A12204W1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C5.2
Honour School of Physics Part C: Paper C5.2

ELASTICITY AND PLASTICITY

TRINITY TERM 2015

THURSDAY, 4 JUNE 2015, 2.30pm to 4.00pm

You may submit answers to as many questions as you wish but only the best two will count for the total mark.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

Do not turn this page until you are told that you may do so

1. (a) [18 marks] An elastic rod with circular cross-section and density ρ, Young modulus E, constant radius a and length L is clamped horizontally at one end ($x = 0$) and is free at the other ($x = L$). The rod sags (deflects) under the influence of its weight, with g denoting the acceleration due to gravity.

   (i) Starting from first principles, show that a small transverse displacement, $w(x)$, satisfies

$$\frac{\mathrm{d}N}{\mathrm{d}x} + T\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - \pi a^2 \rho g = 0$$
$$\frac{\mathrm{d}M}{\mathrm{d}x} - N = 0,$$

   where M is the bending moment, N is the shear force and T is the tension within the rod.

   (ii) Why is $T = 0$?

   (iii) You are given that

$$M = -EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}$$

   where

$$I = \iint_{y^2+z^2 \leqslant a^2} z^2 \; \mathrm{d}y\,\mathrm{d}z$$

   is the moment of inertia of the cross-section of the rod.
   Calculate I in terms of the radius of the cross-section a.
   Write down an explicit differential equation for the transverse displacement $w(x)$ and give, with justification, the appropriate boundary conditions.

   (iv) Solve this problem for $w(x)$ explicitly and give an expression for the vertical deflection of the end of the rod, $w(L)$.

   (v) What is $N(0)$? Interpret your result physically.

(b) [7 Marks] The branches of trees sag under their weight causing a bending moment to be applied at the join between the branch and the tree. If this bending moment becomes too large, the branch may snap off the tree. However, branches are also slightly tapered, i.e. they narrow with distance from the trunk of the tree. To model this, we shall use the ideas developed in part (a), but now accounting for a spatially varying branch radius, $a(x) = a_0(1 - \epsilon x/L)$, and $\epsilon < 1$. The dimensionless parameter ϵ measures the tapering of the branch.

   (i) Write down the differential equation for the bending moment $M(x)$ together with the appropriate boundary conditions.

   (ii) Solve this differential equation and determine the bending moment at the point at which the branch joins the tree. For given a_0 and L, does tapering make it more or less likely that a branch will snap off?

2. (a) [3 marks] Navier's equation for time-dependent motions of an elastic medium with constant density ρ reads

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \tau_{ij}}{\partial x_j}.$$

Here $\mathbf{u} = (u_i)$ is the displacement field and $\mathcal{T} = (\tau_{ij}) = (\tau_{ji})$ is the stress tensor, which satisfies the constitutive equation

$$\tau_{ij} = \lambda(\nabla \cdot \mathbf{u})\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

with λ and μ the (constant) Lamé coefficients.

Show that anti-plane displacements of the form $\mathbf{u} = w(x, y, t)\mathbf{k}$ satisfy a wave equation with wave speed $c_s^2 = \mu/\rho$.

(b) [13 Marks] Now consider a material of density ρ with $\mu = \mu_1$, occupying the region $-h \leqslant y \leqslant h$, $-\infty < x < \infty$. Another material with the same density but $\mu = \mu_2 > \mu_1$ occupies the region $|y| > h$, $-\infty < x < \infty$. Denote the displacements in each of the three regions by $w_1(x, y, t)$ for $|y| \leqslant h$, $w_2^+(x, y, t)$ for $y > h$ and $w_2^-(x, y, t)$ for $y < -h$.

(i) Write down the equations governing w_1, w_2^+ and w_2^-. What are the corresponding boundary conditions?

(ii) By seeking wave solutions travelling in the x-direction, with wavenumber k and angular frequency ω, show that either

$$\tan mh = \frac{\mu_2}{\mu_1} \frac{\ell}{m}$$

or

$$\cot mh = -\frac{\mu_2}{\mu_1} \frac{\ell}{m},$$

where

$$\ell^2 = k^2 - \frac{\rho\omega^2}{\mu_2}, \quad m^2 = \frac{\rho\omega^2}{\mu_1} - k^2.$$

(c) [9 Marks] Consider now the problem from part (b) but with the additional restriction that $\mu_1/\mu_2 = \delta \ll 1$.

(i) Show that if $\delta = 0$, then the minimum phase speed of these waves satisfies

$$\frac{c_{\text{min}}^2}{c_1^2} = 1 + \frac{\pi^2}{4k^2h^2},$$

where $c_1^2 = \mu_1/\rho$.

(ii) Show that the leading order correction to this result is

$$\frac{c_{\text{min}}^2}{c_1^2} - 1 - \frac{\pi^2}{4k^2h^2} \approx -\frac{\pi^2}{2k^3h^3}\delta.$$

[*You may make use of the result that* $\tan(\pi/2 + \theta) \approx -\theta^{-1}$ *for* $\theta \ll 1$.]

3. (a) [5 Marks] Assuming plane strain in the x-y plane, calculate the shear stress on a surface with unit normal $\mathbf{n} = (\cos\theta, \sin\theta, 0)^T$ and show that the maximum shear stress (as θ varies) is

$$S = \sqrt{\tau_{xy}^2 + \frac{(\tau_{yy} - \tau_{xx})^2}{4}}.$$

(b) [10 Marks] A linear elastic material occupies the annulus $a < r < b$ in plane polar coordinates (r, θ). The inner surface, $r = a$, is stress free, while the outer surface $r = b$ is subject to a radial displacement $u(r = b) = -U$.

(i) Calculate the stress and displacement fields within the annulus.

(ii) Suppose that the material satisfies the Tresca condition, i.e. that $S \leqslant \tau_y$ with τ_y the yield stress. Find the critical displacement, U_c, at which yield first occurs and the radial position at which it occurs.

(c) [10 Marks] For $U > U_c$, assume that the material is perfectly plastic within some region $a < r < s < b$, i.e. $S = \tau_y$ there.

(i) Evaluate the stress components and show that the unknown edge of the plastic region, s, satisfies

$$U = \frac{\tau_y s^2}{2\mu b} + \frac{\tau_y b}{2(\lambda + \mu)} \left[2\log(s/a) + 1\right]. \qquad (\dagger)$$

(ii) Calculate the radial stress that must be imposed at $r = b$ to impose a displacement $U > U_c$.

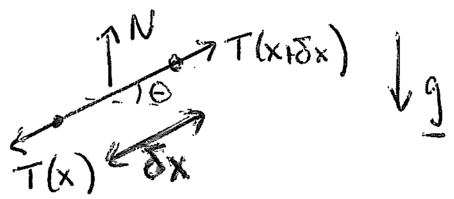
[*In this question, you may use without proof the steady momentum equation together with the constitutive relations for purely radial displacement* $u(r)$ *of a linearly elastic solid, namely*

$$\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0, \quad \tau_{rr} = (\lambda + 2\mu)\frac{\mathrm{d}u}{\mathrm{d}r} + \lambda\frac{u}{r}, \quad \tau_{\theta\theta} = \lambda\frac{\mathrm{d}u}{\mathrm{d}r} + (\lambda + 2\mu)\frac{u}{r}$$

where (r, θ) *are plane polar coordinates and* λ *and* μ *are the Lamé constants.*]

Initial Marks | Comments

9) i)



Weight per unit length is $\sigma = \rho \cdot g A$ (x-sectional area.)

Bookwork

Then balance of forces gives:

$$\underline{0} = \begin{pmatrix} 0 \\ -\rho g A \end{pmatrix} \delta x + \left[\begin{pmatrix} T\cos\theta \\ T\sin\theta \end{pmatrix} \right]_x^{x+\delta x} + \left[\begin{pmatrix} 0 \\ N \end{pmatrix} \right]_x^{x+\delta x}$$

Letting $\delta x \to 0$, we find: $\frac{d}{dx}(T\cos\theta) = 0$

and: $0 = \frac{dN}{dx} + \frac{d}{dx}(T\sin\theta) - \rho g A$

Displacements are small, $|\theta| \ll 1 \Rightarrow \cos\theta \simeq 1$

$\sin\theta \simeq \tan\theta \approx \frac{dw}{dx}$.

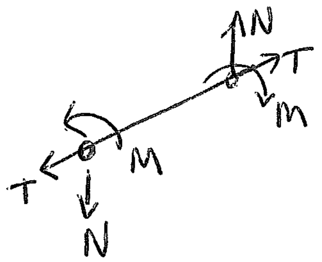
$$\Rightarrow \frac{dT}{dx} = 0$$

and so:

$$0 = -\rho g A + \frac{dN}{dx} + T\frac{d^2w}{dx^2}$$

(Explicitly covered in lectures)

To determine N, we use torque balance on the segment



M = bending moment exerted by each segment of beam on its neighbour.

Taking moments about x:

$$M(x+\delta x) - M(x) - N(x+\delta x)\delta x = 0$$

$$\Rightarrow \frac{dM}{dx} = N.$$

5

with marks

Ib.

Comments

$$\Rightarrow \quad \frac{d^2M}{dx^2} + T\frac{d^2w}{dx^2} - \rho g \pi a^2 = 0.$$

ii) We already saw that $\frac{dT}{dx} = 0$.

2 However $T @ x = L = 0$ (no force applied transversely)

$$\Rightarrow T = 0 \ \forall x.$$

S (T=0 req, but related to free end in lectures)

iii) Given that $M = -EI\frac{d^2w}{dx^2}$

so that $N = -\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)$

B

$$I = \int z^2 dy\, dz = \int_0^a dr \cdot r\int_0^{2\pi} d\theta\left[(z = r\sin\theta)^2\right]$$

$$= \frac{1}{2} 2\pi \int_0^a r^3 dr = \pi a^4/4,$$

$$\therefore \quad N = -\frac{\pi a^4 E}{4}\frac{d^3w}{dx^3}$$

and final eqn is:

$$0 = -\rho g \pi a^2 - \frac{\pi a^4 E}{4}\frac{d^4w}{dx^4}$$

$$\Rightarrow \quad \frac{d^4w}{dx^4} = -\frac{4\rho g}{Ea^2}$$

6. Need 4 bcs:

clamped horizontally $@ x = 0 \Rightarrow w(0) = w'(0) = 0$

Free $@ x = L \Rightarrow M(L) = N(L) = 0 \Rightarrow w''(L) = w'''(L) = 0$

Marks

Ic

Comments

S

iv) We have:

$$\frac{d^3 w}{dx^3} = \frac{4\rho g}{Ea^2}(L-x) \qquad \left(\because w''(L)=0\right)$$

$$\Rightarrow \frac{d^2 w}{dx^2} = -\frac{2\rho g}{Ea^2}\left[(x-L)^2\right]. \qquad \left(w''(L)=0.\right)$$

$$\Rightarrow \frac{dw}{dx} = \frac{2\rho g}{3Ea^2}\left[(L-x)^3 - L^3\right]$$

$$\left(w'(0)=0\right)$$

$$\Rightarrow w(x) = \frac{2\rho g}{3Ea^2}\left[\frac{L^4-(L-x)^4}{4} - L^3 x\right]$$

$$\left(w(0)=0\right)$$

3

$$W(L) = \frac{2\rho g}{3Ea^2}\left[\frac{L^4}{4} - L^4\right] = -\frac{\rho g L^4}{2Ea^2}$$

v) $N(0) = -\frac{\pi a^4 E}{4} w'''(0)$

$$= -\frac{\pi a^4 E}{4} \cdot \frac{4\rho g}{Ea^2} L$$

$$= -\pi a^2 \cdot \rho g \cdot L$$

$= -$ weight of ~~beam~~ rod

2

Shear force exactly balances weight of rod (as it must since it is holding the rod up!)

Details new but similar to problems in lectures + sheets

Id

Comments

N

b) i) Still have: $\dfrac{d^2M}{dx^2} = \rho g \pi a^2$
$$= \rho g \pi a_0^2 (1-\varepsilon x/L)^2$$

(but not $\dfrac{d^4 w}{dx^4} = -\dfrac{4\rho g}{Ea^2}$ since I is inside M).

BCs are $M(L) = N(L) = 0$
$$N(L) = M'(L)$$
$$\Rightarrow M(L) = M'(L) = 0.$$

3

ii) $$\frac{dM}{dx} = \rho g \pi a_0^2 \frac{L}{3\varepsilon}\left[(1-\varepsilon)^3 - (1-\varepsilon x/L)^3\right] \quad (M'(L)=0)$$

$$\Rightarrow M = \rho g \pi a_0^2 \frac{L}{3\varepsilon}\left[(1-\varepsilon)^3(x-L) + \frac{L}{4\varepsilon}\left[(1-\varepsilon x/L)^4 - (1-\varepsilon)^4\right]\right] \quad (M(L)=0).$$

Hence moment where branch joins tree is:
$$M(0) = \rho g \pi a_0^2 \frac{L}{3\varepsilon}\left[-L(1-\varepsilon)^3 + \frac{L}{4\varepsilon}\left[1-(1-\varepsilon)^4\right]\right]$$

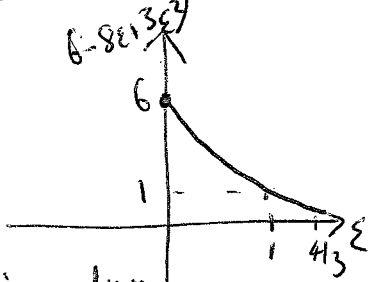
$$= \rho g \pi a_0^2 \frac{L^2}{12\varepsilon^2}\left[1-(\cancel{1}-4\varepsilon+6\varepsilon^2-4\varepsilon^3+\varepsilon^4) - 4\varepsilon(\cancel{1}-3\varepsilon+3\varepsilon^2-\varepsilon^3)\right]$$

$$= \rho g \pi a_0^2 \frac{L^2}{12}\left(6-8\varepsilon+3\varepsilon^2\right)$$

Now when $\varepsilon = 0$
$M(0) = \rho g \pi a_0^2 L^2/2$ (also from a).

with $\varepsilon < 4/3$ $M(0) < \rho g \pi a_0^2 L^2/2$

but in practice must have $\varepsilon < 1$ to keep radius finite → tapering reduces bending torque on tree & less likely to snap



+

Previously shown as solved for w → calculations!, only now a_0 is non-constant cross section.

II a

a/ We are given that:

$$\underline{u} = w(x,y,t)\,\underline{k}$$

with: $\rho \dfrac{\partial^2 u_i}{\partial t^2} = \lambda \dfrac{\partial}{\partial x_i}(\nabla\cdot\underline{u}) + \mu \dfrac{\partial^2 u_i}{\partial x_j \partial x_j} + \mu \dfrac{\partial}{\partial x_i}\left(\dfrac{\partial u_j}{\partial x_j}\right)$

$$\Rightarrow \rho \frac{\partial^2 \underline{u}}{\partial t^2} = (\lambda+\mu)\nabla(\nabla\cdot\underline{u}) + \mu \nabla^2 \underline{u}$$

But $\nabla\cdot\underline{u} = 0$ here

and $\nabla^2 \underline{u} = (\nabla^2 w)\,\underline{k}$

so we have:

$$\rho \frac{\partial^2 w}{\partial t^2} = \mu \nabla^2 w$$

3

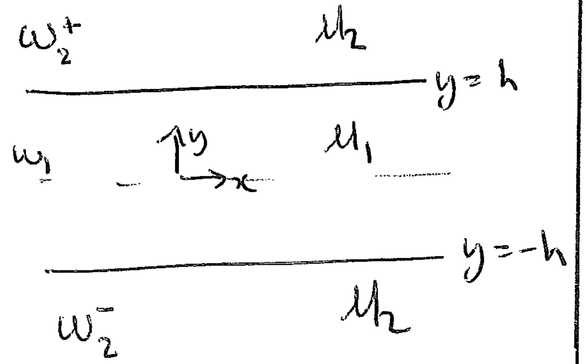
or $\dfrac{\partial^2 w}{\partial t^2} = c_S^2 \nabla^2 w$ with $c_S^2 = \mu/\rho$.

i.e. the wave eqn with speed of sound c_S.

b)

i)

Now specialize to the problem of a seam:



here:

$$\rho \frac{\partial^2 w}{\partial t^2} = \mu_i \nabla^2 w.$$

require continuity of displacement

at $y = \pm h \Rightarrow w_1(h) = w_2^+(h)$

and $w_1(-h) = w_2^-(-h)$.

3

Also, require continuity of $\tau\,\underline{n} = \tau\cdot\underline{k} = \tau_{yz} = \mu_i \dfrac{\partial w}{\partial y}$.

$\left[\tau_{ij} = \lambda(\nabla\cdot\underline{u})\delta_{ij} + \mu\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right)\right]$.

so we have:

$$\mu_2 \frac{\partial w_2^+}{\partial y}\Big|_{y=h} = \mu_1 \frac{\partial w_1}{\partial y}\Big|_h$$

and $\mu_2 \dfrac{\partial w_2^-}{\partial y}\Big|_{y=-h} = \mu_1 \dfrac{\partial w_1}{\partial y}\Big|_h$

3

Also require $w_2^+ \to 0$ as $y \to +\infty$

$w_2^- \to 0$ as $y \to -\infty$.

Comm

Background

B

S

Case $\frac{\mu_2}{\mu_1} = \infty$ done in lectures

B

II b

Connell S

b)ii) Look for a solution of the form:

$$W_1 = f_1(y)\, e^{i(kx-\omega t)}$$
$$W_2^+ = f_2^+(y)\, e^{i(kx-\omega t)}$$
$$W_2^- = f_2^-(y)\, e^{i(kx-\omega t)}$$

then:

$$-\rho\omega^2 f_1 = \mu_1\left[f_1'' - k^2 f_1\right]$$

and $$-\rho\omega^2 f_2^{\pm} = \mu_2\left[\frac{d^2 f_2^{\pm}}{dy^2} - k^2 f_2^{\pm}\right].$$

Letting $$l^2 = k^2 - \frac{\rho\omega^2}{\mu_2}$$

and $$m^2 = \frac{\rho\omega^2}{\mu_1} - k^2$$

we have:

$$f_1'' = -m^2 f_1$$ (want sinusoidal solutions in finite region

and $$f_2^{\pm\,''} = l^2 f_2$$ + exponential solutions in ∞ regions).

$$\Rightarrow \quad f_1 = B\cos my + C\sin my$$
$$f_2^+ = A_+ e^{-ly}$$
$$f_2^- = A_- e^{ly}.$$

Continuity of w

$$\Rightarrow \quad B\cos mh + C\sin mh = A_+ e^{-lh}$$

and $$B\cos mh - C\sin mh = A_- e^{-lh}$$

$$\Rightarrow \quad 2B\cos mh = (A_+ + A_-)e^{-lh}.$$

Covered in printed notes under assumption that modes are symmetric or anti-symmetric.

Unit Marks

II c

Continuity of $\tau_{yz} \Rightarrow$

$$m\mu_1\left(-B\sin mh + C\cos mh\right) = -\mu_2\, l\, A_+ e^{-lh}$$

$$m\mu_1\left(B\sin mh + C\cos mh\right) = \mu_2 l A_- e^{-lh}.$$

$$\Rightarrow \quad m\mu_1 2B\sin mh = \mu_2 l (A_+ + A_-)e^{-lh}.$$

$$= \mu_2 l \cdot 2B\cos mh$$

$$\Rightarrow \boxed{B=0} \text{ or: } \boxed{\tan mh = \frac{\mu_2}{\mu_1}\frac{l}{m}}.$$

(1) (2).

In case (1), immediately have $A_+ = -A_- = Ce^{lh}\sin mh$

$\Rightarrow$

$$m\mu_1 \cos mh = -\mu_2 l \sin mh.$$

$$\Rightarrow \boxed{\cot mh = -\frac{\mu_2}{\mu_1}\frac{l}{m}}.$$

4

(Case (1) is odd solution, (2) is even).

c)

i) In the limit $\mu_2/\mu_1 \gg 1$ we have, at first order,

Case (2): $\tan mh \gg 1$

ie $mh \simeq (n+1/2)\pi$.

$$\Rightarrow kh\sqrt{\frac{c^2}{c_1^2}-1} \simeq (n+1/2)\pi$$

$$\Rightarrow c^2/c_1^2 = 1 + (n+1/2)^2\pi^2/k^2h^2$$

so min speed is $c_{min}^2/c_1^2 = 1 + \pi^2/4k^2h^2$.

4

Case (1): $\cot mh \gg 1 \Rightarrow mh \simeq n\pi$.

$$\Rightarrow c^2/c_1^2 = 1 + n^2\pi^2/k^2h^2 . \quad n = 1, 2, \ldots$$

$\Rightarrow c_{min}^2/c_1^2$ second min!

Comm

B

Comments

II d.

(iii) Envisage that with $\varepsilon = \frac{\mu_1}{\mu_2} \ll 1$ will have results close to those above.

N

Case 2 (This is the one with min speed at $\varepsilon = 0$).

$$\varepsilon \cancel{k}h\sqrt{\frac{c^2}{c_1^2}-1} \times \tan kh\sqrt{\frac{c^2}{c_1^2}-1}$$
$$= \cancel{k}h\sqrt{1-\frac{c^2}{c_1^2}\cdot\varepsilon}$$
$$\simeq \cancel{k}h\left(1-\frac{1}{2}\varepsilon\frac{c^2}{c_1^2}\right).$$

Natural then to let:
$$\frac{c^2}{c_1^2} = 1 + \pi^2/4k^2h^2 + \varepsilon\Delta c^2, \text{ say}$$

$$\Rightarrow \varepsilon \cancel{k}h\left[\frac{\pi^2}{4k^2h^2}+\varepsilon\Delta c^2\right]^{1/2} \cdot \tan\left[kh\sqrt{\frac{\pi^2}{4k^2h^2}+\varepsilon\Delta c^2}\right] \simeq \cancel{k}h\,(1-O(\varepsilon)).$$

$$\tan\left[kh\sqrt{\frac{\pi^2}{4k^2h^2}+\varepsilon\Delta c^2}\right] \simeq \tan\left[\frac{\pi}{2} + \frac{1}{4}\frac{\varepsilon\Delta c^2\, 4k^2h^2}{\pi}\right]$$
$$\simeq -\frac{\pi}{\varepsilon\Delta c^2 k^2h^2}$$

hence @ $O(1)$: $1 = -\frac{\pi}{2kh}\cdot\frac{\pi}{\Delta c^2 k^2h^2}$

$$\Rightarrow \boxed{\Delta c^2 = -\frac{\pi^2}{2k^3h^3}}.$$

Hence
$$\frac{c_{min}^2}{c_1^2} = 1 + \frac{\pi^2}{4k^2h^2} - \frac{\varepsilon\pi^2}{2k^3h^3}.$$

5

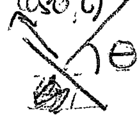
Not covered in lectures/printed notes/sheets

Draft Marks

III a. Comm.

a) B

$\underline{t} = (-\sin\theta, \cos\theta, 0)^T$ $\quad \underline{n} = (\cos\theta, \sin\theta, 0)^T$



In plane strain, have $\tau = \begin{pmatrix} \tau_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \tau_{yy} & 0 \\ 0 & 0 & \tau_{zz} \end{pmatrix}$

So the shear stress is:

$$\underline{t}\,\tau\,\underline{n} = (-\sin\theta, \cos\theta, 0)\begin{pmatrix} \tau_{xx}\cos\theta + \tau_{xy}\sin\theta \\ \tau_{xy}\cos\theta + \tau_{yy}\sin\theta \\ 0 \end{pmatrix}$$

$$= \frac{(\tau_{yy} - \tau_{xx})}{2}\sin 2\theta + \tau_{xy}\cos 2\theta .$$

To maximize this write it as $A\sin(2\theta + \phi)$

with $A = \sqrt{\frac{(\tau_{yy} - \tau_{xx})^2}{4} + \tau_{xy}^2}$

A is clearly the maximum value and so:

5 $$S = \sqrt{\tau_{xy}^2 + \frac{(\tau_{yy} - \tau_{xx})^2}{4}} .$$

Discussed in lectures - detailed in 2nd Yr Lab Sheet.

b) i) B

We have:

$$\frac{d\tau_{rr}}{dr} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0$$

with: $\tau_{rr} = (\lambda + 2\mu)\frac{du}{dr} + \lambda u/r$

$\tau_{\theta\theta} = \lambda \frac{du}{dr} + (\lambda + 2\mu) u/r .$

Boundary conditions $\Rightarrow$ $\tau_{rr}|_{r=a} = 0$

$u(b) = -U .$

Substituting these in, we have:

$$u'' + u'/r - u/r^2 = 0 .$$

III b

Solutions are of the form:

$$u = Ar + B/r$$

so that:

$$-U = Ab + B/b.$$

and $$0 = (\lambda+2\mu)\left(A - B/a^2\right) + \lambda\left(A + B/a^2\right)$$

$$= 2(\lambda+\mu)A - 2\mu B/a^2$$

$$\Rightarrow B = a^2 \frac{\lambda+\mu}{\mu} A$$

$$\therefore -U = \frac{A}{b}\left[b^2 + a^2 \frac{\lambda+\mu}{\mu}\right]$$

$$= \frac{A}{\mu b}\left[\mu b^2 + a^2(\lambda+\mu)\right]$$

$$\Rightarrow A = -\frac{U\mu b}{\mu b^2 + (\lambda+\mu)a^2}.$$

ie: $$u(r) = -\frac{U\mu b}{\mu b^2 + (\lambda+\mu)a^2}\left[r + \frac{\lambda+\mu}{\mu}\frac{a^2}{r}\right].$$

$$\tau_{rr} = -\frac{U\mu b}{\mu b^2+(\lambda+\mu)a^2}\left[(\lambda+2\mu)\left(1 - \frac{\lambda+\mu}{\mu}a^2/r^2\right) + \lambda\left(1 + \frac{\lambda+\mu}{\mu}a^2/r^2\right)\right]$$

$$= -\frac{U\mu b}{\mu b^2+(\lambda+\mu)a^2}\left[2(\lambda+\mu) - 2(\lambda+\mu)a^2/r^2\right]$$

$$= -\frac{2U\mu(\lambda+\mu)b}{\mu b^2+(\lambda+\mu)a^2}\left[1 - a^2/r^2\right]$$

$$\tau_{\theta\theta} = -\frac{2U\mu(\lambda+\mu)b}{\mu b^2+(\lambda+\mu)a^2}\left[1 + a^2/r^2\right]$$

2

Seen in lectures (though with 2x bcs, τ_{rr}).

III c

ii) Here $\tau_{r\theta} = 0$ by assumption

$$\Rightarrow S = \frac{|\tau_{rr} - \tau_{\theta\theta}|}{2}$$

$$= \frac{1}{2} \cdot 2 \frac{a^2}{r^2} \frac{2U\mu(\lambda+\mu)b}{\mu b^2 + (\lambda+\mu)a^2}.$$

Tresca condition requires $S \leq \tau_y$.

Clearly S decreases with r, so largest at $r = a$.

$\therefore$ Yield will occur first at $r=a$ with:

$$\tau_y = \frac{a^2}{a^2} \frac{2U_c\mu(\lambda+\mu)b}{\mu b^2 + (\lambda+\mu)a^2}$$

i.e. $$\boxed{U_c = \frac{\mu b^2 + (\lambda+\mu)a^2}{2\mu(\lambda+\mu)b} \cdot \tau_y}$$

3

c) ~~In this case, material disintegrates when~~

i) Once the material has yielded, we have:

$$|\tau_{rr} - \tau_{\theta\theta}| = 2\tau_y \quad \text{in } a<r<s$$

and: $$\tau_{rr} = A - B/r^2 \quad \text{in } s<r<b$$
$$\tau_{\theta\theta} = A + B/r^2$$

with $\tau_{rr}(r=a) = 0$
$u_r(r=b) = -U$
and τ_{rr} cts at $r=s$.

Now in $s<r$, $\tau_{rr} - \tau_{\theta\theta} = -2B/r^2 \gtrless 0$ (judging from earlier soln)

Hence expect $\tau_{rr} - \tau_{\theta\theta} = +2\tau_y$ in $a<r<s$.

$$\Rightarrow \frac{d\tau_{rr}}{dr} = -\frac{2\tau_y}{r} \Rightarrow \boxed{\tau_{rr} = -2\tau_y \log(r/a)}$$

($\because \tau_{rr}|_{r=a} = 0$)

III d.

Now at $r = s_i^+$

have $\tau_{rr} - \tau_{\theta\theta} = -2B/s^2$

$= +2\tau_y$

$\Rightarrow \boxed{B = -\tau_y s^2}$

But also, in $r > s$:

$(\lambda+2\mu)\frac{du}{dr} + \lambda u/r = \tau_{rr} = A - B/r^2$

$\Rightarrow u = \frac{A}{2(\lambda+\mu)} r + \frac{1}{2\mu}\frac{B}{r}$

and $\boxed{-U = \frac{A}{2(\lambda+\mu)} b + \frac{B}{2\mu}\frac{1}{b}}.$

while $A - B/s^2 = -2\tau_y \log s/a$.

$\Rightarrow A = -2\tau_y \log s/a - \tau_y$

$\therefore +U = \frac{\tau_y[2\log(s/a) + 1]}{2(\lambda+\mu)} b + \frac{\tau_y s^2}{2\mu b}$ (*)

[Check with $s = a$:

$+U_c \stackrel{?}{=} \frac{\tau_y}{2(\lambda+\mu)} b + \frac{\tau_y s^2}{2\mu b}$.

$= \frac{\tau_y}{2\mu(\lambda+\mu)b}\left[\mu b^2 + (\lambda+\mu)a^2\right]$

as earlier.]

ii) Now, $\tau_{rr}|_{r=b} = A - B/b^2$

$= -2\tau_y \log s/a - \tau_y + \tau_y s^2/b^2$

But from (*), $2\tau_y \log s/a = \frac{2(\lambda+\mu)}{b}\left[U - \frac{\tau_y s^2}{2\mu b}\right] - \tau_y$

$\therefore \tau_{rr}(r=b) = \cancel{\tau_y} - \frac{2(\lambda+\mu)}{b}\left[U - \frac{\tau_y s^2}{2\mu b}\right] - \cancel{\tau_y} + \tau_y s^2/b^2$

$= -\frac{2(\lambda+\mu)}{b} U + \tau_y s^2/b^2 \left(1 + \frac{\lambda+\mu}{\mu}\right) = -\frac{2(\lambda+\mu)}{b} U + \tau_y \left(\frac{\lambda+2\mu}{\mu}\right) \times s^2/b^2$

increasing τ_c decreases stress needed to impose given $U > U_c$

N