↳ typically needed when there are two time or length scales in a differential equation.

(Processes have their own scales which act simultaneously...)

5.1 van der Pol oscillator

$$\ddot{x} + \varepsilon \dot{x}(x^2-1) + x = 0 \quad \text{with } 0<\varepsilon \ll 1 \quad \text{with } x=1,\ \dot{x}=0 \text{ @ } t=0.$$

Treating the problem as a regular perturbation expansion:

$x \sim x_0 + \varepsilon x_1 + \dots$ — expand and collect terms at each order:

$O(1)$: $\ddot{x}_0 + x_0 = 0$ with $x_0(0)=1,\ \dot{x}_0(0)=0 \Rightarrow x_0(t) = \cos t.$

$O(\varepsilon)$: $\ddot{x}_1 + x_1 = -(x_0^2-1)\dot{x}_0 = (1-\cos^2 t)(-\sin t) = \frac{1}{4}\sin 3t - \underbrace{\frac{3}{4}\sin t}$

$x_1(t) = \frac{3}{8}(t\cos t - \sin t) - \frac{1}{32}(\sin 3t - 3\sin t)$

will generate a resonant term...

Putting the terms together:

$x(t;\varepsilon) \sim \cos t + \varepsilon\left[\frac{3}{8}(t\cos t - \sin t) - \frac{1}{32}(\sin 3t - 3\sin t)\right] + \dots$

valid for fixed t as $\varepsilon \to 0$, but breaks down as $t \sim O(\frac{1}{\varepsilon})$ because of the resonant terms.

Problem: damping term only changes the solution by $O(1)$ over a timescale of $O(\frac{1}{\varepsilon})$ — ie it's a slow accumulation of small effects.

ie the processes on the different timescales are <u>fast</u> oscillations and <u>slow</u> damping.

Solution — introduce <u>two</u> time variables: $\tau = t$ — fast time of oscillation

$T = \varepsilon t$ — slow time of amplitude drift.

Seek a solution $x(t;\varepsilon) = x(\tau, T; \varepsilon)$

treat τ and T as independent.

$$\frac{d}{dt} = \frac{d\tau}{dt}\cdot\frac{\partial}{\partial\tau} + \frac{dT}{dt}\frac{\partial}{\partial T} = \frac{\partial}{\partial \tau} + \varepsilon\frac{\partial}{\partial T}$$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial\tau^2} + 2\varepsilon\frac{\partial^2}{\partial\tau\partial T} + \varepsilon^2\frac{\partial^2}{\partial T^2}$$

converts ODEs $\to$ PDEs
ie makes the problem more complicated! (usually we aim to go the other way – to see that this simplifies the problem we keep going...)

Expand: $x(\tau, T;\varepsilon) \sim x_0(\tau, T) + \varepsilon x_1(\tau, T) + \ldots$ as $\varepsilon \to 0^+$.

$O(\varepsilon^0)$: $\dfrac{\partial^2 x_0}{\partial\tau^2} + x_0 = 0$ with $x_0 = 1$, $\dfrac{\partial x_0}{\partial \tau} = 0$ at $t=0$.

$$\Rightarrow \quad x_0(\tau, T) = R(T)\cos(\tau + \theta(T)).$$

amplitude and phase constant as far as the fast timescale τ concerned, but vary on slow timescale T.

ICs $\Rightarrow R(0) = 1$ and $\theta(0) = 0$. $\Leftarrow$ o/w $R(T), \theta(T)$ as yet undetermined.

$O(\varepsilon^1)$:
$$\frac{\partial^2 x_1}{\partial\tau^2} + x_1 = -\frac{\partial x_0}{\partial\tau}(x_0^2 - 1) - 2\frac{\partial^2 x_0}{\partial\tau\partial T}$$

$$= 2R\frac{d\theta}{dT}\cos(\tau+\theta) + \left(2R_T + \tfrac{1}{4}R^3 - R\right)\sin(\tau+\theta) + \tfrac{1}{4}R^3\sin(3(\tau+\theta))$$

will resonate ⊛

with $x_1 = 0$, $\underbrace{\dfrac{\partial x_1}{\partial \tau} = -\dfrac{\partial x_0}{\partial T}}$ $= -\dfrac{dR}{dT}$ at $t = 0$

(comes from the $O(\varepsilon^1)$ term at $t=0$: $\dfrac{\partial x_1}{\partial\tau} + \dfrac{\partial x_0}{\partial T} = 0$)

⊛ So we need to remove them or the expansion will cease to be valid for $t \sim O(\frac{1}{\varepsilon})$ (again!).

↳ use the freedom in $R(T), \theta(T)$ to do this...

Secularity conditions: $2R\frac{d\theta}{dT} = 0 \Rightarrow \frac{d\theta}{dT} = 0$

$$2\frac{dR}{dT} + \frac{1}{4}R^3 - R = 0 \Rightarrow \frac{dR}{dT} = \frac{1}{8}R(4-R^2)$$

Solving: $\theta(T) \equiv 0$

(using $\theta(0) = 0$ and $R(0) = 1$)

$$R(T) = 2(1+3e^{-T})^{-\frac{1}{2}}$$

$$\rightarrow 2 \text{ as } T \to \infty$$

(is a stable limit cycle)

$$\therefore \quad x(t) \sim x_0(\tau, T) = \frac{2}{(1+3e^{-\varepsilon t})^{\frac{1}{2}}} \cos t + O(\varepsilon)$$

Can evaluate x_1 as $x_1(\tau, T) = -\frac{1}{32}R^3 \sin 3\tau + S(T)\sin(\tau + \varphi(T))$

suppressing the resonant terms again.

amplitude + phase fns. will be determined by a secularity condition on x_2...

At higher orders, we would infact find that resonant forcing is impossible to avoid – eg in solving for x_1 we cannot avoid resonance in x_2. Can be mitigated by introducing another "slow-slow" timescale: $T_2 = \varepsilon^2 t$.

NB could do everything using exponentials:

generally, it simplifies the algebra!

$$x_0 = R(T)\cos(\tau + \theta(T)) = \frac{1}{2}(Ae^{i\tau} + \bar{A}e^{-i\tau}) \quad \text{with } A = A(R,\theta) = Re^{i\theta}$$

$O(\varepsilon^1)$:

$$\frac{\partial^2 x_1}{\partial \tau^2} + x_1 = -2\frac{\partial^2 x_0}{\partial \tau \partial T} - (x_0^2 - 1)\frac{\partial x_0}{\partial \tau}$$

$$= -i\left[\frac{dA}{dT}e^{i\tau} + \frac{d\bar{A}}{dT}e^{-i\tau}\right] - \left[\frac{1}{4}(Ae^{i\tau} + \bar{A}e^{-i\tau})^2 - 1\right] \cdot \frac{i}{2}\left[Ae^{i\tau} - \bar{A}e^{-i\tau}\right]$$

$$= \left[-i\left(\frac{dA}{dT} - \frac{1}{8}A(4-|A|^2)\right)e^{i\tau} + c.c.\right] + \text{nonsecular terms.}$$

Suppressing resonant terms: $\frac{dA}{dT} = \frac{1}{8}A(4-|A|^2)$

$$\Rightarrow \frac{dR}{dT}e^{i\theta} + iR\frac{d\theta}{dT}e^{i\theta} = Re^{i\theta} \cdot \frac{1}{8}(4-R^2)$$

Im: $R\frac{d\theta}{dT} = 0$

Re: $\frac{dR}{dT} = \frac{1}{8}R(4-R^2)$

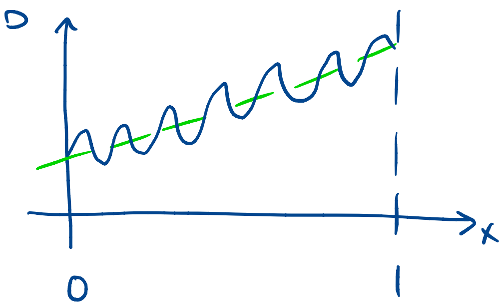
same eqns as before!

Homogenisation

Example $\frac{d}{dx}\Big(\underbrace{D(x,\tfrac{x}{\varepsilon})}\frac{du}{dx}\Big) = f(x)$ $\quad x\in(0,1)$ with $u(0)=a$, $u(1)=b$ and $0<\varepsilon\ll 1$.

↳ $0 < D_-(x) < D(x,\tfrac{x}{\varepsilon}) < D_+(x)$ with $D_\pm$ continuous.

For eg $D(x,\tfrac{x}{\varepsilon}) = 10 + x + \tfrac{1}{4}\sin(\tfrac{x}{\varepsilon})$



Question – can we approximate by $\frac{d}{dx}\Big(\bar{D}(x)\frac{du}{dx}\Big) = f(x)$ $\quad x\in(0,1)$, $u(0)=a$, $u(1)=b$?

Use the method of multiple scales: let $u(x;\varepsilon) = u(x,X;\varepsilon)$ with $X = x/\varepsilon$.

(NB here I'll write $x = x$ and $X = \frac{x}{\varepsilon}$ to make clear which x is which!)

$$\frac{d}{dx} \mapsto \frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial X} \quad\Rightarrow\quad \Big(\frac{\partial}{\partial x}+\frac{1}{\varepsilon}\frac{\partial}{\partial X}\Big)\Big(D(x,X)\Big(\frac{\partial}{\partial x}+\frac{1}{\varepsilon}\frac{\partial}{\partial X}\Big)u\Big) = f(x)$$

$$\text{ie}\quad \Big(\varepsilon\frac{\partial}{\partial x}+\frac{\partial}{\partial X}\Big)\Big(D(x,X)\Big(\varepsilon\frac{\partial}{\partial x}+\frac{\partial}{\partial X}\Big)u\Big) = \underbrace{\varepsilon^2 f(x)}$$

Let $u(x,X;\varepsilon) = u_0(x,X) + \varepsilon u_1(x,X) + \dots$

assume the u_i bdd – as is physically sensible

ε^2 here $\Rightarrow$ will need to go to higher order in our calculations!

$$O(\varepsilon^0):\quad \big(D(x,X)\,u_{0X}\big)_X = 0$$

$$O(\varepsilon^1):\quad \big(D(x,X)[u_{1X}+u_{0x}]\big)_X + \big(D(x,X)\,u_{0X}\big)_x = 0$$

$$O(\varepsilon^2):\quad \big(D(x,X)[u_{2X}+u_{1x}]\big)_X + \big(D(x,X)[u_{1X}+u_{0x}]\big)_x = f(x)$$

Integrating at $O(\varepsilon^0)$: $\quad D(x,X)\,u_{0X} = c_1(x)$

$$\Rightarrow u_0(x,X) = c_2(x) + c_1(x) + \int_0^X \frac{1}{D(x,s)}\,ds$$

Note that $\int_0^X \frac{1}{D(x,s)}\,ds \sim \text{ord}(X)$ as $X \to \infty$ since $D(x,X)$ is bounded.

Recall that $X = \frac{x}{\varepsilon}$ so as we take $\varepsilon \to 0$, $X \to \infty$ and the integral blows up

$\Rightarrow$ Need $G(x) \equiv 0$ to keep the solution bounded ie. $u_0 = u_0(x)$.

$$O(\varepsilon^1):\quad \underbrace{\left(D(x,X)[u_{1X} + u_{0x}]\right)_X + (D(x,X)\underbrace{u_{0X}}_{=0 \text{ since } u_0 = u_0(x)})_X}_{=0} = 0$$

$$\therefore\quad D(x,X)[u_{1X} + u_{0x}] = d_1(x)$$

$$u_1(x,X) = d_2(x) + d_1(x)\underbrace{\int_0^X \frac{1}{D(x,s)}\,ds}_{\text{ord}(X)} - \underbrace{X u_{0x}}_{\text{ord}(X)}$$

Then, similarly, for $u_1(x,X)$ to be bounded, we need the ord(X) terms to balance ie.

$$d_1(x) = \lim_{X\to\infty}\left[X \Big/ \int_0^X \frac{1}{D(x,s)}\,ds\right] u_{0x} \;:=\; D_H(x)\,u_{0x}$$

$$O(\varepsilon^2):\quad \left(D(x,X)[u_{2X} + u_{1x}]\right)_X + \underbrace{\left(D(x,X)[u_{1X} + u_{0x}]\right)_x}_{= d_1(x)} = f(x)$$

$$\Rightarrow\quad \left(D(x,X)[u_{2X} + u_{1x}]\right)_X = f(x) - d_1'(x) = d_{1x}$$

$$\Rightarrow\quad D(x,X)[u_{2X} + u_{1x}] = e_1(x) + (f(x) - d_1'(x))X$$

$$u_{2X} = \frac{1}{D(x,X)}\left[e_1(x) + (f(x) - d_1'(x))X\right] - u_{1x}$$

$$\therefore\quad u_2(x,X) = e_2(x) + \underbrace{e_1(x)\int_0^X \frac{1}{D(x,s)}\,ds}_{\text{ord}(X) \text{ as } X\to\infty}$$

$$+\ \underbrace{(f(x) - d_{1x})\int_0^X \frac{s}{D(x,s)}\,ds}_{\text{ord}(X^2) \text{ as } X\to\infty} + \underbrace{\int_0^X u_{1x}\,ds}_{\text{ord}(X) \text{ as } X\to\infty}$$

$\hookrightarrow$ has nothing to balance it as $X \to \infty$

Recall that $d_1(x) = \lim_{X\to\infty}\left[X \Big/ \int_0^X \frac{1}{D(x,s)}\,ds \right] u_{0x} = D_H(x)\, u_{0x}$

Then we need $f(x) = d_{1x} \Rightarrow \frac{d}{dx}\left[D_H(x)\frac{du_0}{dx}\right] = f(x)$

↑ This is the homogenised eqn!

NB If $D(x,X)$ is periodic, say with period 1, then D_H simplifies by taking $X \in \mathbb{N}$ so that

$$D_H = \lim_{X\to\infty}\left[X \Big/ \underbrace{X\int_0^1}_{\text{from periodicity}} \frac{1}{D(x,s)}\,ds \right] = \left[\int_0^1 \frac{1}{D(x,s)}\,ds\right]^{-1}$$

Chapter 6 The WKB method (Wentzel, Kramers and Brillouin)

Singular perturbation problem that does not have boundary layers:

$$\varepsilon^2 y'' + y = 0 \quad (0<\varepsilon \ll 1)$$

- Oscillatory solutions of the form $y = A\cos\left(\frac{x}{\varepsilon}+\theta\right)$

high frequency oscillations.

- Typical of many problems arising in wave propagation.
- Need a method to deal asymptotically ($\varepsilon \to 0^+$) with these problems.

The WKB method is such a method for <u>linear</u> wave propagation problems.

Consider $\varepsilon^2 y'' + q(x) y = 0$ with $q(x)>0$ in the region of interest.

- Leads to the question of what happens when the frequency of oscillations is modulated on the slow scale.
  - ↳ For the van der Pol oscillator, in the M of MS example, we saw that the amplitude was modulated on the slow scale. So, here we expect that the frequency will be modulated on the slow scale.

First - try the M of MS - and see that it fails to capture the dynamics.

Let $\varepsilon X = x \Rightarrow \dfrac{d^2y}{dX^2} + q(\varepsilon X) y = 0$ ← oscillator with slowly varying frequency.

We might be tempted to then write $y = y(x, X)$ so that

$$\frac{\partial^2 y}{\partial X^2} + 2\frac{\partial^2 y}{\partial x \partial X} + \varepsilon^2 \frac{\partial^2 y}{\partial x^2} + q(x) y = 0 \qquad \left(\frac{\partial y}{\partial X} = \frac{\partial y}{\partial X} + \varepsilon \frac{\partial y}{\partial x}\right)$$

Expand as $y = y_0 + \varepsilon y_1 + \cdots$ and collect terms to give

$O(1)$: $\frac{\partial^2 y_0}{\partial X^2} + q(x) y_0 = 0 \Rightarrow y_0 = A(x)\cos\left(q(x)^{1/2} X + \theta(x)\right)$

arbitrary functions of x – determined by secularity conditions at $O(\varepsilon)$.

$O(\varepsilon)$: $\frac{\partial^2 y_1}{\partial X^2} + 2\frac{\partial^2 y_0}{\partial x \partial X} + q(x) y_1 = 0$

$$\Rightarrow \frac{\partial^2 y_1}{\partial X^2} + q(x) y_1 = 2\frac{\partial}{\partial x}\left(A(x) q(x)^{1/2} \sin\left(q(x)^{1/2} X + \theta(x)\right)\right)$$

$$= \underline{2\frac{d}{dx}\left(Aq^{1/2}\right)} \sin\left(q^{1/2} X + \theta\right)$$

$$- \underline{2Aq^{1/2}\left(X\frac{dq^{1/2}}{dx} + \frac{d\theta}{dx}\right)} \cos\left(q^{1/2} X + \theta\right)$$

Need both coefficients to be zero to avoid resonant terms (secularity condition)

$$\Rightarrow \frac{d}{dx}\left(Aq^{1/2}\right) = 0 \quad \text{and} \quad \underline{X\frac{dq^{1/2}}{dx} + \frac{d\theta}{dx}} = 0$$

requires $A = 0$

Since $q = q(x)$ and $\theta = \theta(x)$ this cannot be satisfied for $\forall X$

↳ will happen whenever the frequency of the fast oscillation depends on the slow scale.

Instead of the MofMS, we need to use a WKB expansion of the form

$$y(x) = e^{i\varphi(x)/\varepsilon} \underline{A(x;\varepsilon)}$$

will ultimately expand $A(x;\varepsilon)$ using an asymptotic expansion (but not φ).

NB we ultimately want a real solution – but this is OK since we can just add the c.c. ✓

Then $\dfrac{dy}{dx} = e^{i\varphi(x)/\varepsilon}\left[\dfrac{i\varphi'(x)}{\varepsilon}A + A'\right]$

$$\frac{d^2y}{dx^2} = e^{i\varphi(x)/\varepsilon}\left[-\frac{(\varphi')^2}{\varepsilon^2}A + \frac{2i\varphi'}{\varepsilon}A' + \frac{i\varphi''}{\varepsilon}A + A''\right]$$

$$\therefore\ \cancel{e^{i\varphi(x)/\varepsilon}}\left[-\frac{(\varphi')^2}{\varepsilon^2}A + \frac{2i\varphi'}{\varepsilon}A' + \frac{i\varphi''}{\varepsilon}A + A''\right] + q(x)\cancel{e^{i\varphi(x)/\varepsilon}} = 0$$

$$\Rightarrow \quad \varepsilon^2 A'' + 2i\varepsilon\varphi' A' + [-\varphi'^2 + i\varepsilon\varphi'' + q]A = 0.$$

Now, expand A by writing $A = A_0 + \varepsilon A_1 + \cdots$, substitute and collect terms:

$O(1)$: $\quad [-\varphi'^2 + q_0]A_0 = 0 \quad \Rightarrow \quad \varphi'(x)^2 = q_0(x)$

ie $\quad \varphi'(x) = \pm\sqrt{q_0(x)}$

$O(\varepsilon)$: $\quad 2\varphi' A_0' + \varphi'' A_0 + \underbrace{\{-\varphi'^2 + q_0\}}_{=0} A_1 = 0$

$$\Rightarrow \quad 2\varphi' A_0' + \varphi'' A_0 = 0$$

ie $\quad \dfrac{2A_0'}{A_0} + \dfrac{\varphi''}{\varphi'} = 0 \quad \Rightarrow \quad \log(A_0^2\varphi') = \text{constant}$

recall $q(x)$ input

$$\Rightarrow \quad A_0 = \frac{\alpha}{\sqrt{\varphi'}} \quad \alpha \in \mathbb{C}.$$

Hence, at leading order,

$$y = e^{i\varphi(x)/\varepsilon} A(x;\varepsilon) \sim e^{i\varphi(x)/\varepsilon} A_0(x)$$

ie $\quad y(x) = \dfrac{\alpha_+}{[q(x)]^{1/4}}\, e^{\frac{i}{\varepsilon}\int^x \sqrt{q(s)}\,ds} + \dfrac{\alpha_-}{[q(x)]^{1/4}}\, e^{-\frac{i}{\varepsilon}\int^x \sqrt{q(s)}\,ds}$

lower limits can be absorbed into $\alpha_\pm$ – as is easiest for a given problem

NB insisting $y \in \mathbb{R}$ – since we have $e^{\pm i/\varepsilon}$ then we need $\alpha_+ = \overline{\alpha_-}$ ie c.c.s.

At higher order:

$$O(\varepsilon^{n+1}): \quad A_{n-1}'' + 2i\varphi' A_n' + i\varphi'' A_n = 0$$ (first order, linear eqns)

$$2i\sqrt{\varphi'}\left((\varphi')^{\frac{1}{2}} A_n\right)' = -A_{n-1}''$$

using integrating factors

$$\Rightarrow \quad A_n = \frac{i}{2\sqrt{\varphi'}} \int \frac{A_{n-1}''}{\sqrt{\varphi'}}\, dx$$

Q – what happens if we have a $q(x)$ instead that has $q(x) = 0$ for some x. (eg as we go from $q(x) < 0 \to q(x) > 0$)?

NB change from sin/cos $\to e^{\pm}$ ...

Example

↳ semi-classical quantum turning points.

1D steady state Schrodinger eqn:

$$\psi'' - x^2\psi = -E\psi \text{ with } \psi \to 0 \text{ as } x \to \infty \text{ and } \psi'(0) = 0 \quad (x \in \mathbb{R}, \psi \in \mathbb{C})$$

Take an even reflection of ψ to generate an even wave function for $x \in \mathbb{R}$.

Problem – find the large ($E \gg 1$) eigenvalues

NB – can only find a solution for some values of E – known as the energy levels.

Rescale: $y = \psi$, $x = \frac{\bar{x}}{\varepsilon}$ with $\varepsilon = \frac{1}{E}$

(dropping bars)

$$\Rightarrow \quad \varepsilon y'' + (1 - x^2) y = 0 \quad \text{and} \quad y(\infty) = 0, \; y'(0) = 0 \quad 0 < \varepsilon \ll 1.$$

Proceeding exactly as before: $y = e^{i\varphi(x)/\varepsilon} A(x;\varepsilon) \sim e^{i\varphi(x)/\varepsilon} \sum \varepsilon^n A_n(x)$

(NB problem of the same form but domain is different.)

$$O(\varepsilon^0): \quad \varphi'(x) = \pm\sqrt{1 - x^2}$$

$$O(\varepsilon^1): \quad A_0(x) = \frac{\text{constant}}{(1 - x^2)^{1/4}}$$

Blows up as $x \to 1$ - an indication the WKB will not work close to $x = 1$, but it will work either side (outer regions).

Using the WKB expansion:

0 < x < 1

$$y \sim \frac{M_0}{|1-x^2|^{1/4}} e^{i/\varepsilon \int_0^x \sqrt{1-s^2}\,ds} + \frac{N_0}{|1-x^2|^{1/4}} e^{-i/\varepsilon \int_0^x \sqrt{1-s^2}\,ds}$$

$$\sim \frac{P_0}{|1-x^2|^{1/4}} \cos\left(\frac{1}{\varepsilon}\int_0^x \sqrt{1-s^2}\,ds\right) + \frac{k_0}{|1-x^2|^{1/4}} \sin\left(\frac{1}{\varepsilon}\int_0^x \sqrt{1-s^2}\,ds\right)$$

Then $y'(0) = \frac{d}{dx}\left(\frac{P_0}{|1-x^2|^{1/4}}\right)\Big|_{x=0} + \frac{k_0}{\varepsilon} + o(1) = 0$ (LH OUTER)

hence, at leading order, $k_0 = 0$

$$\Rightarrow y \sim \frac{P_0}{|1-x^2|^{1/4}} \cos\left(\frac{1}{\varepsilon}\int_0^x \sqrt{1-s^2}\,ds\right)$$

x > 1

$$y \sim \frac{Q_0}{|1-x^2|^{1/4}} e^{-\frac{1}{\varepsilon}\int_1^x \sqrt{s^2-1}\,ds}$$

No $e^{\frac{1}{\varepsilon}\int}$ term as we want $y(\infty) = 0$.

absorb i into the integral $= \sqrt{i^2(1-s^2)} = \sqrt{1-s^2}$ (RH OUTER)

Solutions break down in a region around $x=1 \Rightarrow$ need to consider an inner region @ $x=1$ and then match to the LH/RH outer solutions.

Inner region (around $x=1$) Let $x = 1+\delta_1(\varepsilon)X$

anticipate $\delta_1(\varepsilon)$ small so that $X \to \infty$ matches the RH outer and $X \to -\infty$ matches the LH outer.

We don't know if the solutions (inner/outer) will be on the same scale - since $|1-x^2|^{-\frac{1}{2}}$ blows up as we go from outer to inner

$\Rightarrow$ scale $Y(X) = \delta_2(\varepsilon) y$

$$\therefore \varepsilon^2 y'' + (1-x^2)y = 0 \Rightarrow \frac{\varepsilon^2}{\delta_1^2} Y''(X) + (1-(1+\delta_1 X)^2)Y = 0$$

$$\underset{1}{\frac{\varepsilon^2}{\delta_1^2} Y''(X)} - (\underset{2}{2\delta_1 XY} + \underset{3}{\delta_1^2 X^2 Y}) = 0$$

NB 3 $\ll$ 2

Hence dominant balance is $\frac{\varepsilon^2}{\delta_1^2} = 2\delta_1 \Rightarrow \delta_1 = \left(\frac{\varepsilon^2}{2}\right)^{1/3}$ ie small let $Y(X) = Y_0(X) + \underline{o(1)}$

small correction – won't evaluate the scale!

Then $\underline{Y_0''(X) - XY_0(X) = 0}$ – Boundary conditions will come from matching with the LH/RH outer solns.

two linearly independent solns. → Ai / Bi – Airy functions

$$Y_0(X) = R_0 \, Ai(X) + S_0 \, Bi(X)$$

$$Ai(X) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3 + Xt\right) dt \sim \frac{1}{2\sqrt{\pi}X^{1/4}} e^{-2/3 X^{3/2}} \quad \text{as } X \to \infty$$

$$\sim \frac{1}{\sqrt{\pi}(-X)^{1/4}} \sin\left(\frac{2}{3}(-X)^{3/2} + \frac{\pi}{4}\right) \quad \text{as } X \to -\infty.$$

$$Bi(X) = \frac{1}{\pi}\int_0^\infty e^{\left(-\frac{1}{3}t^3 + Xt\right)} dt \sim \frac{1}{\sqrt{\pi}X^{1/4}} e^{2/3 X^{3/2}} \quad \text{as } X \to \infty$$

$$\sim \frac{1}{\sqrt{\pi}(-X)^{1/4}} \cos\left(\frac{2}{3}(-X)^{3/2} + \frac{\pi}{4}\right) \quad \text{as } X \to -\infty.$$

(Deriving these expansions is a whole other exercise!)

Matching – inner $(X \to \infty)$ with RH outer $(x \to 1+)$.

In the inner solution: $Bi(X) \to \infty$ as $X \to \infty \Rightarrow S_0 = 0$.

Otherwise everything scales with $\frac{1}{X^{1/4}} e^{-2/3 X^{3/2}}$ (using VDMR or the intermediate scaling)

So, we require that the coefficients match. Use an intermediate variable: recall first

$0 < x < 1$ $\quad y \sim \dfrac{P_0}{(1-x^2)^{1/4}} \cos\left(\frac{1}{\varepsilon}\int_0^x \sqrt{1-s^2}\, ds\right)$ $\quad$ LH OUTER

$x > 1$ $\quad y \sim \dfrac{Q_0}{(1-x^2)^{1/4}} e^{-1/\varepsilon \int_1^x \sqrt{s^2-1}\, ds}$ $\quad$ RH OUTER

$y \sim \dfrac{Y_0(X)}{\delta_2(\varepsilon)} = \dfrac{R_0 \, Ai(X)}{\delta_2(\varepsilon)}$ $\quad x = 1 + \delta_1(\varepsilon) X, \quad \delta_1(\varepsilon) = \dfrac{\varepsilon^{2/3}}{2^{1/3}}$ $\quad$ INNER

Matching $x-1 = \delta_1^{\beta}\hat{x} = \delta_1 X \quad \beta \in (0,1)$

Then $\varepsilon \to 0^+$ gives $\delta_1^{\beta} \to 0 \Rightarrow x \to 1$ and $X \to \pm\infty$ (depending on $\mathrm{sign}(\hat{x})$)

Take $\hat{x} > 0$ to match inner with RH outer:

(↙ $Ai(X)$ as $X \to \infty$)

$$Y_0 = R_0 Ai(X) = R_0 Ai\left(\hat{x}/\delta_1^{1-\beta}\right) \sim R_0 \frac{1}{2\sqrt{\pi}} \left(\frac{\delta_1^{1-\beta}}{\hat{x}}\right)^{1/4} e^{-2/3 \frac{\hat{x}^{3/2}}{(\delta_1^{1-\beta})^{3/2}}}$$

$$y \sim \frac{1}{[(x-1)(x+1)]^{1/4}} e^{-\frac{1}{\varepsilon}\int_1^x \sqrt{s^2-1}\,ds}$$

↑ let $s = 1+\eta$ with η small since x is close to 1.

$$\int_0^x \sqrt{s^2-1}\,ds = \int_0^{x-1} \underbrace{\eta^{1/2}\, 2^{1/2}}_{\text{just this}} \underbrace{(1+\tfrac{1}{2}\eta)^{1/2}}_{\text{neglect at leading order}} d\eta$$

$$= \sqrt{2}\,\frac{2}{3}(x-1)^{3/2}$$

Then, at leading order,

$$\delta_2 y_0 \sim \frac{\delta_2 \varphi_0}{2^{1/4}\,\delta_1^{\beta/4}\,\hat{x}^{1/4}} \exp\left[\frac{1}{\sqrt{2}\,\delta_1^{3/2}} \sqrt{2}\cdot\frac{2}{3}(\delta_1^{\beta})^{3/2}\hat{x}^{3/2}\right]$$

$$\sim \frac{\delta_2 \varphi_0}{2^{1/4}\,\delta_1^{\beta/4}\,\hat{x}^{1/4}} \exp\left[-\frac{2}{3}\frac{\hat{x}^{3/2}}{(\delta_1^{1-\beta})^{3/2}}\right]$$

And $Y = \delta_2 y \Rightarrow \quad \sim \frac{R_0 \delta_1^{1/4}}{2\sqrt{\pi}} \frac{1}{\delta_1^{\beta/4}} \frac{1}{\hat{x}^{1/4}} e^{-\frac{2}{3}\frac{\hat{x}^{3/2}}{(\delta_1^{1-\beta})^{3/2}}}$

For coefficients to be equal: $\frac{R_0 \delta_1^{1/4}}{2\sqrt{\pi}} \frac{1}{\delta_1^{\beta/4}} = \frac{\delta_2 \varphi_0}{2^{1/4}\,\delta_1^{\beta/4}}$ ↙ can now establish δ_2 scaling!

$$\therefore \quad \delta_2 = \delta_1^{1/4} = \left(\frac{\varepsilon^{2/3}}{2^{1/3}}\right)^{1/4} = \frac{\varepsilon^{1/6}}{2^{1/12}} \quad \text{and} \quad \varphi_0 = \frac{R_0}{2^{3/4}\sqrt{\pi}}$$

Matching – inner ($X \to -\infty$) with LH outer ($x \to 1+$).

As before: $x - 1 = \delta_1^{\gamma}\hat{x} = \delta_1 X$ with $\gamma \in (0,1)$, $\hat{x} \sim \mathrm{ord}(1)$, $\hat{x} < 0$.

$$Y_0(x) = R_0 Ai\Big(\underbrace{\frac{\hat{x}}{\delta_1^{1-\gamma}}}_{\to -\infty \text{ as } \delta_1 \to 0}\Big) \sim \frac{R_0 \delta_1^{(1-\gamma)/4}}{\sqrt{\pi}\,(-x)^{1/4}} \sin\left(\frac{2}{3}\frac{(-\hat{x})^{3/2}}{(\delta_1^{1-\gamma})^{3/2}} + \frac{\pi}{4}\right)$$

↗ relevant expansion.

$$y \sim \frac{P_0}{|(1-x)(1+x)|^{1/4}} \cos\left(\frac{1}{\varepsilon}\int_0^x \sqrt{1-s^2}\,ds\right)$$

$\int_0^x = \underbrace{\int_0^1}_{=\pi/4} - \int_x^1$

$$\sim \frac{P_0}{2^{1/4}\delta_1^{1/4}(-\hat{x})^{1/4}} \cos\left(\frac{\pi}{4\varepsilon} - \frac{1}{\varepsilon}\int_x^1 \sqrt{1-s^2}\,ds\right)$$

$$\sim \frac{P_0}{2^{1/4}\delta_1^{1/4}(-\hat{x})^{1/4}} \cos\left(\frac{\pi}{4\varepsilon} - \frac{1}{\varepsilon}\frac{2\sqrt{2}}{3}(1-x^2)^{3/2} + \dots\right)$$

convert $x \to \hat{x}$

$$\sim \frac{P_0}{2^{1/4}\delta_1^{1/4}(-\hat{x})^{1/4}} \cos\left(\frac{\pi}{4\varepsilon} - \frac{2}{3}\frac{1}{(\delta_1^{1-\gamma})^{3/2}}(-\hat{x})^{3/2}\right)$$

$$y_0 \sim \frac{Y_0}{\delta_2} \Rightarrow \sim \frac{R_0\delta_1^{-1/4}}{\sqrt{\pi}\,(-\hat{x})^{1/4}} \sin\left(\frac{\pi}{4} + \frac{2}{3}\frac{(-\hat{x})^{3/2}}{(\delta_1^{1-\gamma})^{3/2}}\right)$$

Note that $\delta_1^{1/4}$ and $(-\hat{x})^{1/4}$ terms match ✓✓

$\Rightarrow$ We require

$$\frac{P_0}{2^{1/4}}\cos\left(\frac{\pi}{4\varepsilon} - \underbrace{\frac{2}{3}\frac{1}{(\delta_1^{1-\gamma})^{3/2}}(-\hat{x})^{3/2}}_{W}\right) \sim \frac{R_0}{\sqrt{\pi}}\sin\left(\frac{\pi}{4} + \underbrace{\frac{2}{3}\frac{(-\hat{x})^{3/2}}{(\delta_1^{1-\gamma})^{3/2}}}_{W}\right)$$

Expanding the sin/cos terms:

$$\frac{P_0}{2^{1/4}}\left[\cos\left(\frac{\pi}{4\varepsilon}\right)\cos W + \sin\left(\frac{\pi}{4\varepsilon}\right)\sin W\right] \sim \frac{R_0}{\sqrt{\pi}}\left[\sin\left(\frac{\pi}{4}\right)\cos W + \cos\left(\frac{\pi}{4}\right)\sin W\right]$$

Note that W contains $\hat{x}$ which varies (recall – not matching at a single point)

↳ hence equality must hold $\forall W$

$\therefore$ coefficients of $\cos W$ term must be equal: $\frac{P_0}{2^{1/4}}\cos\left(\frac{\pi}{4\varepsilon}\right) = \frac{R_0}{\sqrt{2\pi}}$

——— " ——— $\sin W$ term ——— " ———: $\frac{P_0}{2^{1/4}}\sin\left(\frac{\pi}{4\varepsilon}\right) = \frac{R_0}{\sqrt{2\pi}}$

$\therefore \tan\left(\frac{\pi}{4\varepsilon}\right) = 1$ as $\varepsilon \to 0 \Rightarrow \frac{\pi}{4\varepsilon_n} = \frac{\pi}{4} + n\pi$ and as $\varepsilon \to 0$ we need $n \to \infty$

$\therefore E_n = \frac{1}{\varepsilon_n} = 1 + 4n$ ← energy levels (this was the objective!)

Also, $\cos\left(\frac{\pi}{4\varepsilon_n}\right) = \cos\left(\frac{\pi}{4} + n\pi\right) = \frac{1}{\sqrt{2}}(-1)^n$

$$\Rightarrow \quad P_0 = \frac{2^{1/4}(-1)^n R_0}{\sqrt{\pi}} = 2(-1)^n \varphi_0$$

$\therefore$ Have determined all the coefficients in terms of φ_0

– We can't determine φ_0 as we are dealing with a linear diff. equation with homogeneous BCs $y(\infty)=0$, $y'(0)=0$ so multiplying any soln by a constant gives another solution.

All together:

$$y_n \sim \frac{1}{(x^2-1)^{1/4}} e^{-\frac{1}{\varepsilon_n}\int_1^x \sqrt{s^2-1}\,ds} \qquad \text{RH OUTER } x>1,\ x \not\approx 1$$

$$\sim \frac{2^{1/2}}{2^{1/6}} 2^{3/4}\sqrt{\pi}\,\varphi_0 \,\mathrm{Ai}\left(\frac{2^{1/3}(x-1)}{\varepsilon_n^{2/3}}\right) \qquad \text{INNER}$$

$$\sim \frac{2(-1)^n\varphi_0}{(x^2-1)^{1/4}} \cos\left(\frac{1}{\varepsilon_n}\int_0^x \sqrt{1-s^2}\,ds\right) \qquad \text{LH OUTER } x<1,\ x \not\approx 1.$$