SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C5.2

Master of Science in Mathematical Sciences: Paper C5.2

ELASTICITY AND PLASTICITY

TRINITY TERM 2020

Monday 01 June

Opening time: 14:30 (BST)

You have 2 hours 45 minutes to complete the paper and upload your answer file

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you observe the following points:

1. Write with a black or blue pen OR with a stylus on tablet (colour set to black or blue).
2. On the first page, write
   - your candidate number
   - the paper code
   - the paper title
   - and your course title (e.g. FHS Mathematics and Statistics Part B)
   - but ***do not*** enter your name or college.
3. For each question you attempt,
   - start writing on a new sheet of paper,
   - indicate the question number clearly at the top of each sheet of paper,
   - number each page
4. Before scanning and submitting your work,
   - add to the first page, in numerical order, the question numbers attempted,
   - cross out all rough working and any working you do not want to be marked,
   - and orient all scanned pages in the same way.
5. Submit a single PDF document with your answers for this paper.

If you do not attempt any questions at all on this paper, you should still submit a single page indicating that you have opened the exam but not attempted any questions. Please make sure to write your candidate number on this single page.

1. Consider an elastic beam of length L and bending stiffness B in equilibrium and subject to negligible body force. The beam undergoes two-dimensional deformations in the (x, y)-plane, such that its centre-line makes an angle $\theta(s)$ with the x-axis, where s is arc-length. Compressive forces $(P, 0)$ and $(-P, 0)$ are applied at the two ends of the beam $s = 0$ and $s = L$, respectively. The end $s = 0$ is clamped parallel to the x-axis, while the end $s = L$ is clamped at a small angle α to the x-axis.

   (a) [4 marks] Derive the *beam equation*

   $$B\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + P\sin\theta = 0,$$

   explaining clearly any assumptions that you make.

   Obtain the dimensionless model

   $$\frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2} + \pi^2\lambda\sin(\theta) = 0, \qquad \theta(0) = 0, \qquad \theta(1) = \alpha,$$

   and define the normalised compressive load λ in terms of P, B and L.

   (b) [5 marks] Assuming that $|\alpha| \ll 1$ and that θ remains small enough for the problem to be linearised, obtain the approximate solution

   $$\theta(\xi) \sim A\sin\left(\pi\xi\sqrt{\lambda}\right),$$

   and derive an expression for the amplitude A, in terms of λ and α.

   Show that, if the applied load is gradually increased from zero, the linearisation fails as λ approaches 1.

   Explain why nonlinearity becomes important when $\lambda - 1 = O\left(|\alpha|^{2/3}\right)$ and $\theta = O\left(|\alpha|^{1/3}\right)$.

   (c) [10 marks] Now assume that $\alpha = \epsilon^3\gamma$ and $\lambda = 1 + 3\epsilon^2/2$, where $0 < \epsilon \ll 1$ and $\gamma = O(1)$. Show that

   $$\theta(\xi) \sim \epsilon A_1\sin(\pi\xi) + O\left(\epsilon^3\right),$$

   where

   $$A_1\left(A_1^2 - 12\right) = \frac{16\gamma}{\pi}.$$

   [*You may use without proof the identity* $\sin^3(z) \equiv \big(3\sin(z) - \sin(3z)\big)/4$.]

   (d) [6 marks] Sketch a bifurcation diagram of A_1 versus γ, and describe qualitatively how the beam would behave if γ were gradually increased from -2π to 2π and then gradually decreased back to -2π.

2. Consider steady antiplane strain of a uniform linear elastic solid, with displacement field given by $\mathbf{u} = \big(0, 0, w(x, y)\big)^{\mathrm{T}}$.

   (a) [5 marks] Explain why w may be expressed as $w(x, y) = \mathrm{Im}\big[f(Z)\big]$ where f is a holomorphic function of $Z = x + \mathrm{i}y$.

   Show that the non-vanishing components of the stress tensor $\mathcal{T}$ are given by

$$\tau_{yz} + \mathrm{i}\tau_{xz} = \mu f'(Z).$$

   Deduce that $\mathrm{Re}\big[f(Z)\big]$ must be constant on any stress-free boundary.

   (b) [14 marks] Consider a Mode III crack whose surface S is given by the ellipse

$$\frac{x^2}{c^2 \cosh^2 \epsilon} + \frac{y^2}{c^2 \sinh^2 \epsilon} = 1,$$

   where $c > 0$ and $0 < \epsilon \ll 1$. The boundary of the crack is stress-free. The crack is inside an infinite medium and is subject to a linear stress field in the far field, such that

$$\tau_{xz} \sim bx + ay + o\left((x^2 + y^2)^{-1}\right), \qquad \tau_{yz} \sim ax - by + o\left((x^2 + y^2)^{-1}\right)$$

   as $x^2 + y^2 \to \infty$, where a and b are constants.

   Show that, in the limit as $\epsilon \to 0$, the displacement is given up to a constant by

$$w(x, y) = \frac{1}{2\mu} \,\mathrm{Im}\left[aZ\sqrt{Z^2 - c^2} + \mathrm{i}bZ^2\right],$$

   carefully defining the square root in the above expression.

   [*Hint: note that S is the image of the circle $|\zeta| = \mathrm{e}^{\epsilon}$ under the Joukowski conformal mapping*

$$Z = \frac{c}{2}\left(\zeta + \frac{1}{\zeta}\right).]$$

   (c) [6 marks] The crack will propagate if $K_{\mathrm{III}} > K^{\star}$, where the stress intensity factor is defined by

$$K_{\mathrm{III}} = \lim_{x \searrow c} \left(\tau_{yz}(x, 0)\sqrt{2\pi(x - c)}\right).$$

   Show that the crack propagates if a exceeds a critical value, which is to be determined in terms of $K^{\star}$ and c.

3. A perfectly plastic material undergoes plain strain in the region outside a circular cavity of radius a. The material is unstressed in the far field, while the surface of the cavity at $r = a$ is subject to a pressure $P = -\tau_{rr}$ and shear stress $\sigma = \tau_{r\theta}$, in terms of plane polar coordinates (r, θ). The applied pressure P is gradually increased from zero, while the applied shear stress is held at a constant value $\sigma = k\tau_Y$, where τ_Y is the yield stress and $0 \leqslant k < 1$.

(a) [2 marks] Evaluate the shear stress on an infinitesimal surface element with unit normal $\mathbf{n} = \mathbf{e}_r \cos\alpha + \mathbf{e}_\theta \sin\alpha$, and show that the maximum shear stress over all inclination angles α is given by the Tresca yield function

$$f = \sqrt{\frac{(\tau_{rr} - \tau_{\theta\theta})^2}{4} + \tau_{r\theta}^2}.$$

(b) [4 marks] Derive the compatibility condition

$$\frac{\mathrm{d}}{\mathrm{d}r}(\tau_{rr} + \tau_{\theta\theta}) = 0$$

which holds when the material is elastic.

(c) [10 marks] Assume that the material remains elastic while $f < \tau_Y$, and the stress satisfies $f = \tau_Y$ when the material is plastic.

Show that plastic yield first occurs when

$$\frac{P}{\tau_Y} = \sqrt{1 - k^2}.$$

Show that as P increases further, the material is plastic in a region $a < r < s$, in which the radial stress satisfies

$$\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} = \frac{2\tau_Y}{r^3}\sqrt{r^4 - k^2 a^4}.$$

(d) [9 marks] Deduce that the position $r = s$ of the elastic–plastic boundary is determined by the relation

$$\frac{P}{\tau_Y} - \sqrt{1 - k^2} = \cosh^{-1}\left(\frac{s^2}{ka^2}\right) - \cosh^{-1}\left(\frac{1}{k}\right).$$

[*You may use without proof the radially symmetric Navier equations:*

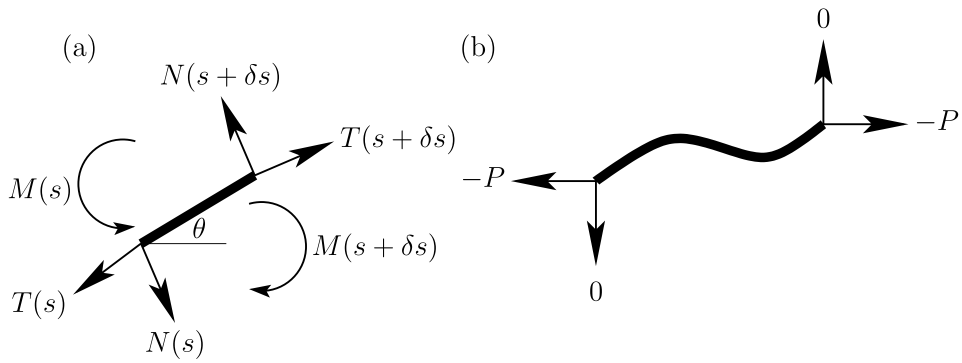
$$\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0, \qquad \frac{\mathrm{d}\tau_{r\theta}}{\mathrm{d}r} + \frac{2\tau_{r\theta}}{r} = 0,$$

and the linear elastic constitutive relations:

$$\tau_{rr} = (\lambda + 2\mu)\frac{\mathrm{d}u_r}{\mathrm{d}r} + \lambda\frac{u_r}{r}, \qquad \tau_{r\theta} = \mu\frac{\mathrm{d}u_\theta}{\mathrm{d}r}, \qquad \tau_{\theta\theta} = \lambda\frac{\mathrm{d}u_r}{\mathrm{d}r} + (\lambda + 2\mu)\frac{u_r}{r},$$

where λ, μ are the Lamé constants.]

Question 1



(a) Balance forces and moments on a small segment of the beam as shown in (a) above, neglecting inertia and body forces to get

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(T\cos\theta - N\sin\theta\right) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}s}\left(N\cos\theta + T\sin\theta\right) = 0, \qquad \frac{\mathrm{d}M}{\mathrm{d}s} - N = 0,$$

where T, N and M are the tension, shear force and bending moment. Apply boundary conditions as in (b) above to get

$$T\cos\theta - N\sin\theta = -P, \qquad N\cos\theta + T\sin\theta = 0,$$

and therefore

$$T = -P\cos\theta, \qquad N = P\sin\theta.$$

Now impose constitutive relation of the bending moment being proportional to the curvature, i.e.

$$M = -B\frac{\mathrm{d}\theta}{\mathrm{d}s},$$

where B is the bending stiffness. Combine all the above to get the beam equation

$$B\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + P\sin\theta = 0.$$

3 Bookwork

Non-dimensionalise $\xi = s/L$ to get the given model with

$$\lambda = \frac{L^2 P}{\pi^2 B}.$$

1 Straightforward manipulation

(b) Linearisation leads to

$$\theta'' + \pi^2\lambda\theta \sim 0,$$

and the solution subject to the given boundary conditions is

$$\theta(\xi) \sim A\sin\left(\pi\xi\sqrt{\lambda}\right), \qquad \text{where} \quad A = \frac{\alpha}{\sin\left(\pi\sqrt{\lambda}\right)}.$$

1 Straightforward manipulation

The amplitude increases without bound, so the linearisation must fail, as $\lambda \nearrow 1$. Note that

$$A = \frac{\alpha}{\sin\left(\pi\left[1-\sqrt{\lambda}\right]\right)} = O\left(\frac{\alpha}{1-\lambda}\right).$$

Furthermore, writing the beam equation in the form

$$\theta'' + \pi^2\theta \sim \pi^2(1-\lambda)\theta + \frac{\pi^2\theta^3}{6},$$

we see that nonlinearity balances excess load on the right-hand side when $A^2 = O(1-\lambda)$. We estimate A from the above to get

$$\frac{\alpha^2}{(1-\lambda)^2} = O(1-\lambda)$$

and therefore $\lambda - 1 = O\left(|\alpha|^{2/3}\right)$, $A = O\left(|\alpha|^{1/3}\right)$, as required.

4 Unfamiliar scaling using boundary value of θ

(c) Now make the suggested scalings and also scale $\theta = \epsilon\phi$ to get

$$\phi'' + \pi^2\left(1+3\epsilon^2/2\right)\frac{\sin(\epsilon\phi)}{\epsilon} = 0, \qquad \phi(0) = 0, \qquad \phi(1) = \epsilon^2\gamma$$

or, expanding in powers of ϵ:

$$\phi'' + \pi^2\phi \sim \epsilon^2\pi^2\left(\frac{\phi^3}{6} - \frac{3\phi}{2}\right) + O\left(\epsilon^4\right).$$

Now seek the solution as an asymptotic expansion in powers of ϵ^2, i.e. $\phi \sim \phi_0 + \epsilon^2\phi_1 + \cdots$. At leading order:

$$\phi_0'' + \pi^2\phi_0 = 0, \qquad \phi_0(0) = 0, \qquad \phi_0(1) = 0,$$

whose solution is

$$\phi_0(\xi) = A_1 \sin(\pi\xi),$$

where A_1 is (as yet) arbitrary.

At $O(\epsilon^2)$:

$$\phi_1'' + \pi^2\phi_1 = \pi^2\left(\frac{\phi_0^3}{6} - \frac{3\phi_0}{2}\right), \qquad \phi_1(0) = 0, \qquad \phi_1(1) = \gamma.$$

Substitute in for ϕ_0 and use the hint:

$$\begin{aligned}\phi_1'' + \pi^2\phi_1 &= \pi^2\left[\frac{A_1^3\sin^3(\pi\xi)}{6} - \frac{3A_1\sin(\pi\xi)}{2}\right]\\ &= \pi^2\left[\left(\frac{A_1^3}{8} - \frac{3A_1}{2}\right)\sin(\pi\xi) - \frac{A_1^3}{24}\sin(3\pi\xi)\right].\end{aligned}$$

Set $v(\xi) = \sin(\pi\xi)$ and note that $v'' + \pi^2 v = 0$, and hence

$$\int_0^1 \left(\phi_1'' + \pi^2\phi_1\right) v \,\mathrm{d}\xi = \int_0^1 \left[\left(\phi_1'' + \pi^2\phi_1\right) v - \left(v'' + \pi^2 v\right)\phi_1\right] \mathrm{d}\xi = \left[\phi_1' v - v'\phi_1\right]_0^1$$

$$\Rightarrow \quad \pi^2\int_0^1 \left[\left(\frac{A_1^3}{8} - \frac{3A_1}{2}\right)\sin(\pi\xi) - \frac{A_1^3}{24}\sin(3\pi\xi)\right]\sin(\pi\xi)\,\mathrm{d}\xi = \pi\left[\phi_1(0) + \phi_1(1)\right]$$

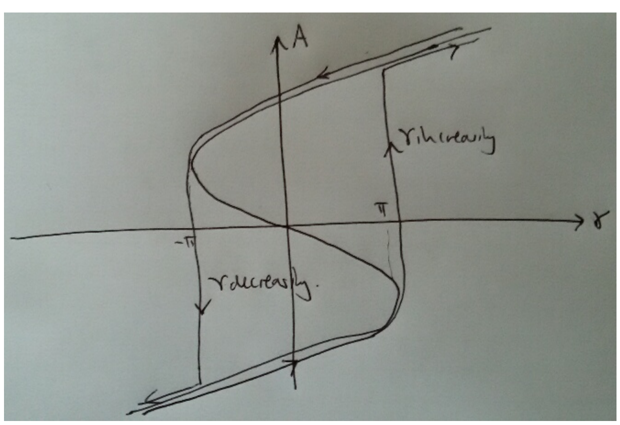
$$\Rightarrow \quad \frac{\pi}{2}\left(\frac{A_1^3}{8} - \frac{3A_1}{2}\right) = \gamma$$

$$\Rightarrow \quad A_1\left(A_1^2 - 12\right) = \frac{16\gamma}{\pi}$$

as required.

10 Generalisation of problem sheet

(d) Left-hand side is an odd cubic function of A_1 with stationary points at $A_1 = \pm 2$, $\gamma = \mp\pi$:



When γ starts at -2π, the solution starts on the lower branch with A_1 negative and the beam bending downwards. As γ increases through the critical value π, the beam "snaps through" to the other branch where $A_1 > 0$ and the beam bends upwards. As γ then decreases it stays on the upper branch until $\gamma = -\pi$ when it then snaps back onto the lower branch again.

6 New example

Question 2

(a) With the given displacement field, the only onzero stress components are τ_{xz} and τ_{yz}, and the steady Navier equation reduces to

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0.$$

It follows that there exists a potential function $\phi(x, y)$ such that

$$\mu \frac{\partial \phi}{\partial y} = \tau_{xz} = \mu \frac{\partial w}{\partial x}, \qquad -\mu \frac{\partial \phi}{\partial x} = \tau_{yz} = \mu \frac{\partial w}{\partial y}.$$

We observe that $(-\phi)$ and w satisfy the Cauchy–Riemann equations and can therefore be expressed as

$$w = \operatorname{Im}\big[f(Z)\big], \qquad \phi = -\operatorname{Re}\big[f(Z)\big],$$

where f is a holomorphic function of $Z = x + \mathrm{i}y$.

2 Bookwork

Since $f(Z) = -\phi + \mathrm{i}w$, we have

$$\mu f'(Z) = -\mu \frac{\partial \phi}{\partial x} + \mu \mathrm{i} \frac{\partial w}{\partial x} = \tau_{yz} + \mathrm{i}\tau_{xz},$$

as required.

1 Bookwork

Consider a stress-free boundary $(x(s), y(s), z)^{\mathrm{T}}$ parameterised by arc-length s. The unit normal (suitably oriented) is given by $\mathbf{n} = (y'(s), -x'(s), 0)^{\mathrm{T}}$ and so the zero stress boundary condition reduces to

$$0 = \tau_{xz} \frac{\mathrm{d}y}{\mathrm{d}s} - \tau_{yz} \frac{\mathrm{d}x}{\mathrm{d}s} = \mu \frac{\partial \phi}{\partial y} \frac{\mathrm{d}y}{\mathrm{d}s} + \mu \frac{\partial \phi}{\partial x} \frac{\mathrm{d}x}{\mathrm{d}s} = \mu \frac{\mathrm{d}\phi}{\mathrm{d}s}.$$

So $\operatorname{Re}[f] = -\phi =$ constant at a stress-free boundary.

2 Bookwork

(b) From the far-field stress condition we have

$$\mu f'(Z) = \tau_{yz} + \mathrm{i}\tau_{xz} \sim (ax - by) + \mathrm{i}(bx + ay) = (a + \mathrm{i}b)Z \quad \text{as } Z \to \infty$$

and therefore

$$\mu f(Z) \sim (a + \mathrm{i}b)\frac{Z^2}{2} + O(1) \quad \text{as } Z \to \infty.$$

Use the hint: define

$$g(\zeta) = \frac{c}{2}\left(\zeta + \frac{1}{\zeta}\right), \qquad\qquad F(\zeta) = f\big(g(\zeta)\big).$$

Then

- $F(\zeta)$ is holomorphic in $|\zeta| > \mathrm{e}^{\epsilon}$;
- $\mu F(\zeta) \sim (a + \mathrm{i}b)\dfrac{c^2\zeta^2}{8}$ as $\zeta \to \infty$;
- $\mathrm{Re}\big[F(\zeta)\big] = 0$ (without loss of generality) when $|\zeta| = \mathrm{e}^{\epsilon}$.

$\boxed{5}$ Generalisation of problem sheet

E.g. by separating the variables in polars, or by using the Circle Theorem, or just by inspection, find the solution

$$\begin{aligned}\mu F(\zeta) &= \frac{c^2}{8}\left((a + \mathrm{i}b)\zeta^2 - (a - \mathrm{i}b)\frac{\mathrm{e}^{4\epsilon}}{\zeta^2}\right)\\ &\to \frac{c^2}{8}\left((a + \mathrm{i}b)\zeta^2 - \frac{(a - \mathrm{i}b)}{\zeta^2}\right) \quad \text{as } \epsilon \to 0.\end{aligned}$$

Now invert to get the solution in terms of Z. Note that

$$\zeta^2 + \frac{1}{\zeta^2} = \frac{4Z^2}{c^2} - 2, \qquad \text{and} \qquad \zeta^2 = \frac{2Z\zeta}{c} - 1$$

to get

$$\mu f(Z) = \frac{acZ\zeta}{2} - \frac{(a - \mathrm{i}b)Z^2}{2} - \frac{ibc^2}{4}.$$

Solve quadratic equation for ζ:

$$\zeta = \frac{Z + \sqrt{Z^2 - c^2}}{c},$$

where we take the positive square root to map the outside of S to $|\zeta| > \mathrm{e}^{\epsilon}$. Plug this in to get

$$\mu f(Z) = \frac{aZ\sqrt{Z^2 - c^2}}{2} + \frac{ibZ^2}{2} - \frac{ibc^2}{4},$$

and so

$$w(x, y) = \frac{1}{2\mu}\,\mathrm{Im}\left[aZ\sqrt{Z^2 - c^2} + \mathrm{i}bZ^2\right] + \text{constant}.$$

7 New example

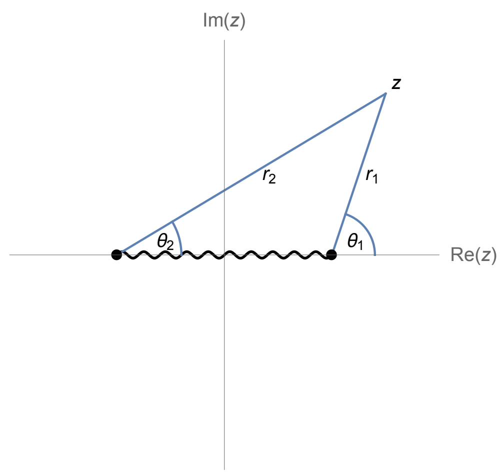
We define the square root as

$$\sqrt{Z^2 - c^2} = \sqrt{r_1 r_2}\,\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)/2}$$

where

$$r_1 = |Z - c|, \qquad r_2 = |Z + c|, \qquad \theta_1 = \arg(Z - c), \qquad \theta_2 = \arg(Z + c),$$

and we choose the branch where θ_1, $\theta_2 \in (-\pi, \pi]$ so the branch cut runs along the real-Z axis from $-c$ to c as shown.



2 Bookwork

(c) Calculate stress components from

$$\tau_{yz} + \mathrm{i}\tau_{xz} = \mu f'(Z) = \frac{1}{2}\left(a\sqrt{Z^2 - c^2} + \frac{aZ^2}{\sqrt{Z^2 - c^2}} + 2\mathrm{i}bZ\right)$$
$$= \frac{a\,(2Z^2 - c^2)}{2\sqrt{Z^2 - c^2}} + \mathrm{i}bZ$$

so, when $Z = x > c$,

$$\tau_{yz} = \frac{a\,(2x^2 - c^2)}{2\sqrt{x^2 - c^2}}, \qquad \tau_{xz} = bx.$$

We find

$$K_{\mathrm{III}} = \lim_{x \searrow c}\left(\frac{\sqrt{2\pi}\,a\,(2x^2 - c^2)}{2\sqrt{x + c}}\right) = \frac{ac^{3/2}\sqrt{\pi}}{2},$$

so the critical value of a is

$$a^\star = \frac{2K^\star}{c^{3/2}\sqrt{\pi}}.$$

6 Generalisation of problem sheet

Question 3

(a) The shear stress on an infinitesimal line element with normal $\mathbf{n} = \mathbf{e}_r \cos\alpha + \mathbf{e}_\theta \sin\alpha$ and tangent $\mathbf{t} = -\mathbf{e}_r \sin\alpha + \mathbf{e}_\theta \cos\alpha$ is given by

$$\begin{aligned}
\sigma &= \begin{pmatrix} -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \tau_{rr} & \tau_{r\theta} \\ \tau_{r\theta} & \tau_{\theta\theta} \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \\
&= -\sin\alpha(\tau_{rr}\cos\alpha + \tau_{r\theta}\sin\alpha) + \cos\alpha(\tau_{r\theta}\cos\alpha + \tau_{\theta\theta}\sin\alpha) \\
&= \tau_{r\theta}\cos(2\alpha) - \frac{\tau_{rr} - \tau_{\theta\theta}}{2}\sin(2\alpha). \\
&= \sqrt{\frac{(\tau_{rr} - \tau_{\theta\theta})^2}{4} + \tau_{r\theta}^2}\, \cos(2\alpha + \phi),
\end{aligned}$$

where $\tan\phi = (\tau_{rr} - \tau_{\theta\theta})/(2\tau_{r\theta})$. The maximum shear stress over all angles α is therefore given by the Tresca yield function

$$f = \sqrt{\frac{(\tau_{rr} - \tau_{\theta\theta})^2}{4} + \tau_{r\theta}^2}.$$

2 Standard result

(b) Use given Navier equations and elastic constitutive relations to get

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}r}\left((\lambda + 2\mu)\frac{\mathrm{d}u_r}{\mathrm{d}r} + \lambda\frac{u_r}{r}\right) + \frac{2\mu}{r}\left(\frac{\mathrm{d}u_r}{\mathrm{d}r} - \frac{u_r}{r}\right) \\
&= (\lambda + 2\mu)\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}u_r}{\mathrm{d}r} + \frac{u_r}{r}\right),
\end{aligned}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}r}(\tau_{rr} + \tau_{\theta\theta}) = 2(\lambda + \mu)\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}u_r}{\mathrm{d}r} + \frac{u_r}{r}\right) = 0.$$

4 From lectures

(c) Impose boundary conditions

$$\begin{aligned}
\tau_{rr} = -P, \quad \tau_{r\theta} = k\tau_Y \qquad & \text{at } r = a, \\
\tau_{rr},\ \tau_{\theta\theta} \to 0 \qquad & \text{as } r \to \infty
\end{aligned}$$

to deduce that $\tau_{rr} + \tau_{\theta\theta} = 0$ (while the material remains elastic) so we have to solve

$$\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} + \frac{2\tau_{rr}}{r} = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\tau_{rr}\right) = 0.$$

Applying the boundary condition at $r = a$ we get

$$\tau_{rr} = -\frac{Pa^2}{r^2}, \qquad \tau_{\theta\theta} = \frac{Pa^2}{r^2},$$

and similarly

$$\tau_{r\theta} = \frac{k\tau_Y a^2}{r^2}.$$

The yield function is given by

$$f^2 = \frac{(\tau_{rr} - \tau_{\theta\theta})^4}{4} + \tau_{r\theta}^2 = \frac{\left(P^2 + k^2\tau_Y^2\right) a^4}{r^4},$$

which is a decreasing function of r, so yield first occurs at $r = a$ when

$$P^2 + k^2\tau_Y^2 = \tau_Y^2, \qquad \text{i.e.} \qquad \frac{P}{\tau_Y} = \sqrt{1-k^2}.$$

5 New example

As P increases further, the material yields in a neighbourhood of $r = a$, say $a < r < s$. The second Navier equation holds everywhere, so we have

$$\tau_{r\theta} = \frac{k\tau_Y a^2}{r^2} \quad \text{for all } r > a.$$

In the plastic region $a < r < s$ we apply the yield condition

$$\begin{aligned} (\tau_{rr} - \tau_{\theta\theta})^2 + 4\tau_{r\theta}^2 &= 4\tau_Y^2 \\ \Rightarrow \qquad \tau_{\theta\theta} - \tau_{rr} &= 2\sqrt{\tau_Y^2 - \tau_{r\theta}^2} \\ &= 2\tau_Y\sqrt{1 - \frac{k^2a^4}{r^4}}. \end{aligned}$$

The radial Navier equation therefore becomes

$$\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} = \frac{\tau_{\theta\theta} - \tau_{rr}}{r} = \frac{2\tau_Y}{r^3}\sqrt{r^4 - k^2a^4}.$$

5 New example

(d) In $r > s$ the material is still elastic, so we have the solution

$$\tau_{rr} = -\frac{A}{r^2}, \quad \tau_{\theta\theta} = \frac{A}{r^2} \qquad\qquad \text{for } r > s,$$

but with A now unknown *a priori*; $A \geqslant 0$ by continuity with the above unyielded solution.

The yield condition applies at the free boundary $r = s$, i.e.

$$\frac{A^2 + k^2\tau_Y^2 a^4}{s^4} = \tau_Y^2$$
$$\Rightarrow \quad A = \tau_Y\sqrt{s^4 - k^2a^4}.$$

3 New example

Now use the Fundamental Theorem of Calculus and apply continuity of τ_{rr} at $r = s$:

$$\begin{aligned}
\big[\tau_{rr}\big]_a^s &= \int_a^s \frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r}\,\mathrm{d}r \\
\Rightarrow \quad P - \frac{A}{s^2} &= \int_a^s \frac{2\tau_Y}{r^3}\sqrt{r^4 - k^2a^4}\,\mathrm{d}r \\
\Rightarrow \quad \frac{P}{\tau_Y} &= \frac{\sqrt{s^4 - k^2a^4}}{s^2} + 2\int_a^s \frac{\sqrt{r^4 - k^2a^4}}{r^3}\,\mathrm{d}r.
\end{aligned}$$

Now integrate by parts:

$$\begin{aligned}
2\int_a^s \frac{\sqrt{r^4 - k^2a^4}}{r^3}\,\mathrm{d}r &= \left[-\frac{\sqrt{r^4 - k^2a^4}}{r^2}\right]_a^s + \int_a^s \frac{1}{r^2}\frac{2r^3}{\sqrt{r^4 - k^2a^4}}\,\mathrm{d}r \\
&= \sqrt{1-k^2} - \frac{\sqrt{s^4 - k^2a^4}}{s^2} + \int_{a^2}^{s^2} \frac{\mathrm{d}z}{\sqrt{z^2 - k^2a^4}} \\
&= \sqrt{1-k^2} - \frac{\sqrt{s^4 - k^2a^4}}{s^2} + \left[\cosh^{-1}\left(\frac{z}{ka^2}\right)\right]_{a^2}^{s^2}.
\end{aligned}$$

Now just assemble all the pieces to get

$$\frac{P}{\tau_Y} - \sqrt{1-k^2} = \cosh^{-1}\left(\frac{s^2}{ka^2}\right) - \cosh^{-1}\left(\frac{1}{k}\right).$$

6 New example