1}{2}(A(T)e^{it} + \bar{A}(T)e^{-it})$

$\therefore$ $x \sim \frac{1}{2}(A(T)e^{it} + \bar{A}(t)e^{-it})$ as $\varepsilon \to 0^+$ with $T = \varepsilon t = O(1)$.

$O(\varepsilon^1)$: $x_{1tt} + x_1 = -2x_{0tT} - x_{0t}$

$$= -(iA_T e^{it} - i\bar{A}_T e^{-it}) - \frac{1}{2}(iAe^{it} - i\bar{A}e^{-it})$$

$$= -i(A_T + \tfrac{1}{2}A)e^{it} + \text{c.c.}$$

Suppress secular terms ($e^{\pm it}$) $\Leftrightarrow$ $A_T + \frac{1}{2}A = 0$

Let $A = Re^{i\theta}$ so that $R_T e^{i\theta} + Ri\theta_T e^{i\theta} + \frac{1}{2}Re^{i\theta} = 0$

$\therefore$ $\theta_T = 0 \Rightarrow \theta = \text{constant} = \theta_0$ $\quad(\theta_0, R_0 \in \mathbb{R})$

$R_T = -\frac{1}{2}R \Rightarrow R = R_0 e^{-\frac{1}{2}T}$

$\therefore$ $x_0 = R_0 e^{-\frac{1}{2}T}\cos(t + \theta_0)$

Exact solution: $x = r_0 e^{-\frac{1}{2}\varepsilon t}\cos\left((1 - \frac{\varepsilon^2}{4})^{1/2} t + \theta_0\right)$ $\quad(r_0, \theta_0 \in \mathbb{R})$

$$\sim r_0 e^{-\frac{1}{2}T}\cos(t + \theta_0 - \tfrac{1}{8}\varepsilon^2)$$

$\therefore$ $x - x_0 \sim O(\varepsilon)$ for $t = O(\frac{1}{\varepsilon})$.

(b) $\ddot{x} + x = \varepsilon x^3$ as $\varepsilon \to 0^+$.

Let $x = x(t, T)$ with $T = \varepsilon t$, and write $x \sim x_0(t,T) + \varepsilon x_1(t,T) + \dots$

$$x_{tt} + 2\varepsilon x_{tT} + \varepsilon^2 x_{TT} + x = \varepsilon x^3$$

Substitute and collect terms.

$O(\varepsilon^0)$: $x_{0tt} + x_0 = 0 \Rightarrow x_0(t,T) = \frac{1}{2}\left(A(T)e^{it} + \bar{A}(t)e^{-it}\right)$

$O(\varepsilon^1)$: $x_{1tt} + x_{1t} = -2x_{0tT} - x_0^3$

$$= -\left(iA_T e^{it} - i\bar{A}_T e^{-it}\right) - \frac{1}{8}\left(Ae^{it} + \bar{A}e^{-it}\right)^3$$

$$= \left[-iA_T + \frac{3}{8}A^2\bar{A}\right]e^{it} + \text{c.c.} + \text{non-secular terms.}$$

Hence to suppress secular terms we need $iA_T = \frac{3}{8}A^2\bar{A}$

Let $A = Re^{i\theta} \Rightarrow i(R_T + iR\theta_T) = \frac{3}{8}R^3$

$\therefore$ $R_T = 0 \Rightarrow R(T) = R_0$

$R\theta_T = -\frac{3}{8}R^3 \Rightarrow \theta_T = -\frac{3}{8}R_0^2 \Rightarrow \theta = -\frac{3}{8}R_0^2 T + \theta_0$

$$\Rightarrow A(T) = R_0 e^{i(\theta_0 - \frac{3}{8}R_0^2 T)}$$

$$= A_0 e^{-\frac{3}{8}|A_0|^2 T}$$

(c) $\ddot{x} + \varepsilon(x^2 - \mu)\dot{x} + x = 0$ as $\varepsilon \to 0^+$

let $x = x(t, T)$ with $T = \varepsilon t$ and write $x = x_0(t,T) + \varepsilon x_1(t,T) + \dots$

$$x_{tt} + 2\varepsilon x_{tT} + \varepsilon^2 x_{TT} + \varepsilon(x^2 - \mu)(x_t + \varepsilon x_T) + x = 0.$$

$O(\varepsilon^0)$: $\quad x_{0_{tt}} + x_0 = 0 \Rightarrow x_0(t,T) = \frac{1}{2}\left(A(T)e^{it} + \bar{A}(T)e^{-it}\right)$

$$\begin{aligned} O(\varepsilon^1): \quad x_{1_{tt}} + x_1 &= -2x_{0_{tT}} - (x_0^2 - \mu)x_{0_t} \\ &= -(iA_T e^{it} - i\bar{A}_T e^{-it}) \\ &\quad - \left[\tfrac{1}{4}(Ae^{it} + \bar{A}e^{-it})^2 - \mu\right](iA_T e^{it} - i\bar{A}_T e^{-it}) \\ &= \left[-iA_T - \tfrac{1}{4}A^2\left(-\tfrac{i}{2}\bar{A}\right) - \left(\tfrac{2}{4}A\bar{A} - \mu\right)\tfrac{iA}{2}\right]e^{it} + \text{c.c.} \\ &\quad + \text{non-secular terms} \end{aligned}$$

Suppress non-secular terms by taking

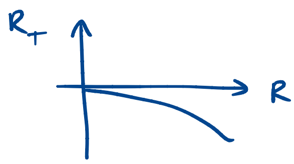
$$-2iA_T + \frac{i}{4}A^2\bar{A} - i\left(\frac{1}{2}A\bar{A} - \mu\right)A = 0$$

ie $\quad 2A_T = \left(\mu - \frac{|A|^2}{4}\right)A$

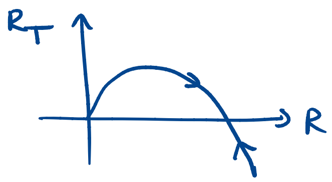
let $A = Re^{i\theta} \Rightarrow \quad 2(R_T + i\theta_T R) = \left(\mu - \frac{1}{4}R^2\right)R$

$\therefore \quad \theta_T = 0 \Rightarrow \theta = \theta_0$

$$2R_T = \left(\mu - \frac{1}{4}R^2\right)R$$



$\mu < 0$
$\Rightarrow R_T < 0$
$\Rightarrow R(T) \to 0$



$\mu > 0$
$\Rightarrow$ $R(T) \to 2\sqrt{\mu}$ as $T \to \infty$

$\therefore$ For $\mu < 0$ – system tends to a steady state with $R = 0$, whilst for $\mu > 0$, solution tends to a periodic orbit with period 2π and amplitude $2\sqrt{\mu}$ (at leading order).

$\Leftarrow$ called a Hopf bifurcation.

PS4 Q5

$\varepsilon y'' + y' + xy = 0$ as $\varepsilon \to 0^+$ with $0 < x < 1$ and $y(0) = 0$, $y(1) = 1$.

(a) WKB expansion: $y(x) = e^{S(x)/\varepsilon}$ with $S(x) = S_0 + \varepsilon S_1 + \dots$

$y'(x) = \dfrac{S'(x)}{\varepsilon} e^{S(x)/\varepsilon}$ and $y''(x) = \left[\dfrac{(S')^2}{\varepsilon^2} + \dfrac{S''}{\varepsilon} \right] e^{S(x)/\varepsilon}$

$$\Rightarrow \quad (S')^2 + S' + \varepsilon(S'' + x) = 0$$

Expanding and collecting terms:

$O(\varepsilon^0)$: $\quad (S_0')^2 + S_0' = 0 \Rightarrow S_0' = 0$ or $S_0' = -1$

$\therefore \; S_0(x) = A_1, \quad S_0(x) = B_1 - x \quad (A_1, B_1 \in \mathbb{R})$

$O(\varepsilon^1)$: $\quad 2S_0' S_1' + S_1' + S_0'' + x = 0$

$S_0(x) = A_1 \Rightarrow S_1' = -x \Rightarrow S_1(x) = A_2 - \frac{1}{2}x^2 \quad (A_2 \in \mathbb{R})$

$S_0(x) = B_1 - x \Rightarrow S_1' = x \Rightarrow S_1(x) = B_2 + \frac{1}{2}x^2 \quad (B_2 \in \mathbb{R})$

$\therefore$ General solution is $y \sim A_3 e^{-\frac{1}{2}x^2} + B_3 e^{-\frac{x}{\varepsilon} + \frac{1}{2}x^2} \quad (A_3, B_3 \in \mathbb{R})$

Boundary conditions: $y(0) = 0 \Rightarrow A_3 \sim B_3$

$y(1) = 1 \Rightarrow A_3 e^{-\frac{1}{2}} + B_3 e^{-\frac{1}{\varepsilon} + \frac{1}{2}} \sim 1$

$$\therefore \; A_3 \sim -B_3 \sim \frac{1}{e^{-1/2} - e^{-1/\varepsilon + 1/2}} = \frac{e^{\frac{1}{2}}}{1 - e^{1 - 1/\varepsilon}}$$

Hence
$$y \sim \frac{e^{(1-x^2)/2} - e^{-x/\varepsilon + (1+x^2)/2}}{1 - e^{1-1/\varepsilon}} \quad \text{as } \varepsilon \to 0^+$$

(b) $\varepsilon y'' + y' + xy = 0$ for $0 < x < 1$ with $y(0) = 0$ and $y(1) = 1$ (10)

- Seek a BL at $x=1$: $x = 1 + \varepsilon X$, $y(x) = Y(X)$ with $X < 0$, $X = \text{ord}(1)$

$$\Rightarrow \quad Y'' + Y' + \varepsilon(1+\varepsilon X)Y = 0 \quad \Rightarrow \quad Y_0(X) = c_1 + c_2 e^{-X} \quad (c_1, c_2 \in \mathbb{R})$$

Then, matching with the outer solution with require $Y_0(-\infty)$ finite and hence $c_2 = 0$ so that $Y_0(X) = c_1 = 1$ and there is no BL at $x=1$.

- Seek a BL at $x=0$: $x = \varepsilon X$, $y(x) = Y(X)$ with $X > 0$, $X = \text{ord}(1)$.

$$\Rightarrow \quad Y'' + Y' + \varepsilon^2 X Y = 0 \quad \text{with} \quad Y(0) = 0$$

Expand as $Y \sim Y_0 + \varepsilon Y_1 + \ldots$ and collect terms:

$O(\varepsilon^0)$ $\quad Y_0'' + Y_0' = 0$ with $Y_0(0) = 0 \Rightarrow Y_0(X) = E_1(1 - e^{-X})$ $\quad (E_1 \in \mathbb{R})$

- Solution in outer region:

Expand as $y \sim y_0 + \varepsilon y_1 + \ldots$ and collect terms:

$O(\varepsilon^0)$: $\quad y_0' + xy_0 = 0$ with $y_0(1) = 1$

$$\Rightarrow \quad \frac{y_0'}{y_0} = -x \quad \Rightarrow \quad \ln|y_0| = D_1 - \tfrac{1}{2}x^2$$

$$\therefore \quad y_0(x) = e^{(1-x^2)/2}$$

Matching: $\quad (1\text{to}) = e^{(1-x^2)/2} = e^{(1-\varepsilon^2 X^2)/2} \sim e^{\frac{1}{2}}$ $\quad | \quad (1\text{ti}) = E_1(1 - e^{-X}) = E_1(1 - e^{-x/\varepsilon}) \sim E_1$

$$\Rightarrow \quad E_1 = e^{\frac{1}{2}}$$

Composite expansion: $\quad y \sim y_0(x) + Y_0(x/\varepsilon) - (1\text{ti})(1\text{to})$

$$= e^{(1-x^2)/2} - e^{\frac{1}{2} - \frac{x}{\varepsilon}} \quad \text{as } \varepsilon \to 0^+$$

PS4 Q6

(a) $\varepsilon^2 y'' + (1-x)y = 0$ as $\varepsilon \to 0^+$ for $x > 0$ with $y(0) = 1$, $y(\infty) = 0$

Let $x = 1 + \varepsilon^{2/3} X$ and $y(x) = Y(X)$ $\Rightarrow$ $Y'' - XY = 0$

for $X > -\varepsilon^{-2/3}$

$\therefore$ $Y(X) = a\,Ai(X) + b\,Bi(X)$ $(a, b \in \mathbb{R})$

with $Y(-\varepsilon^{-2/3}) = 1$, $Y(\infty) = 0$

$Y(\infty) = 0 \Rightarrow b = 0$, $Y(-\varepsilon^{-2/3}) = 1 \Rightarrow a\,Ai(-\varepsilon^{-2/3}) = 1$

$$\therefore \; y(x) = Y(X) = \frac{Ai(X)}{Ai(-\varepsilon^{-2/3})} = \frac{Ai(\varepsilon^{-\frac{2}{3}}(x-1))}{Ai(-\varepsilon^{-\frac{2}{3}})}$$

(b) WKB expansion: $y(x) = A(x)e^{i\varphi(x)/\varepsilon}$

$\Rightarrow$ $y'(x) = \left(\frac{iA\varphi'}{\varepsilon} + A'\right)e^{i\varphi/\varepsilon}$, $y''(x) = \left(-\frac{A(\varphi')^2}{\varepsilon^2} + \frac{2iA'\varphi'}{\varepsilon} + \frac{iA\varphi''}{\varepsilon} + A''\right)e^{i\varphi/\varepsilon}$

$\therefore$ $-A(\varphi')^2 + \varepsilon(2iA'\varphi' + iA\varphi'') + \varepsilon^2 A'' + (1-x)A = 0$

Expand as $A \sim A_0 + \varepsilon A_1 + \dots$ and collect terms:

$O(\varepsilon^0)$: $-A_0(\varphi')^2 + (1-x)A_0 = 0 \Rightarrow \varphi' = \pm(1-x)^{\frac{1}{2}}$ $(A_0 \neq 0)$

$\varphi = \pm\frac{2}{3}(1-x)^{\frac{3}{2}} + c_1$

$O(\varepsilon^1)$: $-A_1(\varphi')^2 + 2iA_0'\varphi' + iA_0\varphi'' + (1-x)A_1 = 0$

$\Rightarrow$ $(A_0^2\varphi')' = 0$ (collecting imaginary parts)

$\Rightarrow$ $A_0^2 = \frac{\tilde{c}_2}{\varphi'}$

$\Rightarrow$ $A_0 = \frac{c_2}{(1-x)^{1/4}}$ character of solution depends on whether $x > 1$ or $x < 1$.

RH outer solution $(x>1)$

$y(\infty)=0 \Rightarrow$ need to eliminate the growing solution and so

$$y(x) \sim \frac{c_1}{(x-1)^{1/4}} e^{-\frac{2}{3\varepsilon}(x-1)^{3/2}} \quad \text{as } \varepsilon \to 0^+ \text{ with } x>1,\ x-1=\text{ord}(1)$$

$(c_1 \in \mathbb{R})$

LH outer solution $(0<x<1)$

- Both roots $\left(\varphi = \pm\frac{2}{3}(1-x)^{3/2}\right)$ admissible. Write solution as

$$y \sim \frac{c_2}{(1-x)^{1/4}} \sin\left(\frac{2}{3\varepsilon}(1-x)^{3/2}+\alpha_2\right) \quad \text{as } \varepsilon\to 0^+ \text{ with } 0<x<1$$

and $x=\text{ord}(1)$.

$(c_2, \alpha_2 \in \mathbb{R})$

BC: $y(0)=1 \Rightarrow c_2 \sin\left(\frac{2}{3\varepsilon}+\alpha_2\right)=1$

$$\therefore \quad y(x) \sim \frac{\operatorname{cosec}\left(\frac{2}{3\varepsilon}+\alpha_2\right)}{(1-x)^{1/4}} \sin\left(\frac{2}{3\varepsilon}(1-x)^{3/2}+\alpha_2\right)$$

Inner region (near $x=1$)

Note that, due to factors $(1-x)^{-1/4}$, both outer solutions are unbounded as $x\to 1^\pm$. So seek an inner solution of the form

$$y=\delta(\varepsilon)^{-1/4}\, Y(X) \quad \text{with} \quad x = 1+\delta(\varepsilon) X$$

$$\Rightarrow \quad Y''-XY=0 \quad \text{provided } \delta(\varepsilon)=\varepsilon^{2/3}$$

$$\therefore \quad Y(X) = c_3 \operatorname{Ai}(X) + c_4 \operatorname{Bi}(X) \qquad (c_3, c_4 \in \mathbb{R})$$

NB $\operatorname{Ai}(X) \sim \frac{1}{2\sqrt{\pi}\,X^{1/4}} e^{-2/3 X^{3/2}}$ and $\operatorname{Bi}(X) \sim \frac{1}{\sqrt{\pi}\,X^{1/4}} e^{2/3 X^{3/2}}$ as $X\to\infty$

Matching inner solution $(X\to\infty)$ with RH outer solution $(x\to 1^+)$

Use an intermediate variable: $x-1=\varepsilon^\alpha \hat{x} = \varepsilon^{2/3} X \quad \left(0<\alpha<\frac{2}{3}\right)$

$(\hat{x}>0)$

$$X = \frac{\hat{x}}{\varepsilon^{2/3-\alpha}} \to \infty \text{ as } \varepsilon \to 0^+ \text{ with } \hat{x} > 0,\ \hat{x} = \text{ord}(1)$$

$$\delta^{-\frac{1}{4}} Y\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right) = \frac{c_3}{\varepsilon^{1/6}} Ai\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right) + \frac{c_4}{\varepsilon^{1/6}} Bi\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right)$$

$$\sim \frac{c_3}{\varepsilon^{1/6}} \frac{1}{2\sqrt{\pi}\left(\hat{x}/\varepsilon^{2/3-\alpha}\right)^{1/4}} e^{-2/3\left(\hat{x}/\varepsilon^{2/3-\alpha}\right)^{3/2}}$$

$$+ \frac{c_4}{\varepsilon^{1/6}} \frac{1}{\sqrt{\pi}\left(\hat{x}/\varepsilon^{2/3-\alpha}\right)^{1/4}} e^{2/3\left(\hat{x}/\varepsilon^{2/3-\alpha}\right)^{3/2}}$$

and

$$y(x) \sim \frac{c_1}{(x-1)^{1/4}} e^{-2/3\varepsilon (x-1)^{3/2}} \qquad x = 1 + \varepsilon^\alpha \hat{x} \to 1^+ \text{ as } \varepsilon \to 0^+$$

with $\hat{x} > 0,\ \hat{x} = \text{ord}(1)$

$$\therefore\ y(1+\varepsilon^\alpha \hat{x}) \sim \frac{c_1}{(\varepsilon^\alpha \hat{x})^{1/4}} e^{-2/3\varepsilon (\varepsilon^\alpha \hat{x})^{3/2}}$$

Matching $\Rightarrow$ $c_4 = 0$ and $c_1 = \frac{c_3}{2\sqrt{\pi}}$

Matching inner solution with LH outer solution
$(X \to -\infty)$ $(x \to 1^-)$

$$X = \frac{\hat{x}}{\varepsilon^{2/3-\alpha}} \to -\infty \text{ as } \varepsilon \to 0^+ \text{ with } \hat{x} < 0,\ \hat{x} = \text{ord}(1).$$

$$\delta^{-\frac{1}{4}} Y\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right) \sim \frac{c_3}{\varepsilon^{1/6}} Ai\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right)$$

$$\sim \frac{c_3}{\varepsilon^{1/6}} \frac{1}{\sqrt{\pi}\left(-\hat{x}/\varepsilon^{2/3-\alpha}\right)} \sin\left(\frac{2}{3}\left(\frac{-\hat{x}}{\varepsilon^{2/3-\alpha}}\right)^{3/2} + \frac{\pi}{4}\right)$$

$$x = 1 + \varepsilon^\alpha \hat{x} \to 1^+ \text{ as } \varepsilon \to 0^+ \text{ with } \hat{x} < 0,\ \hat{x} = \text{ord}(1)$$

$$y(1+\varepsilon^\alpha \hat{x}) \sim \frac{\operatorname{cosec}\left(\frac{2}{3\varepsilon} + \alpha_2\right)}{(-\varepsilon^\alpha \hat{x})^{1/4}} \sin\left(\frac{2}{3\varepsilon}(-\varepsilon^\alpha \hat{x})^{3/2} + \alpha_2\right)$$

Matching $\Rightarrow$ $\alpha_2 = \frac{\pi}{4}$ (WLOG) and $\frac{c_3}{\sqrt{\pi}} = \operatorname{cosec}\left(\frac{2}{3\varepsilon} + \frac{\pi}{4}\right)$