SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C5.2
Honour School of Physics Part C: Paper C5.2

ELASTICITY AND PLASTICITY

TRINITY TERM 2018

TUESDAY, 5 JUNE 2018, 2.30pm to 4.15pm

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you:

- *start a new answer booklet for each question which you attempt.*
- *indicate on the front page of the answer booklet which question you have attempted in that booklet.*
- *cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.*
- *hand in your answers in numerical order.*

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so

1. A thin elastic beam is clamped vertically at its lower base and is subject to the gravitational acceleration, g, in the vertical direction. The length of the beam is L, and its mass per unit length is ρ. Throughout this question, you should assume that all deformations are planar and therefore consider the angular deflection of the beam, $\phi = \phi(s)$, to be measured from the *vertical*, with s the arc length measured from the base of the beam. With this convention, the torque exerted by elements of the beam on each other is $M = B\mathrm{d}\phi/\mathrm{d}s$, with B a constant.

   (a) [9 marks] At its unclamped end, $s = L$, the beam is subject to a force V in the positive vertical direction; there is no horizontal force or applied torque at $s = L$.

      (i) Use considerations of force and torque balance to show that the the angular deflection of the beam from the vertical, $\phi(s)$, satisfies

$$B\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} = [V + \rho g(s - L)]\sin\phi. \qquad (\dagger)$$

      (ii) Give, with justification, appropriate boundary conditions for (†).

      (iii) What is the vertical force that the beam exerts on its clamp at $s = 0$? Justify your answer both in terms of the derivation of (†) and using physical considerations.

   (b) [4 marks] Consider, first, the case of negligible beam weight, $\rho g = 0$, with the beam under compression, $V = -P < 0$.

   Determine the values of P for which the linearization of (†) has non-trivial solutions. What is the smallest such $P > 0$?

   (c) [6 marks] Next, consider the case of no vertical applied force, $V = 0$, but including the beam's weight, $\rho g > 0$.

      (i) Linearise (†) in this case, and render it dimensionless by letting $\xi = (\rho g/B)^{1/3}(s-L)$.

      (ii) Determine an equation satisfied by the parameter $\Lambda = L/(B/\rho g)^{1/3}$ for non-trivial solutions of the linearised equation to exist. Denote the smallest solution of this equation by Λ_c.

      [*Note that the general solution of Airy's equation, $y''(x) = xy(x)$, may be written $y(x) = \alpha\mathrm{Ai}(x) + \beta\mathrm{Bi}(x)$, for constants α and β. You may express your answer in terms of the functions* Ai, Bi, *and their derivatives.*]

   (d) [6 marks] Finally, consider the general case, $V = -P < 0$ and $\rho g > 0$. Assume further that the parameter $\Lambda < \Lambda_c$ with Λ_c as defined in part (c).

      (i) Determine an equation satisfied by the dimensionless buckling load $\mathcal{P} = P/[B^{1/3}(\rho g)^{2/3}]$, for a fixed value of Λ.

      (ii) Based on your answer to part (b), give an approximate expression for $\mathcal{P}(\Lambda)$ as $\Lambda \to 0$.

2. The Navier equation for an elastic displacement $\mathbf{u}(\mathbf{x},t)$ reads

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = (\lambda+2\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu\nabla\times(\nabla\times\mathbf{u})$$

where $\lambda,\mu>0$ are the Lamé coefficients.

(a) [10 marks] Consider harmonic travelling wave solutions of the form

$$\mathbf{u}(\mathbf{x},t) = \mathrm{Re}\left\{\mathbf{a}\exp[\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)]\right\}.$$

(i) Show that given a vector $\mathbf{a}$ and a non-zero vector $\mathbf{k}$, there exists a unique scalar A and vector $\mathbf{B}$ such that $\mathbf{a} = A\mathbf{k} + \mathbf{B}\times\mathbf{k}$ with $\mathbf{B}\cdot\mathbf{k} = 0$.

(ii) Deduce that either:

$$\mathbf{B} = 0 \text{ and } \omega^2 = c_p^2|\mathbf{k}|^2,$$

or:

$$A = 0 \text{ and } \omega^2 = c_s^2|\mathbf{k}|^2,$$

where c_p and c_s are wave-speeds that you should give in terms of λ,μ and ρ.

(b) [9 marks] An elastic material occupies the half-space $x<0$ with the face $x=0$ held fixed (where $\mathbf{x}=(x,y,z)$). A plane S-wave is incident from $x\to-\infty$ with displacement given by

$$\mathbf{u}_{\mathrm{inc}} = \mathrm{Re}\left\{(\sin\alpha, -\cos\alpha, 0)^T\exp\left\{\mathrm{i}\left[k_s(x\cos\alpha + y\sin\alpha) - \omega t\right]\right\}\right\},$$

where $k_s = \omega/c_s$.

(i) Show that the boundary condition at $x=0$ may be satisfied by a reflected wave field that includes an S-wave and a P-wave with angles of reflection β and γ, respectively, where β should be determined and

$$\sin\gamma = \frac{c_p}{c_s}\sin\alpha.$$

(ii) Determine the amplitudes of the reflected waves.

(c) [6 marks] Consider the problem of part (b) in the case that $\sin\alpha > c_s/c_p$.

(i) Explain physically the behaviour of the P-wave in this case, including the significance of the length ℓ that is defined by

$$\ell^{-1} = k_s\left(\sin^2\alpha - c_s^2/c_p^2\right)^{1/2}.$$

(ii) Show that in the limit $\mu/\lambda \ll \sin^2\alpha$, the amplitude of the P-wave is $O((\mu/\lambda)^{1/2})$.

3. (a) [9 marks] (i) Assuming plane strain in the x-y plane, calculate the shear stress on a surface with unit normal $\mathbf{n} = (\cos\theta, \sin\theta, 0)^T$ and show that the maximum shear stress (as θ varies) is

$$S = \left[\tau_{xy}^2 + \frac{(\tau_{xx} - \tau_{yy})^2}{4}\right]^{1/2},$$

where τ_{ij} are the components of the Cauchy stress tensor.

(ii) Show that an axisymmetric plane strain displacement, $u(r)$, must be of the form

$$u(r) = Ar + Br^{-1}$$

for some constants A and B, and determine the corresponding stresses τ_{rr} and $\tau_{\theta\theta}$.

(b) [11 marks] An isotropic material occupies the region $r > a > 0$ in plane polar coordinates (r, θ). The material is subject to a far-field *compressive* stress p_∞ (i.e. $\tau_{rr}, \tau_{\theta\theta} \sim -p_\infty < 0$ as $r \to \infty$). Furthermore, the inner surface, at $r = a$, is traction-free.

(i) Assuming that the material is linearly elastic throughout $r > a$, find the elastic stresses in the material.

(ii) Suppose that the material satisfies the Tresca condition, i.e. that $S \leqslant \tau_y$ with τ_y the yield stress and S the maximum shear stress of part (a). Find the critical far-field pressure, p_∞^c and the radial position at which yield first occurs.

(iii) Assuming that the material is perfectly plastic, i.e. that $S = \tau_y$ where yield occurs, determine the size of the region in which the material yields for $p_\infty > p_\infty^c$.

(c) [5 marks] Consider the problem of part (b) but with a perfectly rigid material filling the domain $r < a$.

Determine the critical far-field pressure and the radial position at which yield first occurs. Compare this critical value with that found in part (b).

[*In this question, you may use, without proof, the steady momentum equation together with the constitutive relations for purely radial displacement $u(r)$ of a linearly elastic solid, namely*

$$\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0, \quad \tau_{rr} = (\lambda + 2\mu)\frac{\mathrm{d}u}{\mathrm{d}r} + \lambda\frac{u}{r}, \quad \tau_{\theta\theta} = \lambda\frac{\mathrm{d}u}{\mathrm{d}r} + (\lambda + 2\mu)\frac{u}{r}$$

where (r, θ) are plane polar coordinates and λ, μ are the Lamé coefficients.]

Draft Marks

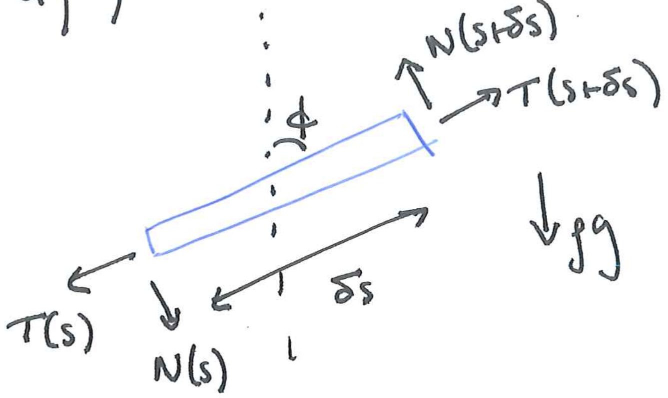
CS.2

Comments

B

Q1

a) i) Consider force balance on an infinitesimal element.



Vertical force balance: $\frac{d}{ds}\left(T\cos\varphi + N\sin\phi\right) - \rho g = 0$

Horizontal force balance: $\frac{d}{ds}\left(T\sin\varphi - N\cos\varphi\right) = 0$

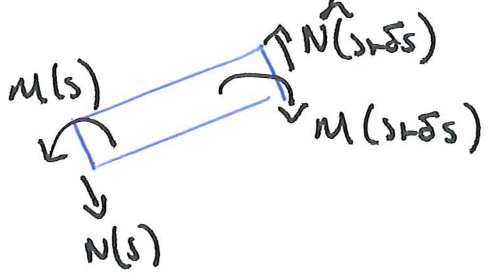
$\therefore$ $T\cos\varphi + N\sin\varphi = V + \rho g(s-L)$, [V is vertical force applied at $s=L$]

and $T\sin\varphi - N\cos\varphi = H = 0$ (no horizontal force applied).

Eliminating T, we have:

$$N = \left[V + \rho g(s-L)\right]\sin\phi \quad (*)$$

Now consider the torque balance on the element:



$N.\delta s - M(s+\delta s) + M(s) = 0$

$\Rightarrow N = \frac{dM}{ds}$.

In question, given constitutive relationship

$$M = B\frac{d\phi}{ds}$$

[$\frac{d\varphi}{ds} > 0$ corresponds to negative curvature so this const. rel. has opposite sign to that in lectures.]

Explicitly covered in lectures for $g=0$ and $\theta = \pi/2 - \varphi$.

Draft Marks

Comments

Substituting into (*) we have:

$$B\frac{d^2\phi}{ds^2} = \left[V + \rho g(s-L)\right]\sin\phi .$$

ii) At the base of the strut, $s=0$, clamped to vertical

$$\Rightarrow \quad \phi(0) = 0 .$$

At the top of the strut, $s=L$, no torque applied

$$\Rightarrow \phi'(L) = 0 .$$

iii) Vertical force on base is $V - \rho g L$, that on top is V.
difference is $\rho g L$, which is the weight of the beam itself.

9

b) In compression, we have: $g=0$, $V = -P$.

Then:

$$B\frac{d^2\varphi}{ds^2} = -P\sin\varphi$$

with $\varphi(0)=0$, $\varphi'(L)=0$

Linearizing, we have:

$$B\varphi'' = -P\varphi$$

with $\varphi(0) = \varphi'(L) = 0$

Non-trivial solution is:

$$\varphi = A\sin\left[\left(\frac{P}{B}\right)^{1/2}s\right] \quad \left(\begin{array}{c}\text{satisfies}\\ \varphi(0)=0\end{array}\right)$$

if $\cos\left[\left(P/B\right)^{1/2}L\right] = 0$

$$\Rightarrow \boxed{P = \pi^2\left(n+1/2\right)^2 B/L^2}$$

Lowest value is $P_c = \dfrac{\pi^2 B}{4L^2}$. ✓

4

B

S relating forces to gravity, not explicitly asked

B

Buckling load of arch with clamped ends correct in lectures.

Draft Marks | Comments

c/ i) When $V=0$, and $g\neq 0$, we have:

$$B\frac{d^2\phi}{ds^2} = \rho g(s-L)\sin\phi$$

Letting: $\xi = \left(\frac{\rho g}{B}\right)^{1/3}(s-L)$, we have (after linearising):

$$\frac{d^2\phi}{d\xi^2} \simeq \xi\phi \qquad (\dagger)$$

with $\phi\left(-L(\rho g/B)^{1/3}\right) = 0$

$\phi'(0) = 0$

ii) Letting $\Lambda = L(\rho g/B)^{1/3}$, first BC is: $\phi(-\Lambda) = 0$.

$(\dagger)$ is Airy's eqn, so solution is:

$$\phi = \alpha Ai(\xi) + \beta Bi(\xi)$$

with α, β such that:

$$0 = \alpha Ai(-\Lambda) + \beta Bi(-\Lambda)$$
$$0 = \alpha Ai'(0) + \beta Bi'(0)$$

6.

$$\det(M) = 0 \Rightarrow Ai(-\Lambda)Bi'(0) - Bi(-\Lambda)Ai'(0) = 0$$

S — Various eigenvalue problems discussed, but not this one.

d) i) When $V\neq 0$, $g\neq 0$, we have: ($V=-P$)

$$B\frac{d^2\phi}{ds^2} \simeq \rho g\left[s-L-P/\rho g\right]\phi$$

Letting: $\xi = (\rho g/B)^{1/3}\left(s-L-P/\rho g\right)$

we have: $\frac{d^2\phi}{d\xi^2} = \xi\phi$ with $\phi[-\Lambda-P] = 0$

and $\phi'(-P) = 0$

where: $\Lambda = \left(\frac{\rho g}{B}\right)^{1/3}L$ and $P = P/_{B^{1/3}(\rho g)^{2/3}}$.

N — Idea of having two parameters controlling buckling new

Draft Notes

1.4

Comments

Following same argument as before, we have:

$$0 = Ai(-\Lambda - P_c)\,Bi'(-P_c) - Bi(-\Lambda - P_c)\,Ai'(-P_c).$$

With $P_c = P_c(\Lambda)$.

ii) As $\Lambda \to 0$, expect role of gravity to be less important.

$\therefore$ expect critical (dimensional) compression force

$$P_c \to \frac{\pi^2}{4} B / L^2$$

$$\Rightarrow \quad P_c \cdot B^{1/3}(\rho g)^{2/3} \to \frac{\pi^2}{4} \cdot \frac{B}{\Lambda^2 B^{2/3}} (\rho g)^{2/3}$$

6. $\therefore \boxed{P_c \sim \pi^2 / 4\Lambda^2 .}$

N

Insight required to see how limit $\Lambda \to 0$ connected to part (b).

Draft Marks

2.1

Comments

B

a/ We have the Navier equations:

$$\rho \frac{\partial^2 \underline{u}}{\partial t^2} = (\lambda + 2\mu)\nabla(\nabla\cdot\underline{u}) - \mu\nabla\wedge(\nabla\wedge\underline{u}).$$

and seek solutions $\underline{u} = \underline{a}\exp[i(\underline{k}\cdot\underline{x} - \omega t)]$

i) First, consider $\underline{a} = A\underline{k} + \underline{B}\wedge\underline{k}$

Dotting with $\underline{k}$: $\underline{a}\cdot\underline{k} = A|\underline{k}|^2 \Rightarrow A = \frac{\underline{a}\cdot\underline{k}}{|\underline{k}|^2}$.

Crossing with $\underline{k}$: $\underline{a}\wedge\underline{k} = -\underline{k}\wedge(\underline{B}\wedge\underline{k})$

$$= -|\underline{k}|^2\underline{B} + (\underline{B}\cdot\underline{k})\underline{k},$$

If we specify $\underline{k}\cdot\underline{B} = 0$ then $\underline{B} = -\frac{\underline{a}\wedge\underline{k}}{|\underline{k}|^2}$

(uniquely).

Explicitly covered in Problem Sheet 2 (Q 2.5)

ii) Substituting $\underline{u} = \underline{a}\exp[i(\underline{k}\cdot\underline{x} - \omega t)]$ into Navier's eqn we have:

$$-\rho\omega^2\underline{a} = -(\lambda+2\mu)\underline{k}(\underline{a}\cdot\underline{k}) + \mu\,\underline{k}\wedge(\underline{k}\wedge\underline{a})$$

$$= -(\lambda+2\mu)\underline{k}(\underline{a}\cdot\underline{k}) + \mu\left[(\underline{a}\cdot\underline{k})\underline{k} - \underline{a}|\underline{k}|^2\right],$$

$$\Rightarrow \rho\omega^2\underline{a} = (\lambda+\mu)\underline{k}(\underline{a}\cdot\underline{k}) + \mu\underline{a}|\underline{k}|^2.$$

i.e. $$(\rho\omega^2 - \mu|\underline{k}|^2)\underline{a} = (\lambda+\mu)\underline{k}(\underline{a}\cdot\underline{k})$$

But we also have: $\underline{a} = A\underline{k} + \underline{B}\wedge\underline{k}$

$$\Rightarrow (\rho\omega^2 - \mu|\underline{k}|^2)\left[A\underline{k} + \underline{B}\wedge\underline{k}\right] = (\lambda+\mu)A|\underline{k}|^2\underline{k}$$

$$\Rightarrow (\rho\omega^2 - \mu|\underline{k}|^2)\underline{B}\wedge\underline{k} + \left[\rho\omega^2 - (\lambda+2\mu)|\underline{k}|^2\right]A\underline{k} = 0.$$

Draft Marks

Comments

Hence we have:

$$\left[\rho\omega^2 - (\lambda+2\mu)|\underline{k}|^2\right] A = 0 \quad , \quad [\cdot\, \underline{k}]$$

and $$\left[\rho\omega^2 - \mu|\underline{k}|^2\right] \underline{B} = \underline{0} \quad , \quad [\wedge\, \underline{k}]$$

B

so either: ① $\cancel{\rho}\omega^2 = \overbrace{\dfrac{\lambda+2\mu}{\rho}}^{c_p^2}|\underline{k}|^2$ and $\underline{B} = \underline{0}$ $\Big] \rightarrow$ P wave

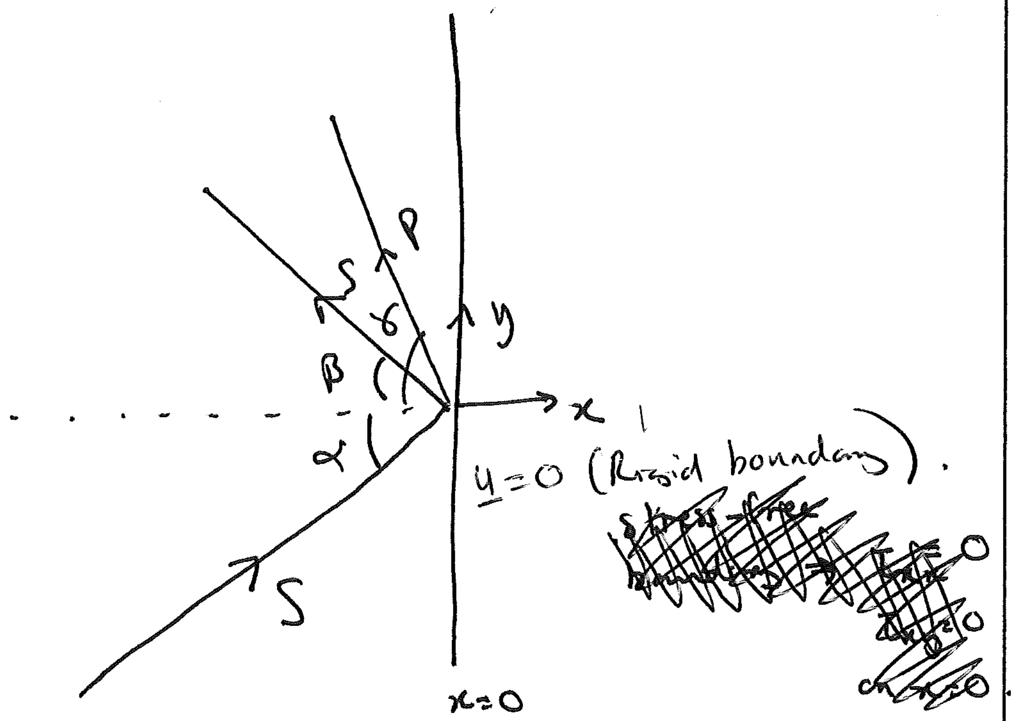
or ② $\omega^2 = \overbrace{\dfrac{\mu}{\rho}}^{c_s^2}|\underline{k}|^2$ and $A = 0$ $\Big] \rightarrow$ S wave

($\lambda, \mu > 0$ means cannot satisfy both dispersion relationships at once!).

10

S

b)



Lectures covered reflection of 'P' wave

For S wave (incoming) $\underline{k} = k_s(\cos\alpha, \sin\alpha)$

and $\underline{B} \perp \underline{k}$

$\Rightarrow \underline{B} \propto (-\sin\alpha, \cos\alpha)$

Hence (setting magnitude = 1): $\underline{u}_{inc} = \begin{pmatrix} +\sin\alpha \\ -\cos\alpha \end{pmatrix} \exp\left\{ i\left[k_s(x\cos\alpha + y\sin\alpha) - \omega t\right]\right\}$

Draft Marks

2.3

Comments

i) For the reflected wave we write:

$$\underline{u}_{ref} = R_s \begin{pmatrix} \sin\beta \\ \cos\beta \end{pmatrix} \exp\left\{ i\left[k_s(-x\cos\beta + y\sin\beta) - \omega t \right] \right\}$$

$$+ R_p \begin{pmatrix} -\cos\gamma \\ \sin\gamma \end{pmatrix} \exp\left\{ i\left[k_p(-x\cos\gamma + y\sin\gamma) - \omega t \right] \right\}$$

S

The boundary condition on $x=0$ is $\underline{u}=0$

To apply this $\forall y$, must have:

$$k_s \sin\alpha = k_s \sin\beta = k_p \sin\gamma$$

The first equality $\Rightarrow$ $\alpha = \beta$ (S-wave reflection is 'specular').

Also: $c_s = \frac{\omega}{k_s}$, $c_p = \frac{\omega}{k_p}$

$$\Rightarrow \quad \sin\gamma = \frac{k_s}{k_p} \sin\alpha = \frac{c_p}{c_s} \sin\alpha$$

(This is Snell's law).

B

Covered in lectures for P-waves

ii) To have $\underline{u}=0$ on $x=0$, need $\underline{u}_{inc} + \underline{u}_{ref} = 0$ there, i.e.:

$$\begin{pmatrix} +\sin\alpha \\ -\cos\alpha \end{pmatrix} + R_s \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix} + R_p \begin{pmatrix} -\cos\gamma \\ \sin\gamma \end{pmatrix} = 0$$

$$\Rightarrow \quad R_s \sin\alpha - R_p \cos\gamma = -\sin\alpha$$

$$R_s \cos\alpha + R_p \sin\gamma = +\cos\alpha .$$

S

Draft Marks

2.4

Comments:

$$\therefore \quad R_s\left[\sin\alpha \sin\gamma + \cos\alpha\cos\gamma\right] = \cos(\alpha+\gamma)$$

$$\Rightarrow R_s = \frac{\cos(\alpha+\gamma)}{\cos(\alpha-\gamma)}.$$

and $$R_p\left[\sin\gamma\sin\alpha + \cos\gamma\cos\alpha\right] = \sin 2\alpha$$

$$\Rightarrow R_p = \frac{\sin 2\alpha}{\cos(\alpha-\gamma)}.$$

9

Given in online notes for P-waves (not explicitly covered)

c/ i) If $\sqrt{\frac{\lambda+2\mu}{\mu}}\sin\alpha > 1$ then $\sin\gamma > 1$

$\Rightarrow \gamma \in \mathbb{C}$. Let: $\gamma = \frac{\pi}{2} - i\varphi$

then: $\cos\gamma = i\sinh\varphi$, $\sin\gamma = \cosh\varphi$

$$\Rightarrow \cosh\varphi = \sqrt{\frac{\lambda}{\mu}+2}\,\sin\alpha$$

~~and~~ The x-dependence in the reflected P-wave is:

$$\exp\left[-i k_p x\cdot\cos\gamma\right]$$

$$= \exp\left[k_p\cdot\sinh\varphi\cdot x\right].$$

so expect exponential decay in $x<0$ with e-folding length:

$$l_e = \frac{1}{k_p\sinh\varphi} = \frac{c_p/c_s}{k_s\sinh\varphi}$$

$$= \frac{1}{k_s}\frac{c_p/c_s}{\sqrt{\frac{c_p^2}{c_s^2}\sin^2\alpha - 1}}$$

$$= \frac{1}{k_s}\frac{1}{\sqrt{\sin^2\alpha - c_s^2/c_p^2}}.$$

(P-wave trapped near boundary)

ii) ~~If $\frac{\mu}{\lambda} \ll \sin^2\alpha < 1$, $c_s^2/c_p^2 \simeq \mu/\lambda \ll \sin^2\alpha$~~

~~$\Rightarrow l_e = \frac{1}{k_s\sin\alpha}$ → e-folding length determined just by wavelength and angle – not materials~~

Evanescent wave not covered in lectures or problem sheets

Draft Marks | ii) | 2.5 | Comment

If $\mu/\lambda \ll \sin^2\alpha < 1$

Then $\cosh\phi \simeq \left(\frac{\lambda}{\mu}\right)^{1/2} \sin\alpha \gg 1$

Hence $\sinh\phi \simeq \cosh\phi \simeq \left(\frac{\lambda}{\mu}\right)^{1/2} \sin\alpha$

Further: $\cos\gamma \simeq i\left(\frac{\lambda}{\mu}\right)^{1/2} \sin\alpha$

$\sin\gamma \simeq \left(\frac{\lambda}{\mu}\right)^{1/2} \sin\alpha$ 1

Hence:

$$R_s \sin\alpha - R_p\, i\left(\frac{\lambda}{\mu}\right)^{1/2} \sin\alpha = -\sin\alpha$$

$$R_s \cos\alpha + R_p \left(\frac{\lambda}{\mu}\right)^{1/2} \sin\alpha = \cos\alpha.$$

$$\Rightarrow R_p \left[\sin^2\alpha + i\cos\alpha\sin\alpha\right]\left(\frac{\lambda}{\mu}\right)^{1/2} = \sin 2\alpha$$

$$\Rightarrow R_p = O\left((\mu/\lambda)^{1/2}\right).$$ 1

6

N

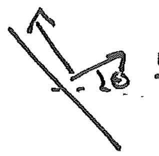
Draft Marks

Comments

B

Q3

a) $\underline{t} = (-\sin\theta, \cos\theta, 0)^T$

i)  $\underline{n} = (\cos\theta, \sin\theta, 0)^T$

In plane strain, we have: $\underline{\tau} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \tau_{yy} & 0 \\ 0 & 0 & \tau_{zz} \end{pmatrix}$

So the shear stress is:

$$\underline{t}\,\underline{\tau}\,\underline{n} = (-\sin\theta, \cos\theta, 0)\begin{pmatrix} \tau_{xx}\cos\theta + \tau_{xy}\sin\theta \\ \tau_{xy}\cos\theta + \tau_{yy}\sin\theta \\ 0 \end{pmatrix}$$

$$= \frac{(\tau_{yy} - \tau_{xx})}{2}\sin 2\theta + \tau_{xy}\cos 2\theta .$$

To Maximise this, write RHS as $A\sin(2\theta + \varphi)$

with $A = \sqrt{\dfrac{(\tau_{yy} - \tau_{xx})^2}{4} + \tau_{xy}^2}$

A is clearly the max, so $S = \sqrt{\tau_{xy}^2 + \dfrac{(\tau_{yy} - \tau_{xx})^2}{4}}$,

as required.

Explicitly covered in lectures

ii) We have $\dfrac{d\tau_{rr}}{dr} + \dfrac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0 \quad (*)$

with $\tau_{rr} = (\lambda + 2\mu)\dfrac{du}{dr} + \lambda u/r$, $\tau_{\theta\theta} = \lambda\dfrac{du}{dr} + (\lambda + 2\mu)u/r$

$(*) \Rightarrow (\lambda + 2\mu)u'' + \lambda\dfrac{u'}{r} - \dfrac{\lambda u}{r^2} + \dfrac{2\mu}{r}\left(u' - u/r\right) = 0$

i.e. $u'' + \dfrac{u'}{r} - \dfrac{u}{r^2} = 0 \Rightarrow u = Ar + B/r$

(trying solutions of the form $u = r^k \Rightarrow k = \pm 1$).

Also: $\tau_{rr} = A\left[\lambda + 2\mu + \lambda\right] + \dfrac{B}{r^2}\left[\lambda - \lambda - 2\mu\right]$

$\tau_{rr} = 2(\lambda + \mu)A - 2\mu B/r^2$

and $\tau_{\theta\theta} = 2(\lambda + \mu)A + 2\mu B/r^2$.

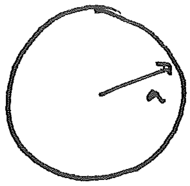
9

Draft Marks.

b) $\qquad \tau_{rr}, \tau_{\theta\theta} \to -p_\infty$ as $r \to \infty$ $\quad \underline{3.2}$

Comments

S



$\tau_{rr} = 0$ @ $r = a$

Different BCs to lectures

i)

From part (a), general elastic solution is:

$$\tau_{rr} = A + B/r^2$$

$$\tau_{\theta\theta} = A - B/r^2 .$$

$\tau_{r\theta} = 0$ by assumption

Here $A = -p_\infty$ by BCs at $r = \infty$.

Further, $\tau_{rr}(r=a) = 0 \Rightarrow B = a^2 p_\infty$.

$$\therefore \quad \tau_{rr} = p_\infty\left(-1 + a^2/r^2\right)$$

$$\tau_{\theta\theta} = p_\infty\left(-1 - a^2/r^2\right).$$

while the material remains elastic.

ii)

The Tresca condition is:

$$2S = |\tau_{rr} - \tau_{\theta\theta}| \leq 2\tau_y . \qquad (\because \tau_{r\theta} = 0).$$

B Follows lectures closely

Here $\tau_{rr} > \tau_{\theta\theta} \Rightarrow |\tau_{rr} - \tau_{\theta\theta}| = \tau_{rr} - \tau_{\theta\theta}$

$$= 2p_\infty a^2/r^2 .$$

so yields when $p_\infty = \tau_y = p_\infty^c$

Find that yield occurs first @ $r = a$

[Even though $\tau_{rr}, \tau_{\theta\theta} \to -\tau_y$, the difference $|\tau_{rr} - \tau_{\theta\theta}| \to 0$ as $r \to \infty$, so no plastic yield far away].

Draft Marks

Comments

S

iii) 3.3

For $p_\infty > \tau_y$, expect yielding in $a < r < s$ and elastic in $r \geq s$.

Hence in $r \geq s$:
$$\tau_{rr} = -p_\infty + B/r^2$$
$$\tau_{\theta\theta} = -p_\infty - B/r^2$$

(from elastic solution) with $\left| \tau_{rr} - \tau_{\theta\theta} \right| \Big|_{r=s} = 2B/s^2 = 2\tau_y$

$$\Rightarrow B = \tau_y \cdot s^2 .$$

In $a < r < s$ expect $\underbrace{\left|\tau_{rr} - \tau_{\theta\theta}\right|}_{= \tau_{rr} - \tau_{\theta\theta}} = 2\tau_y$

$$\Rightarrow \frac{d\tau_{rr}}{dr} = \frac{\tau_{\theta\theta} - \tau_{rr}}{r} = -\frac{2\tau_y}{r}$$

$$\therefore \tau_{rr} = -2\tau_y \log r/a \quad (\because \tau_{rr}(r=a) = 0)$$

Using continuity of τ_{rr} @ $r = s$, we have:

$$-p_\infty + \tau_y = -2\tau_y \log s/a .$$

$$\Rightarrow s = a \exp\left[\frac{p_\infty - \tau_y}{2\tau_y}\right] .$$

[Grows exponentially as p_∞ increases beyond critical value $p_\infty = \tau_y$.].

Different BCs to lectures (Final result is the same, however)

Draft Marks

Comments

N

c/ Now, we have $\tau_{rr}, \tau_{\theta\theta} \to -p_\infty$ as $r \to \infty$ (as before)

but, in addition, $u(r=a) = 0$

In the notation of part (a) we have:

$$2(\lambda+\mu)A = -p_\infty \Rightarrow A = -\frac{p_\infty}{2(\lambda+\mu)}$$ 1

and $0 = Aa + B/a$

$$\Rightarrow B = -Aa^2 = \frac{p_\infty a^2}{2(\lambda+\mu)}.$$ 1

Hence while elastic: $\tau_{rr} = -p_\infty - \frac{2\mu}{2(\lambda+\mu)} \cdot p_\infty \frac{a^2}{r^2}$

$$\tau_{\theta\theta} = -p_\infty + \frac{\mu}{\lambda+\mu} p_\infty \frac{a^2}{r^2}.$$ 1

Now: $\tau_{\theta\theta} > \tau_{rr} \Rightarrow |\tau_{rr} - \tau_{\theta\theta}| = \tau_{\theta\theta} - \tau_{rr}$

$$= \frac{2\mu}{\lambda+\mu} p_\infty a^2/r^2.$$ 1

Tresca $\Rightarrow$ $\frac{2\mu}{\lambda+\mu} p_\infty a^2/r^2 \le 2\tau_y$.

Hence, yield first occurs at $r=a$ with:

$$p_\infty = \tau_y \frac{\lambda+\mu}{\mu} \ge \tau_y. \quad (\lambda>0)$$

Hence presence of rigid inclusion strengthens hole. 1

5

Displacement boundary conditions here are novel.