A12204W1

**SECOND PUBLIC EXAMINATION**

**Honour School of Mathematics Part C: Paper C5.2**
**Master of Science in Mathematical Sciences Paper C5.2**

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# ELASTICITY AND PLASTICITY

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**TRINITY TERM 2019**

**THURSDAY, 13 JUNE 2019, 2.30pm to 4.15pm**

*You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.*

*You should ensure that you:*

- *start a new answer booklet for each question which you attempt.*
- *indicate on the front page of the answer booklet which question you have attempted in that booklet.*
- *cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.*
- *hand in your answers in numerical order.*

*If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.*

**Do not turn this page until you are told that you may do so**

1. A linear elastic material undergoes plane strain in the half-space $\{(x,y) : x < 0,\, y \in \mathbb{R}\}$, with zero stress applied at the boundary $x = 0$. The wave-speeds of $S$-waves and $P$-waves in the material are denoted by $c_s$ and $c_p$, respectively. An $S$-wave making an angle $\alpha$ with the negative $x$-axis is incident from $x,\, y \to -\infty$, with displacement field given by

$$\begin{pmatrix} u(x,y,t) \\ v(x,y,t) \end{pmatrix} = \mathbf{u}_{\text{inc}}(x,y,t) = A \begin{pmatrix} \sin\alpha \\ -\cos\alpha \end{pmatrix} \mathrm{e}^{\mathrm{i}k_s(x\cos\alpha + y\sin\alpha) - \mathrm{i}\omega t},$$

where $\omega \in \mathbb{R}$ is the frequency and $k_s = \omega/c_s$.

(a) [3 marks] Derive the boundary conditions

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 = c_p^2 \frac{\partial u}{\partial x} + \left(c_p^2 - 2c_s^2\right) \frac{\partial v}{\partial y}$$

at $x = 0$.

(b) [7 marks] Explain why the reflected wave-field consists of an $S$-wave and a $P$-wave, making angles $\alpha$ and $\beta$, respectively, with the negative $x$-axis, where

$$\frac{\sin\alpha}{c_s} = \frac{\sin\beta}{c_p}.$$

Show that, if $\sin\alpha > c_s/c_p$, then the $P$-wave reflection angle takes the form $\beta = \pi/2 - \mathrm{i}b$, with $b > 0$. Describe briefly what this behaviour corresponds to physically.

(c) [7 marks] Obtain the general form of the reflected wave-field. Find (but you need not solve) a system of linear algebraic equations satisfied by the $S$-wave and $P$-wave reflection coefficients.

(d) [8 marks] Show that, when the incident amplitude $A$ is equal to zero, there can exist a non-zero reflected wave-field if and only if

$$\left(\frac{c_p^2}{c_s^2} - 1\right)\cos(2\alpha) + \cos\bigl(2(\beta - \alpha)\bigr) = 0.$$

Hence show that it is possible for a wave-field to decay exponentially in the negative $x$-direction while propagating in the $y$-direction with speed $c_R$ satisfying the equation

$$\left(1 - \frac{c_R^2}{2c_s^2}\right)^2 = \sqrt{1 - \frac{c_R^2}{c_p^2}}\,\sqrt{1 - \frac{c_R^2}{c_s^2}}.$$

[*In this question, standard properties of $S$-waves and $P$-waves may be quoted without proof.*]

2. An elastic string of line density $\rho$ is stretched to a tension $T$ along the $x$-axis and subject to a constant gravitational body force $g$ in the negative $z$-direction. The string is fixed at its two ends so that the small transverse displacement $w(x)$ in the $z$-direction satisfies $w(L) = 0 = w(-L)$. A rigid smooth obstacle $z = f(x)$ is brought into contact with the string *from above.*

   (a) [6 marks] Derive the following conditions satisfied by $w(x)$:

   $$(f - w)\left(T\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - \rho g\right) = 0, \qquad (f - w) \geqslant 0, \qquad \left(T\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - \rho g\right) \geqslant 0,$$

   and show that $T$, $w$ and $\mathrm{d}w/\mathrm{d}x$ must all be continuous at points where the string makes or loses contact with the obstacle.

   (b) [6 marks] Let $\mathcal{V} = \left\{v \in C^1[-L, L] : v \leqslant f \text{ on } [-L, L], v(-L) = v(L) = 0\right\}$ and define $U : \mathcal{V} \to \mathbb{R}$ by

   $$U[v] = \int_{-L}^{L} \left[\frac{T}{2}\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2 + \rho g v\right] \mathrm{d}x.$$

   Show that

   $$T\int_{-L}^{L} \frac{\mathrm{d}w}{\mathrm{d}x}\left(\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{\mathrm{d}w}{\mathrm{d}x}\right) \mathrm{d}x \geqslant \rho g \int_{-L}^{L} (w - v)\, \mathrm{d}x \quad \text{for all } v \in \mathcal{V},$$

   and deduce that $U[w] \leqslant U[v]$ for all $v \in \mathcal{V}$. Interpret this result physically.

   (c) [6 marks] Suppose that the obstacle is given by $f(x) = -\delta + \kappa x^2/2$, where $\delta$ and $\kappa$ are positive parameters.

   Assuming that $\kappa > \rho g/T$, show that the string makes contact with the obstacle if $\rho g L^2/(2T) \leqslant \delta < \kappa L^2/2$, in a region $-s \leqslant x \leqslant s$, where

   $$\delta = \frac{\rho g L^2}{2T} + \frac{1}{2}\left(\kappa - \frac{\rho g}{T}\right) s(2L - s).$$

   (d) [7 marks] Now consider a system of two strings. One string with density $\rho$ and tension $T_1$ is pinned at its ends such that its displacement $w_1(x)$ satisfies $w_1(L) = 0 = w_1(-L)$. The second string with identical density $\rho$ but smaller tension $T_2 < T_1$ is pinned above the first string, such that its displacement $w_2(x)$ satisfies $w_2(L) = H = w_2(-L)$, where $H > 0$.

   Show that, if

   $$H \leqslant \frac{\rho g L^2}{2}\left(\frac{1}{T_2} - \frac{1}{T_1}\right),$$

   then the two strings make contact in a region $-s \leqslant x \leqslant s$, and find an expression for $s$.

3. (a) [8 marks] The stress tensor in a two-dimensional granular material is denoted by

$$\mathcal{T} = \begin{pmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{xy} & \tau_{yy} \end{pmatrix}.$$

Calculate the stress on a line element with unit normal $\mathbf{n} = (\cos\theta, \sin\theta)^{\mathrm{T}}$ and hence show that the normal stress $N$ and shear stress $F$ lie on the *Mohr circle*

$$F^2 + \left(N - \frac{1}{2}(\tau_{xx} + \tau_{yy})\right)^2 = \frac{(\tau_{xx} - \tau_{yy})^2}{4} + \tau_{xy}^2.$$

Hence show that the Coulomb yield condition $|F| \leqslant -N\tan\phi$, where $\phi$ is the angle of friction, leads to the condition

$$-(\tau_{xx} + \tau_{yy})\sin\phi \geqslant \sqrt{(\tau_{xx} - \tau_{yy})^2 + 4\tau_{xy}^2}.$$

(b) [17 marks] A granular material occupies the region $r > a$ outside a cavity of radius $a$, where $r$ is the radial polar coordinate. The material is subject to a uniform pressure $P_{\text{out}} > 0$ as $r \to \infty$, and an internal pressure $P_{\text{in}} \geqslant P_{\text{out}}$ at the cavity wall $r = a$.

(i) Show that, while the material remains elastic, the stress components satisfy the compatibility condition

$$\frac{\mathrm{d}}{\mathrm{d}r}(\tau_{rr} + \tau_{\theta\theta}) = 0.$$

(ii) Hence show that, as $P_{\text{in}}$ is gradually increased from a starting value of $P_{\text{out}}$, yield first occurs at $r = a$ when $P_{\text{in}} = 2P_{\text{out}}/(1+k)$, where $k = (1 - \sin\phi)/(1 + \sin\phi)$.

(iii) Show that, as $P_{\text{in}}$ is increased further, the material yields in a region $a < r < s$, where

$$\frac{s}{a} = \left(\frac{(1+k)P_{\text{in}}}{2P_{\text{out}}}\right)^{1/(1-k)}.$$

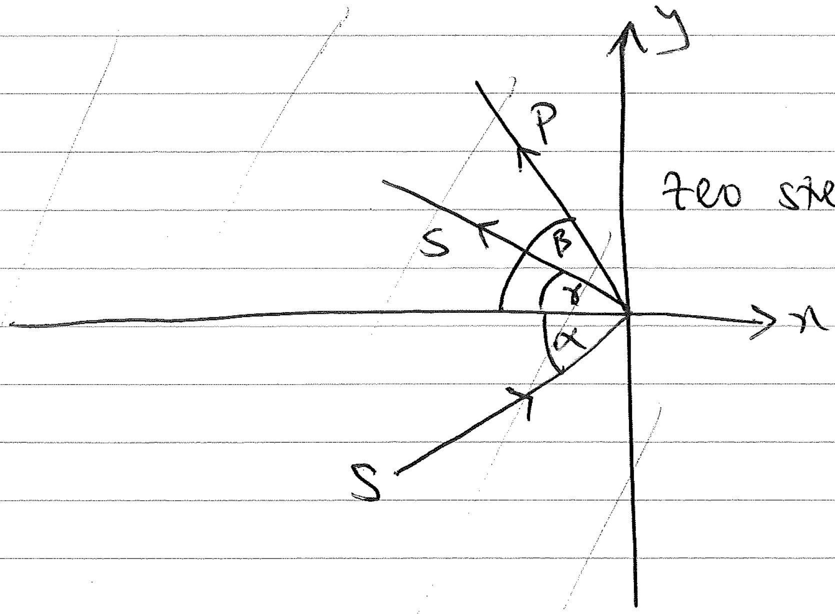
[*You may use without proof the radially symmetric steady Navier equation*

$$\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0,$$

*and the elastic constitutive relations*

$$\tau_{rr} = (\lambda + 2\mu)\frac{\mathrm{d}u_r}{\mathrm{d}r} + \lambda\frac{u_r}{r}, \qquad \tau_{\theta\theta} = \lambda\frac{\mathrm{d}u_r}{\mathrm{d}r} + (\lambda + 2\mu)\frac{u_r}{r},$$

*where $u_r$ is the radial displacement, and $\{\lambda, \mu\}$ are the Lamé constants.*]




Zero stress BCs: $\tau_{xx} = \tau_{xy} = 0$ at $x=0$

$$\therefore \lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\mu\frac{\partial u}{\partial x} = \mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=0$$

Now $c_p^2 = \dfrac{\lambda+2\mu}{\rho}$, $c_s^2 = \dfrac{\mu}{\rho}$ so $\dfrac{\lambda}{\rho} = c_p^2 - 2c_s^2$

& BCs can be written as $\boxed{\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} = 0 = c_p^2\dfrac{\partial u}{\partial x} + (c_p^2-2c_s^2)\dfrac{\partial v}{\partial y}}$

Bookwork at $x=0$

To satisfy both these BCs, the reflected field must consist of a P-wave and an S-wave, with wave-vectors

$$\underset{\sim}{k} = k_p\begin{pmatrix}-\cos\beta\\ \sin\beta\end{pmatrix} \quad \text{and} \quad k_s\begin{pmatrix}-\cos\gamma\\ \sin\gamma\end{pmatrix}, \text{ say}$$

with $k_p = \dfrac{\omega}{c_p}$, $k_s = \dfrac{\omega}{c_s}$.

Then recall · for P-wave, $\underset{\sim}{u}$ is parallel to $\underset{\sim}{k}$

· for S-wave, $\underset{\sim}{u}$ is perpendicular to $\underset{\sim}{k}$

So reflected wave field takes the form

$$\underset{\sim}{u}_{ref} = r_s \begin{pmatrix} \sin\gamma \\ \cos\gamma \end{pmatrix} e^{ik_s(-x\cos\gamma + y\sin\gamma) - i\omega t}$$

$$+ r_p \begin{pmatrix} -\cos\beta \\ \sin\beta \end{pmatrix} e^{ik_p(-x\cos\beta + y\sin\beta) - i\omega t}$$

where reflection coefficients $r_s, r_p$ are to be determined.
Now apply BCs at $x=0$. First require the exponents all to agree, i.e.

$$\underbrace{k_s \sin\alpha = k_s \sin\gamma = k_p \sin\beta}$$

i.e. $\boxed{\gamma = \alpha}$ S-wave reflection is specular

and $\boxed{\frac{\sin\beta}{c_p} = \frac{\sin\alpha}{c_s}}$ (Snell's law)

Bookwork

If $\underline{\sin\alpha > c_s/c_p}$ then $\underline{\sin\beta > 1}$

then $\beta$ is complex, $\underline{\beta = \frac{\pi}{2} - ib}$

with $\underline{\sin\beta = \cosh b, \quad \cos\beta = i\sinh b}$

So this gives a field that propagates in the $y$-direction while decaying exponentially in the $(-x)$-direction (This is an evanescent field analogous to total internal reflection in refraction).

NB exponent $e^{-ik_p x\cos\beta} = \underline{e^{k_p x \sinh b}}$ decays exponentially for $x<0$ provided $\underline{b>0}$.

Similar to problem sheet

Now apply BCs:

(i) $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \Rightarrow ik_s \sin\alpha . A\sin\alpha + ik_s \sin\gamma . r_s \sin\gamma$

$+ ik_p \sin\beta(-r_p\cos\beta) - ik_s\cos\gamma . r_s\cos\gamma$

$- ik_p\cos\beta . r_p\sin\beta - ik_s\cos\alpha . A\cos\alpha = 0$

$$\therefore \boxed{\cos(2\alpha)k_s r_s + \sin(2\beta)k_p r_p + \cos(2\alpha)k_s A = 0}$$

(ii) $c_p^2 \frac{\partial u}{\partial x} + (c_p^2 - 2c_s^2)\frac{\partial v}{\partial y} = 0$

$\Rightarrow c_p^2\left[ik_s\cos\alpha A\sin\alpha - ik_s\cos\gamma r_s\sin\gamma + ik_p\cos\beta r_p\cos\beta\right]$

$+ (c_p^2 - 2c_s^2)\left[-ik_s\sin\alpha A\cos\alpha + ik_s\sin\gamma r_s\cos\gamma + ik_p\sin\beta r_p\sin\beta\right]$

$= 0$

$$\therefore \boxed{\begin{aligned} -c_s^2\sin(2\alpha)k_s r_s &+ \left[c_p^2 - c_s^2 + c_s^2\cos(2\beta)\right]k_p r_p \\ &+ c_s^2\sin(2\alpha)k_s A = 0 \end{aligned}}$$

Similar to problem sheet

i.e. $$\begin{pmatrix} \cos(2\alpha) & \sin(2\beta) \\ -\sin(2\alpha) & \frac{c_p^2}{c_s^2} - 1 + \cos(2\beta) \end{pmatrix}\begin{pmatrix} k_s r_s \\ k_p r_p \end{pmatrix} = -\begin{pmatrix} \cos(2\alpha) \\ \sin(2\alpha) \end{pmatrix} k_s A$$

determinant $\Delta = \cos(2\alpha)\left[\frac{c_p^2}{c_s^2} - 1 + \cos(2\beta)\right] + \sin(2\alpha)\sin(2\beta)$

$$\boxed{\Delta = \left(\frac{c_p^2}{c_s^2} - 1\right)\cos(2\alpha) + \cos(2\beta - 2\alpha)}$$

Note a solution with $A=0$ is possible iff

$$\boxed{\Delta = \left(\frac{c_p^2}{c_s^2} - 1\right)\cos(2\alpha) + \cos(2\beta - 2\alpha) = 0}$$

New

Look for Rayleigh wave by writing

$\alpha = \frac{\pi}{2} - ia, \quad \beta = \frac{\pi}{2} - ib \quad$ with $a, b > 0$

then $\Delta = \left(\frac{c_p^2}{c_s^2} - 1\right)\left(-\cos(2ia)\right) + \cos\left(2i(a-b)\right) = 0$

$$\therefore \left(\frac{c_p^2}{c_s^2} - 1\right)\cosh(2a) = \cosh(2a - 2b)$$

wave-speed is given by $c_R = \frac{\omega}{k_s \sin\alpha} = \frac{\omega}{k_p \sin\beta}$

i.e. $c_R = \frac{c_s}{\cosh a} = \frac{c_p}{\cosh b}$

i.e. $\cosh a = \frac{c_s}{c_R}, \quad \cosh b = \frac{c_p}{c_R}$

$\therefore \cosh(2a) = 2\cosh^2 a - 1 = \frac{2c_s^2}{c_R^2} - 1$

$\cosh(2b) = \frac{2c_p^2}{c_R^2} - 1$

$\sinh(2a) = 2\sqrt{\frac{c_s^2}{c_R^2} - 1} \cdot \frac{c_s}{c_R}$

$\sinh(2b) = 2\sqrt{\frac{c_p^2}{c_R^2} - 1} \cdot \frac{c_p}{c_R}$

So.... $\left(\frac{c_p^2}{c_s^2} - 1\right)\left(\frac{2c_s^2}{c_R^2} - 1\right) = \left(\frac{2c_s^2}{c_R^2} - 1\right)\left(\frac{2c_p^2}{c_R^2} - 1\right)$

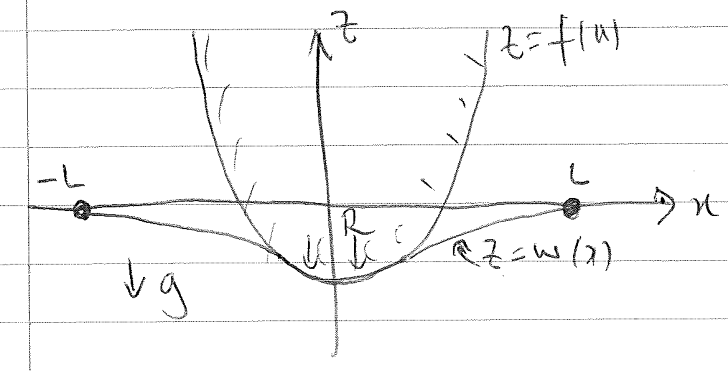
$$- \frac{4c_s c_p}{c_R^2}\sqrt{\frac{c_s^2}{c_R^2} - 1}\sqrt{\frac{c_p^2}{c_R^2} - 1}$$

Rearrange to

$$\left(\frac{2C_s^2}{C_R^2}-1\right)\left(\frac{2C_p^2}{C_R^2}-\frac{C_p^2}{C_s^2}\right)=\frac{4C_s^2C_p^2}{C_R^4}\sqrt{1-\frac{C_R^2}{C_s^2}}\sqrt{1-\frac{C_R^2}{C_p^2}}$$

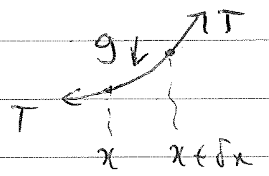
ie.
$$\boxed{\left(1-\frac{C_R^2}{2C_s^2}\right)^2=\sqrt{1-\frac{C_R^2}{C_s^2}}\sqrt{1-\frac{C_R^2}{C_p^2}}}$$

Unfamiliar approach to Rayleigh waves




In contact set: $w = f$

In non-contact set: force balance on a small element of the string




$$\to \frac{d}{dx}\left[T\,\underline{t}\right] = \begin{pmatrix} 0 \\ \rho g \end{pmatrix} \quad \text{where } \underline{t} = \frac{1}{\sqrt{1+(w')^2}}\begin{pmatrix} 1 \\ dw/dx \end{pmatrix}$$

linearise to get $\frac{dT}{dx} = 0$ & $T\frac{d^2w}{dx^2} = \rho g$

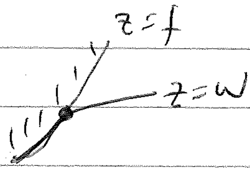
In contact set, include reaction force $R(x) \geq 0$:

$$T\frac{d^2w}{dx^2} - \rho g = R \geq 0$$

In non-contact set, string is below obstacle: $w < f$

Combine all the above to get given linear complementarity conditions.

At edge of contact set:




continuity of w just means that string can't break.

force balance $0 = [T\,\underset{\sim}{t}]_-^+ \sim \left[T\begin{pmatrix}1\\ dw/dx\end{pmatrix}\right]_-^+$

gives continuity of T & $dw/dx$.

Bookwork

From complementary condition

$$0 = \int_{-L}^{L} (f-w)\left(Tw'' - \rho g\right) dx$$

$$= \int_{-L}^{L} \left[ \underbrace{(f-v)}_{\geq 0} \underbrace{(Tw'' - \rho g)}_{\geq 0} + Tw''(v-w) + \rho g(w-v) \right] dx$$

for any $v \in V$

$$\therefore \rho g \int_{-L}^{L} (w-v)\, dx \leq T \int_{-L}^{L} w''(w-v)\, dx = T \int_{-L}^{L} w'(v'-w')dx$$

NB in final step integration by parts is ok because $w'$ and $v'$ are continuous and $w'$ is piecewise differentiable.

Now $\quad U[v] - U[w] = \int_{-L}^{L} \frac{T}{2}\left[(v')^2 - (w')^2\right] + \rho g(v-w)\, dx$

$$\geq \int_{-L}^{L} \frac{T}{2}\left[(v')^2 - (w')^2\right] + Tw'(w'-v')\, dx$$

[from above inequality]

$$= \frac{T}{2} \int_{-L}^{L} (v'-w')^2\, dx \geq 0$$

Thus the displacement is such as to minimise the net elastic plus gravitational potential energy without going through the obstacle.

Bookwork

(i) If there is no contact, then

$$w'' = \rho g/T, \quad w(\pm L) = 0$$

gives $\boxed{w = -\frac{\rho g}{2T}(L^2 - x^2)}$

obstacle makes contact at $x=0$ when $\delta = \frac{\rho g L^2}{2T}$

provided curvature of obstacle > curvature of string
ie $K > \rho g/T$

For $\delta > \frac{\rho g L^2}{2T}$ introduce contact region $x \in (-s, s)$

By symmetry, only need to consider $x > 0$.

Then in $x > s$ we have $w'' = \frac{\rho g}{T}$

with $w(L) = 0, \quad w(s) = f(s), \quad w'(s) = f'(s)$

$\therefore \quad w'(x) = Ks + \frac{\rho g}{T}(x - s)$

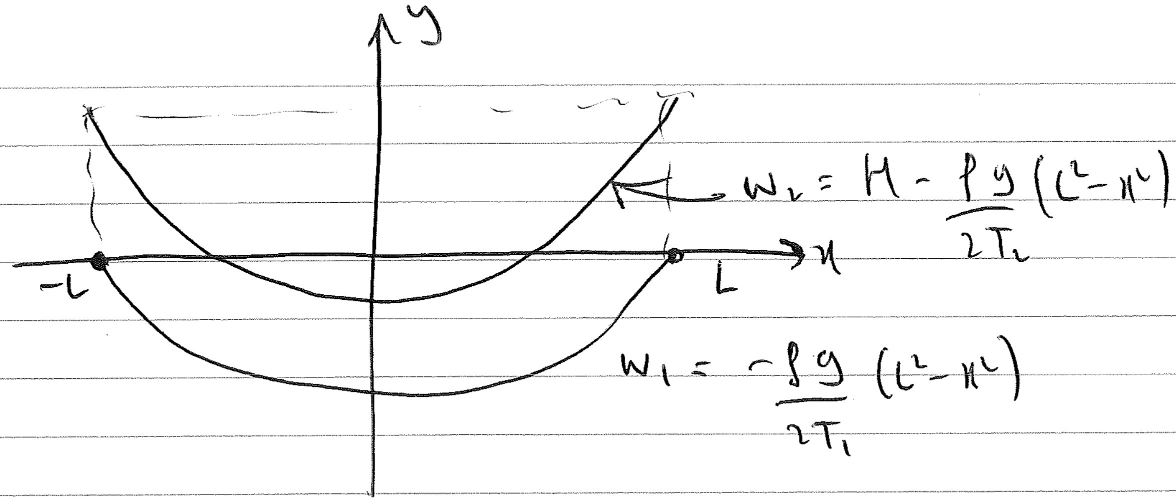
$$\boxed{w(x) = -\delta + \frac{Ks^2}{2} + Ks(x-s) + \frac{\rho g}{2T}(x-s)^2}$$

& $w(L) = 0$ gives

$$\delta = KLs - \frac{Ks^2}{2} + \frac{\rho g}{2T}(L-s)^2$$

ie $\boxed{\delta = \frac{\rho g L^2}{2T} + \frac{1}{2}\left(K - \frac{\rho g}{T}\right)s(2L - s)}$

New example




Before contact, picture is as above. They make contact near $x=0$ if

$$H - \frac{\rho g L^2}{2T_2} < -\frac{\rho g L^2}{2T_1}$$

ie. if

$$\boxed{H < \frac{\rho g L^2}{2}\left(\frac{1}{T_2} - \frac{1}{T_1}\right)}$$

If so, then contact region is $-s<x<s$ by symmetry, and there we have

$$\left.\begin{aligned} T_1 w_1'' &= \rho g + R \\ T_2 w_2'' &= \rho g - R \end{aligned}\right\} \quad \text{where } R = \text{mutual reaction force.}$$

so $w'' = \dfrac{2\rho g}{T_1+T_2} = K$ is the curvature in the contact region.

This plays the role of $K$ in our previous calculations:

$$\delta = \frac{\rho g L^2}{2T_1} + \frac{1}{2}\left(K - \frac{\rho g}{T_1}\right) s(2L-s) \qquad \text{for string 1}$$

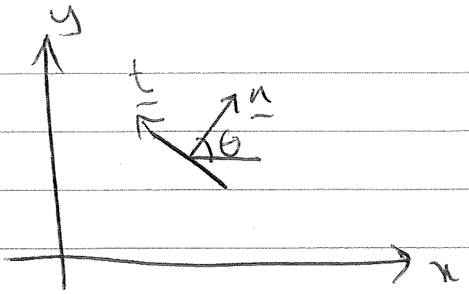
$$\delta + H = \frac{\rho g L^2}{2T_2} + \frac{1}{2}\left(K - \frac{\rho g}{T_2}\right) s(2L-s) \qquad \text{for string 2}$$

Substract to get

$$H = \frac{\rho g}{2}\left(\frac{1}{T_2} - \frac{1}{T_1}\right)(L-s)^2$$

$$\therefore \boxed{\frac{s}{L} = 1 - \left[\frac{2H}{\rho g\left(\frac{1}{T_2} - \frac{1}{T_1}\right)}\right]^{1/2}}$$

New example




Stress on line element

$$\underline{\sigma} = T\underline{n} = \begin{pmatrix}\tau_{xx} & \tau_{xy}\\ \tau_{xy} & \tau_{yy}\end{pmatrix}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$$

$$= \begin{pmatrix}\tau_{xx}\cos\theta + \tau_{xy}\sin\theta\\ \tau_{xy}\cos\theta + \tau_{yy}\sin\theta\end{pmatrix}$$

Normal stress $N = \underline{\sigma}\cdot\underline{n} = \tau_{xx}\cos^2\theta + 2\tau_{xy}\sin\theta\cos\theta + \tau_{yy}\sin^2\theta$

$$\therefore \boxed{N = \tfrac{1}{2}(\tau_{xx}+\tau_{yy}) + \tfrac{1}{2}(\tau_{xx}-\tau_{yy})\cos(2\theta) + \tau_{xy}\sin(2\theta)}$$

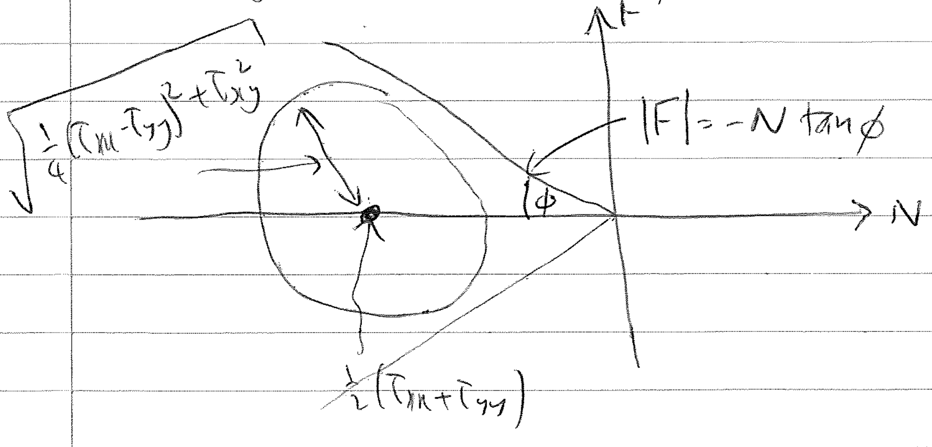
Shear stress $F = \underline{\sigma}\cdot\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} = (\tau_{yy}-\tau_{xx})\sin\theta\cos\theta + \tau_{xy}\cos(2\theta)$

$$\therefore \boxed{F = \tau_{xy}\cos(2\theta) - \tfrac{1}{2}(\tau_{xx}-\tau_{yy})\sin(2\theta)}$$

∴ eliminating $\theta$, get Mohr circle

$$\boxed{\left[N - \tfrac{1}{2}(\tau_{xx}+\tau_{yy})\right]^2 + F^2 = \tau_{xy}^2 + \tfrac{1}{4}(\tau_{xx}-\tau_{yy})^2}$$

In granular medium, need $N \le 0 \ \forall\theta$



The lines $|F| = -N\sin\phi$ are tangent to the Mohr circle when

$$\sin\phi = \frac{\sqrt{\frac{1}{4}(\tau_{xx}-\tau_{yy})^2 + \tau_{xy}^2}}{-\frac{1}{2}(\tau_{xx}+\tau_{yy})}$$

and Coulomb condition is satisfied $\forall\theta$ iff $\sin\phi \geq$ this value, i.e.

$$\boxed{-(\tau_{xx}+\tau_{yy})\sin\phi \geq \sqrt{(\tau_{xx}-\tau_{yy})^2 + 4\tau_{xy}^2}}$$

Bookwork

(i) While material is elastic, constitutive relations give

$$0 = \frac{d\tau_{rr}}{dr} + \frac{\tau_{rr}-\tau_{\theta\theta}}{r} = (\lambda+2\mu)\left[\frac{d^2u_r}{dr^2}\right] + \lambda\left[\frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2}\right] + \frac{2\mu}{r}\left[\frac{du_r}{dr} - \frac{u_r}{r}\right]$$

$$\therefore 0 = (\lambda+2\mu)\left[\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2}\right]$$

while $\frac{d}{dr}\left[\tau_{rr}+\tau_{\theta\theta}\right] = 2(\lambda+\mu)\left[\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2}\right]$

So comparing we have $\boxed{\frac{d}{dr}\left(\tau_{rr}+\tau_{\theta\theta}\right) = 0}$

Unfamiliar approach

(ii) Given boundary conditions: $\boxed{\begin{array}{l}\tau_{rr},\ \tau_{\theta\theta} \to -P_{out} \text{ as } r\to\infty \\ \tau_{rr} = -P_{in} \text{ at } r=a\end{array}}$

So, while material is elastic, $\tau_{rr}+\tau_{\theta\theta} = -2P_{out}$

and $0 = \frac{d\tau_{rr}}{dr} + \frac{\tau_{rr}-\tau_{\theta\theta}}{r} = \frac{d\tau_{rr}}{dr} + \frac{2\tau_{rr}}{r} + \frac{2P_{out}}{r}$

$$\therefore \frac{d}{dr}\left(r^2\tau_{rr}\right) = -2rP_{out}$$

$$r^2\tau_{rr} = (a^2-r^2)P_{out} - a^2P_{in}$$

$$\therefore \boxed{\begin{array}{l}\tau_{rr} = \frac{a^2(P_{out}-P_{in})}{r^2} - P_{out} \\ \tau_{\theta\theta} = \frac{a^2(P_{in}-P_{out})}{r^2} - P_{out}\end{array}}$$

In cylindrical symmetric geometry, Coulomb criterion is

$$-(\tau_{rr}+\tau_{\theta\theta})\sin\phi \geq |\tau_{rr}-\tau_{\theta\theta}|$$

Here $-(\tau_{rr}+\tau_{\theta\theta}) = 2P_{out}$, $\tau_{\theta\theta}-\tau_{rr} = \dfrac{2a^2}{r^2}(P_{in}-P_{out})$

RHS is maximized at $r=a$. Yield first occurs when $\leq$ becomes $=$ at $r=a$, i.e.

$$2P_{out}\sin\phi = 2(P_{in}-P_{out})$$

i.e. when $\boxed{P_{in} = (1+\sin\phi)P_{out} = \dfrac{2P_{out}}{1+k}}$

For $P_{in} > \dfrac{2P_{out}}{1+k}$, introduce plastic region $a<r<s$

In $r>s$: still elastic: $\boxed{\begin{aligned}\tau_{rr} &= -P_{out} - \frac{A}{r^2}\\ \tau_{\theta\theta} &= -P_{out} + \frac{A}{r^2}\end{aligned}}$

with $A$ to be determined.

In $a<r<s$ plastic $-(\tau_{rr}+\tau_{\theta\theta})\sin\phi = \tau_{\theta\theta}-\tau_{rr}$

$$\therefore \tau_{\theta\theta} = \tau_{rr}\left(\frac{1-\sin\phi}{1+\sin\phi}\right) = k\tau_{rr}$$

Then $\boxed{\begin{aligned}&\frac{d\tau_{rr}}{dr} + \frac{(1-k)\tau_{rr}}{r} = 0\\ \text{with}\quad &\tau_{rr} = -P_{in} \text{ at } r=a\end{aligned}}$

gives $\boxed{\begin{aligned} \tau_{rr} &= -p_{in}\left(\frac{a}{r}\right)^{1-k} \\ \tau_{\theta\theta} &= k\,\tau_{rr} \end{aligned}}$ in $a<r<s$

Yield condition $\Rightarrow$ both $\tau_{rr}$ & $\tau_{\theta\theta}$ are continuous across $r=s$, ie.

$$\begin{cases} -p_{in}\left(\dfrac{a}{s}\right)^{1-k} = -p_{out} - \dfrac{A}{s^2} \\ -k\,p_{in}\left(\dfrac{a}{s}\right)^{1-k} = -p_{out} + \dfrac{A}{s^2} \end{cases}$$

$$\Rightarrow \boxed{\frac{s}{a} = \left[\frac{(1+k)\,p_{in}}{2\,p_{out}}\right]^{1/(1-k)}}$$

and $\boxed{A = \dfrac{(1-k)\,s^2}{2}\,p_{in}\left(\dfrac{a}{s}\right)^{(1-k)}}$

ie, $\boxed{A = \dfrac{(1-k)\,p_{out}}{(1+k)\,p_{in}}\cdot s^2}$

New example