2 Elementary steady solutions

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2.1 Isotropic expansion

As a first example, suppose

$$\boldsymbol{u} = \frac{\alpha}{3}\boldsymbol{x}, \tag{2.1}$$

where α is a constant scalar, which must be small for linear elasticity to be valid. When $\alpha > 0$, this corresponds to a *uniform isotropic expansion* of the medium so that, as illustrated in Figure 2.1(a), a unit cube is transformed to a cube with sides of length $1 + \alpha/3$.

The strain and stress tensors corresponding to this displacement field are given by

$$e_{ij} = \frac{\alpha}{3}\delta_{ij} \quad \text{and} \quad \tau_{ij} = \left(\lambda + \frac{2}{3}\mu\right)\alpha\delta_{ij}. \tag{2.2}$$

This is a so-called *hydrostatic* situation, in which the stress is characterised by a scalar isotropic pressure p, and $\tau_{ij} = -p\delta_{ij}$. The pressure is related to the relative volume change by $p = -K\alpha$, where

$$K = \lambda + \frac{2}{3}\mu \tag{2.3}$$

measures the solid's resistance to expansion/compression and is called the *bulk modulus* or *modulus of compression*.

2.2 Simple shear

As our next example, suppose

$$\boldsymbol{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \alpha y \\ 0 \\ 0 \end{pmatrix}, \tag{2.4}$$

where α is again a constant scalar. This corresponds to a *simple shear* of the solid in the x-direction, as illustrated in Figure 2.1(b). The strain and stress tensors are now given by

$$\mathcal{E} = \frac{\alpha}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tau = \alpha\mu\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.5}$$

Note that λ does not affect the stress, so the solid's response to shear is accounted for entirely by μ which is, therefore, called the *shear modulus*.

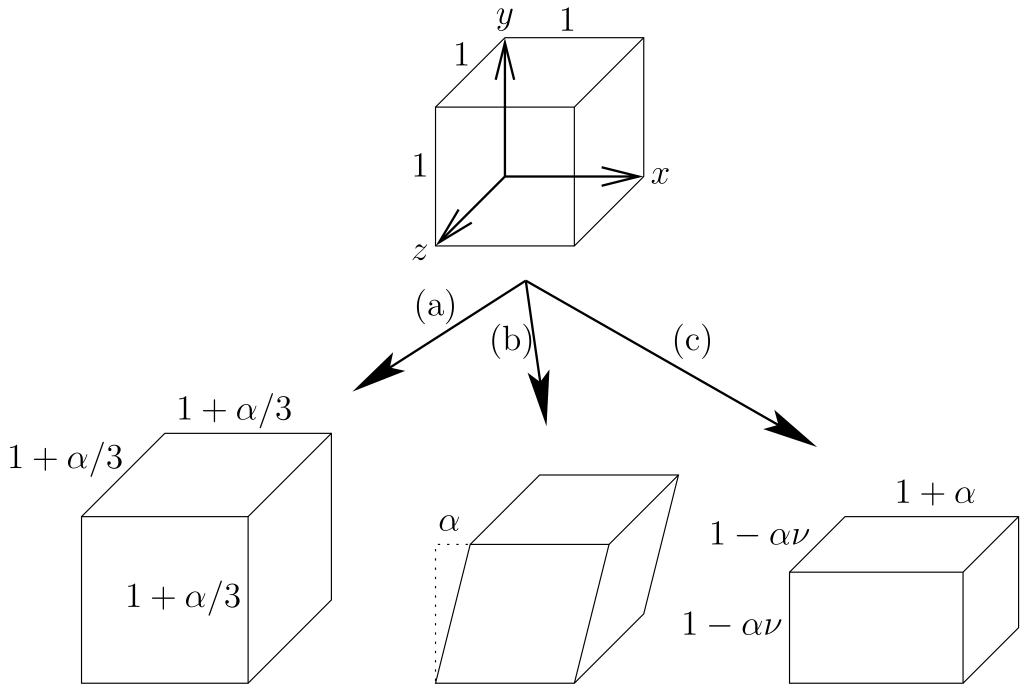

Figure 2.1: A unit cube undergoing (a) uniform expansion, (2.1), (b) one-dimension shear, (2.4), (c) uniaxial stretching, (2.6).

2.3 Uniaxial stretching

Our next example is *uniaxial stretching* in which, as shown in Figure 2.1(c), the solid is stretched by a factor α in (say) the x-direction. We suppose, for reasons that will emerge shortly, that the solid simultaneously shrinks by a factor $\nu\alpha$ in the other two directions. The corresponding displacement, strain and stress are

$$\boldsymbol{u} = \alpha \begin{pmatrix} x \\ -\nu y \\ -\nu z \end{pmatrix}, \qquad \mathcal{E} = \alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & -\nu \end{pmatrix}, \tag{2.6}$$

$$\tau = \alpha \begin{pmatrix} (1-2\nu)\lambda + 2\mu & 0 & 0 \\ 0 & (1-2\nu)\lambda - 2\nu\mu & 0 \\ 0 & 0 & (1-2\nu)\lambda - 2\nu\mu \end{pmatrix}. \tag{2.7}$$

This simple solution may be used to describe a uniform elastic bar that is stretched in the x-direction under a tensile force T. Notice that, since the bar is assumed not to vary in the x-direction, the outward normal $\boldsymbol{n}$ to the lateral boundary always lies in the (y, z)-plane. If the curved surface of the bar is stress-free, then the resulting boundary condition $\tau\boldsymbol{n} = \boldsymbol{0}$ may be satisfied identically by ensuring that $\tau_{yy} = \tau_{zz} = 0$, which occurs if

$$\nu = \frac{\lambda}{2\,(\lambda + \mu)}. \tag{2.8}$$

Hence the bar, while stretching by a factor α in the x-direction, must shrink by a factor $\nu\alpha$ in the two transverse directions; if ν happened to be negative, this would correspond to an

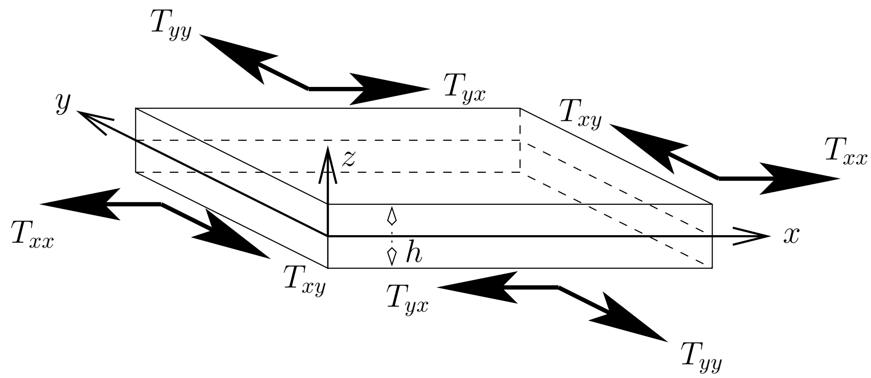

Figure 2.2: Schematic of a plate being strained under tensions T_{xx}, T_{yy} and shear forces T_{xy}, T_{yx}.

expansion. The ratio ν between lateral contraction and longitudinal extension is *Poisson's ratio*.

With ν given by (2.8), the stress tensor has just one nonzero element, namely

$$\tau_{xx} = E\alpha, \tag{2.9}$$

where

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \tag{2.10}$$

is *Young's modulus*. If the cross-section of the bar has area A, then the tensile force T applied to the bar is related to the stress by

$$T = A\tau_{xx} = AE\alpha. \tag{2.11}$$

By measuring T, the corresponding extensional strain α and transverse contraction $\nu\alpha$, one may thus infer the values of E and ν for a particular solid from a bar-stretching experiment. The Lamé constants may then be evaluated using

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}. \tag{2.12}$$

2.4 Biaxial strain

Next consider an elastic plate strained in the (x, y)-plane as illustrated in Figure 2.2. Suppose the plate experiences a linear *in-plane* distortion while shrinking by a factor γ in the z-direction, so the displacement is given by

$$\boldsymbol{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \\ -\gamma z \end{pmatrix}, \tag{2.13}$$

and, as in §2.3, the stress and strain tensors are both constant. Here we choose γ to satisfy the condition $\tau_{zz} = 0$ required on the traction-free upper and lower surfaces of the plate, so that

$$\gamma = \left(\frac{\lambda}{\lambda + 2\mu}\right)(a + d) = \left(\frac{\nu}{1-\nu}\right)(a + d), \tag{2.14}$$

where ν again denotes Poisson's ratio. With this choice, and with E again denoting the Young's modulus, the only nonzero stress components are

$$\tau_{xx} = \frac{E(a+\nu d)}{1-\nu^2}, \qquad \tau_{xy} = \frac{E(b+c)}{2(1+\nu)}, \qquad \tau_{yy} = \frac{E(\nu a + d)}{1-\nu^2}. \tag{2.15}$$

We denote the net in-plane tensions and shear stresses applied to the plate by $T_{ij} = h\tau_{ij}$, as illustrated in Figure 2.2. We can use (2.15) to relate these to the in-plane strain components by

$$T_{xx} = \frac{Eh}{1-\nu^2}\left(e_{xx} + \nu e_{yy}\right), \tag{2.16a}$$

$$T_{xy} = T_{yx} = \frac{Eh}{1+\nu} e_{xy}, \tag{2.16b}$$

$$T_{yy} = \frac{Eh}{1-\nu^2}\left(\nu e_{xx} + e_{yy}\right). \tag{2.16c}$$

These will provide useful evidence when constructing more general models for the deformation of plates.

If no force is applied in the y-direction, that is $T_{xy} = T_{yy} = 0$, then (2.16) reproduces the results of uniaxial stretching, with $d = -\nu a$ and $T_{xx} = Eha$. On the other hand, it is possible for the displacement to be purely in the (x, z)-plane, with

$$b = c = d = 0, \qquad \tau_{yy} = \frac{E\nu a}{1-\nu^2}, \qquad \tau_{xx} = \frac{Ea}{1-\nu^2}. \tag{2.17}$$

Thus a transverse stress τ_{yy} must be applied to prevent the plate from contracting in the y-direction when we stretch it in the x-direction. Notice also that the effective elastic modulus $E/(1-\nu^2)$ is larger than E whenever ν is nonzero, which shows that purely two-dimensional stretching is always more strenuous than uniaxial stretching.

2.5 One-dimensional bending of a beam

The displacement field

$$\boldsymbol{u} = \frac{\kappa}{2}\begin{pmatrix} -2xz \\ 2\nu yz \\ x^2 - \nu y^2 + \nu z^2 \end{pmatrix} \tag{2.18}$$

gives rise to a stress tensor in which the only nonzero component is

$$\tau_{xx} = -E\kappa z. \tag{2.19}$$

This describes *bending* of a beam aligned with the x-axis; the traction-free conditions on the curved surface of the beam are identically satisfied. The net *bending moment* applied about the y-axis is

$$M = \iint_A \tau_{xx} z \,\mathrm{d}y\mathrm{d}z = -E\kappa \iint_A z^2 \,\mathrm{d}y\mathrm{d}z, \tag{2.20}$$

where A is the region of the (y, z)-plane occupied by the bar cross-section. Hence we have discovered a constitutive relation between the bending moment M applied to a beam and its *curvature* $\kappa = \partial^2 w/\partial x^2$, namely

$$M = -EI\frac{\partial^2 w}{\partial x^2}, \tag{2.21}$$

where

$$I = \iint_A z^2 \, \mathrm{d}y\mathrm{d}z \tag{2.22}$$

is the *moment of inertia* of the cross-section about the y-axis. The constant of proportionality EI is known as the *bending stiffness* of the beam.