A12204W1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C5.2
Master of Science in Mathematical Sciences: Paper C5.2

Elasticity and Plasticity

TRINITY TERM 2023

Tuesday 30 May, 2:30pm to 4:15pm

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you observe the following points:

- start a new answer booklet for each question which you attempt.
- indicate on the front page of the answer booklet which question you have attempted in that booklet.
- cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.
- hand in your answers in numerical order.

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so

1. A linear elastic material with density ρ_1 and shear modulus μ_1 occupies a layer of constant thickness h in $0 \leqslant y \leqslant h$, $-\infty < x < \infty$ sandwiched between a rigid plate at $y = 0$ and another linear elastic material with density ρ_2 and shear modulus μ_2 occupying the region $y > h$, $-\infty < x < \infty$. The layered medium undergoes antiplane strain with displacement $\mathbf{u} = w_1(x, y, t)\mathbf{k}$ for $0 \leqslant y \leqslant h$ and displacement $\mathbf{u} = w_2(x, y, t)\mathbf{k}$ for $y > h$, where $\mathbf{k}$ is a unit vector in the z-direction. The displacement $w_1 = 0$ on the plate at $y = 0$, while the displacement and stress are continuous on the boundary between the elastic media at $y = h$.

   (a) [5 marks] Show that the transverse displacement w_j in either material satisfies the two-dimensional wave equation

   $$\frac{\partial^2 w_j}{\partial t^2} = c_j^2 \left(\frac{\partial^2 w_j}{\partial x^2} + \frac{\partial^2 w_j}{\partial y^2} \right),$$

   where the wave speed c_j should be determined for $j = 1$, 2.

   What are the boundary conditions on $y = h$?

   [*You may assume that Navier's equation and the constitutive relations for a linear elastic material with density ρ, first Lamé parameter λ and shear modulus μ are*

   $$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \tau_{ij}}{\partial x_j}, \qquad \tau_{ij} = \lambda(\boldsymbol{\nabla} \cdot \mathbf{u})\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

   where $\mathbf{u} = (u_i)$ is the displacement, (τ_{ij}) is the stress tensor and δ_{ij} is Kronecker's delta.]

   (b) [11 marks] Let $c_1 < c_2$. By seeking solutions of the form $w_j(x, y, t) = f_j(y) \exp\{ik(x - ct)\}$ for $j = 1$, 2, show that the system supports waves travelling in the x-direction with wave speed $c > 0$ and wavenumber $k > 0$, while decaying exponentially as $y \to \infty$, only if c and k are related parametrically to $m > 0$ and $\ell > 0$ by

   $$\frac{c^2}{c_1^2} = 1 + \frac{m^2}{k^2}, \qquad \frac{c^2}{c_2^2} = 1 - \frac{\ell^2}{k^2}, \qquad \tan mh = -\frac{\mu_1 m}{\mu_2 \ell}.$$

   (c) [9 marks] Eliminate c and ℓ to obtain the transcendental equation relating $\theta = mh$ and k given by

   $$\tan\theta = -\frac{\epsilon\theta}{\sqrt{a^2k^2 - \theta^2}}, \qquad (\star)$$

   where $0 < \theta < ak$ and the dependence of the positive constants ϵ and a on ρ_1, ρ_2, μ_1, μ_2 and h should be determined.

   Using a diagram explain why $(\star)$ has $N = \lfloor (ak/\pi) + 1/2 \rfloor$ roots for $\theta \in (0, ak)$, where $\lfloor x \rfloor$ is the largest integer $n \leqslant x$.

   Consider now the limit in which $0 < \epsilon \ll 1$ with ak held fixed. Derive for $N \geqslant 1$ the approximate roots for θ and hence c that arise from neglecting the right-hand side of $(\star)$. Explain why this approximation can fail for the largest wave speed and derive a valid approximation.

2. A thin beam of bending stiffness B undergoes small two-dimensional deformations in the (x, z)-plane. The beam lies along the x-axis and is of length $2L$ in its undeformed state. The ends of the beam are clamped so that its small transverse displacement $w(x)$ in the positive z-direction satisfies $w(\pm L) = 0$ and $w'(\pm L) = \pm\alpha$, where α is an imposed slope. The beam lies above a smooth rigid obstacle at $z = f(x) < 0$ with which it is brought into contact by increasing α from zero while remaining in equilibrium under the action of a body force $p(x)$ in the negative z-direction.

(a) [7 marks] Show that in an open region of non-contact the tension $T(x)$, transverse shear force $N(x)$ and bending moment $M(x)$ about the y-axis satisfy

$$\frac{\mathrm{d}T}{\mathrm{d}x} = 0, \qquad T\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \frac{\mathrm{d}N}{\mathrm{d}x} = p, \qquad \frac{\mathrm{d}M}{\mathrm{d}x} = N.$$

Explaining clearly any assumptions that you make, deduce that $w(x)$ satisfies

$$B\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - T\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + p = 0.$$

How is this equation modified in an open region of contact to account for the upward reaction force $R(x)$ exerted by the obstacle on the beam? State with justification the constraint that $R(x)$ should satisfy for a physically relevant solution.

(b) [3 marks] You may assume that the tension $T(x)$ and displacement $w(x)$ at an end point of an open region of contact or at an isolated point of contact satisfy the jump conditions

$$[T] = 0, \qquad [w] = 0, \qquad \left[\frac{\mathrm{d}w}{\mathrm{d}x}\right] = 0, \qquad \left[\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right] = 0, \qquad \left[B\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}\right] = R_i,$$

where $[g] = g(x_i^+) - g(x_i^-)$ denotes the jump in the quantity $g(x)$ across the point of contact at $x = x_i$ and R_i is the upward point reaction force exerted by the obstacle on the beam at $x = x_i$. State the physical basis for these boundary conditions. State with justification the constraint that R_i should satisfy for a physically relevant solution.

(c) [15 marks] Suppose that $T = 0$, $p(x) = 0$ and $f(x) = -H$, where H is a positive constant.

(i) Show that, as α is gradually increased from zero, the beam does not make contact with the obstacle until $\alpha = 2H/L$.

(ii) Show that, as α is increased further, the beam makes contact with the obstacle at an isolated point until $\alpha = 3H/L$.

(iii) Show that, as α is increased further still, the beam makes contact with the obstacle in a region $-s \leqslant x \leqslant s$, where s should be determined.

(iv) Determine the net upward force exerted by the obstacle on the beam for $\alpha \geqslant 0$.

3. Perfectly plastic material undergoes quasi-steady spherically symmetric strain in the region $r > a$ outside a spherical cavity of radius a, with displacement field given by $\mathbf{u}(r) = u(r)\mathbf{e}_r$, where (r, θ, ϕ) denote spherical polar coordinates and $\mathbf{e}_r$ is a unit vector in the r-direction. The stress tensor is diagonal with entries τ_{rr}, $\tau_{\theta\theta}$ and $\tau_{\phi\phi}$ that satisfy the Navier equations

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\tau_{rr}\right) - \frac{\tau_{\theta\theta}+\tau_{\phi\phi}}{r} = 0, \qquad\qquad \tau_{\theta\theta} = \tau_{\phi\phi}.$$

The *Tresca yield function* may be written in the form $f = \frac{1}{2}|\tau_{rr} - \tau_{\theta\theta}|$. The material remains linearly elastic while $f < \tau_{\mathrm{Y}}$ and the stress satisfies $f = \tau_{\mathrm{Y}}$ when the material is plastic, where $\tau_{\mathrm{Y}} > 0$ denotes the yield stress. The stress vanishes in the far field as $r \to \infty$, while the inner boundary at $r = a$ is subject to a non-negative pressure P.

(a) [5 marks] Show that when the material is elastic the displacement satisfies

$$\frac{\mathrm{d}}{\mathrm{d}r}\left((\lambda+2\mu)\left(\frac{\mathrm{d}u}{\mathrm{d}r} + \frac{2u}{r}\right)\right) = 0$$

and hence derive the *compatibility condition*

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\tau_{rr} + \tau_{\theta\theta} + \tau_{\phi\phi}\right) = 0.$$

[*You may assume the linear elastic constitutive relations*

$$\tau_{rr} = (\lambda+2\mu)\frac{\mathrm{d}u}{\mathrm{d}r} + 2\lambda\frac{u}{r}, \qquad\qquad \tau_{\theta\theta} = \tau_{\phi\phi} = \lambda\frac{\mathrm{d}u}{\mathrm{d}r} + 2(\lambda+\mu)\frac{u}{r},$$

where λ and μ are the Lamé constants.]

(b) [6 marks] First supposing that the material remains elastic, evaluate the stress outside the cavity. Hence show that, as P increases gradually from a starting value of zero, yield first occurs at $r = a$ when P reaches a critical value $P_{\mathrm{c}} = \frac{4}{3}\tau_Y$.

(c) [7 marks] Show that for $P > P_{\mathrm{c}}$, the material is plastic in a region $a < r < s$, where

$$s = a\exp\left(\frac{P - P_{\mathrm{c}}}{4\tau_{\mathrm{Y}}}\right).$$

(d) [7 marks] Suppose that P increases gradually from zero to a maximum value $P_{\mathrm{m}} > P_{\mathrm{c}}$, and then decreases to zero again. Assuming that the material instantaneously reverts to being elastic once the applied pressure starts to decrease, find the stress outside the cavity while it is being unloaded. Under what condition on P_{m} does the material yield again while it is being unloaded? Justify your answer.

C5.2/2023/Q1

(a) $\underline{u} = w_j(x,y,t) \Rightarrow$ nonzero stress components are $\tau_{xz} = \mu_j \frac{\partial w_j}{\partial x}$, $\tau_{yz} = \mu_j \frac{\partial w_j}{\partial y}$

So x- and y- components of Navier equation are satisfied identically, while z-component gives $\rho \frac{\partial^2 w_j}{\partial t^2} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y}$

Substituting $\Rightarrow$ $\boxed{\frac{\partial^2 w_j}{\partial t^2} \overset{(+)}{=} c_j^2 \left(\frac{\partial^2 w_j}{\partial x^2} + \frac{\partial^2 w_j}{\partial y^2} \right), \quad c_j = \sqrt{\frac{\mu_j}{\rho_j}}}$

BCs : $[\underline{u}]_{y=h-}^{y=h+} = \underline{0}$, $[(\tau_{ij})\underline{j}]_{y=h-}^{y=h+} = \underline{0} \Rightarrow$ $\boxed{w_1 = w_2, \ \mu_1 \frac{\partial w_1}{\partial y} = \mu_2 \frac{\partial w_2}{\partial y} \text{ on } y = h}$ B5

$\boxed{5}$

(b) $w_j = f_j(y) e^{ik(x-ct)}$ in (+) and the BCs $\Rightarrow$

$$\boxed{\begin{gathered} f_1'' + \left(\frac{c^2}{c_1^2} - 1\right) k^2 f_1 = 0 \text{ for } 0 < y < h, \quad f_2'' + \left(\frac{c^2}{c_2^2} - 1\right) k^2 f_2 = 0 \text{ for } y > h \\ \text{with } f_1(0) = 0, \ f_1(h) = f_2(h), \ \mu_1 f_1'(h) = \mu_2 f_2'(h), \ f_2 \text{ exp. small as } y \to \infty \end{gathered}}$$

f_2 can only decay exponentially as $y \to \infty$ if $c < c_2$, in which case $\boxed{f_2 = Ae^{-ly}}$ where $A \in \mathbb{C}$ and $l > 0$ s.t. $\boxed{l^2 = \left(1 - \frac{c^2}{c_2^2}\right) k^2}$

B4

Three cases to consider for f_1 : (i) $c > c_1$, (ii) $c = c_1$, (iii) $0 < c < c_1$

In case (i), $f_1(0) = 0 \Rightarrow$ $\boxed{f_1 = B \sin my}$, where $B \in \mathbb{C}$ and $m > 0$ wlog is s.t. $\boxed{m^2 = \left(\frac{c^2}{c_1^2} - 1\right) k^2}$

BCs on $y = h$ $\Rightarrow$

$$\underbrace{\begin{pmatrix} e^{-lh} & -\sin mh \\ \mu_2 l e^{-lh} & \mu_1 m \cos mh \end{pmatrix}}_{M} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\exists$ nontrivial solution $\begin{pmatrix}A\\B\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix} \Leftrightarrow \det(M)=0 \Leftrightarrow$ $\boxed{\tan mh = -\frac{\mu_1 m}{\mu_2 l}}$ (I)

where from above $\boxed{\frac{c^2}{c_1^2} = 1+\frac{m^2}{k^2} \quad \text{and} \quad \frac{c^2}{c_2^2} = 1-\frac{l^2}{k^2}}$ (II)

S4

It therefore remains to rule out cases (ii) and (iii).

In case (ii), $f_1(0)=0 \Rightarrow f_1 = \tilde{B}y$, where $\tilde{B} \in \mathbb{C}$, so that the BCs on $y=h$ give $\tilde{B}h = Ae^{-lh}$, $\mu_1\tilde{B} = -\mu_2 A l e^{-lh}$. So for nontrivial solutions $\mu_1 = -\mu_2 lh$ $\lightning$
[Or recover from case (i) in limit $m \to 0$ with $\tilde{B} = Bm = O(1)$]

In case (iii), the solvability condition is still given by (I) but with $m = i\tilde{m}$, where $\tilde{m} > 0$ wlog is s.t. $\tilde{m}^2 = \left(1-\frac{c^2}{c_1^2}\right)k^2$; but this would require $\tanh \tilde{m}h + \frac{\mu_1 \tilde{m}}{\mu_2 l} = 0$, which is not possible because the LHS >0 for $l, \tilde{m} > 0$.

S3

$\boxed{11}$

(c) (II) $\Rightarrow c_1^2\left(1+\frac{m^2}{k^2}\right) = c^2 = c_2^2\left(1-\frac{l^2}{k^2}\right)$

$$\Rightarrow l^2 = k^2 - \frac{c_1^2}{c_2^2}(k^2+m^2)$$

$$\Rightarrow l^2 = \left(1-\frac{c_1^2}{c_2^2}\right)k^2 - \frac{c_1^2\theta^2}{c_2^2 h^2} \qquad \left(\text{as } m = \frac{\theta}{h}\right)$$

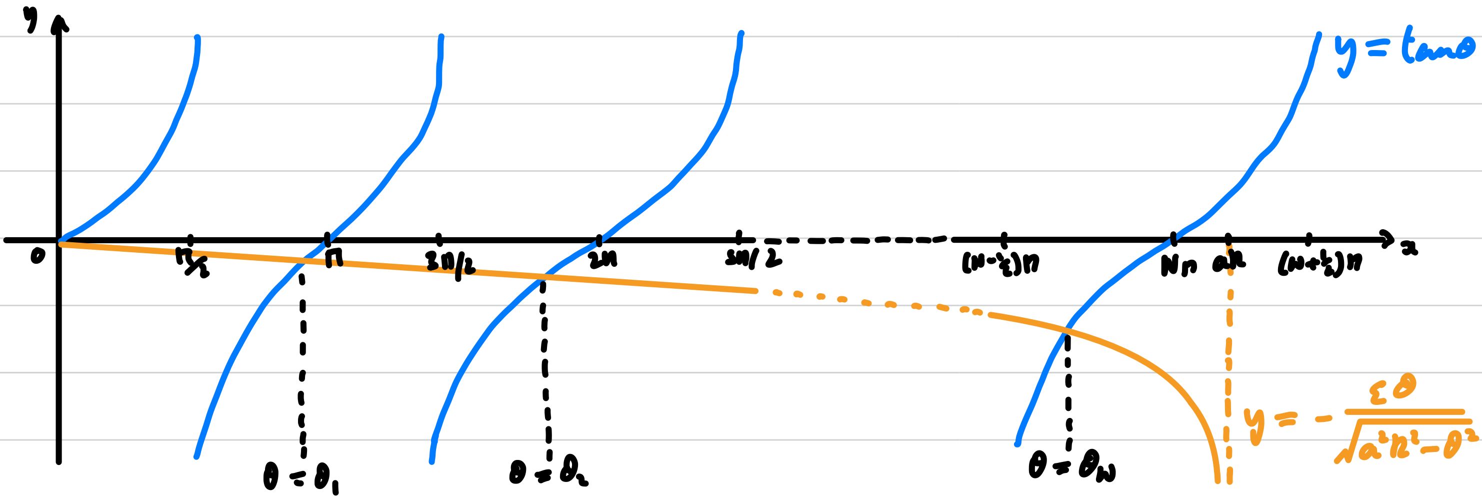
$$\Rightarrow l = \frac{c_1}{c_2 h}\sqrt{a^2k^2 - \theta^2}, \quad \text{where } \boxed{a = h\sqrt{\frac{c_2^2}{c_1^2} - 1}}$$

$$\text{So (I)} \Rightarrow \tan\theta = -\frac{\mu_1 \frac{\theta}{h}}{\mu_2 \frac{c_1}{c_2 h}\sqrt{a^2k^2-\theta^2}}$$

$$\Rightarrow \boxed{\tan\theta \overset{(*)}{=} -\frac{\varepsilon\theta}{\sqrt{a^2k^2-\theta^2}}} \quad \text{where } \boxed{\varepsilon = \sqrt{\frac{\rho_1\mu_1}{\rho_2\mu_2}}}$$

and $l > 0$ provided $\theta \in (0, ak)$.

S/N 3



As illustrated, $\tan\theta$ is monotonic increasing from $-\infty$ to $+\infty$ on $((n-\frac{1}{2})\pi, (n+\frac{1}{2})\pi)$ for $n \in \mathbb{Z}_0^+$, while $-\varepsilon\theta/\sqrt{a^2R^2-\theta^2}$ is monotonic decreasing on $(0, aR)$. So there are N roots $\theta \in (0, aR)$, namely $\theta = \theta_1, \theta_2, \dots, \theta_N$ as illustrated, iff $aR \in [(N-\frac{1}{2})\pi, (N+\frac{1}{2})\pi)$, i.e. $N = \lfloor \frac{aR}{\pi} + \frac{1}{2} \rfloor$.

S/N3

Neglecting RHS ⊛ for $\varepsilon \ll 1$ $\Rightarrow$ $\tan\theta = 0$ $\Rightarrow$ $\theta \sim \theta_n = n\pi$ for $n = 1, 2, \dots, N$.

$\Rightarrow$ $c = c_1\sqrt{1 + \frac{\theta^2}{R^2h^2}} \sim c_1\sqrt{1 + \frac{n^2\pi^2}{R^2h^2}}$ for $n = 1, 2, \dots, N$.

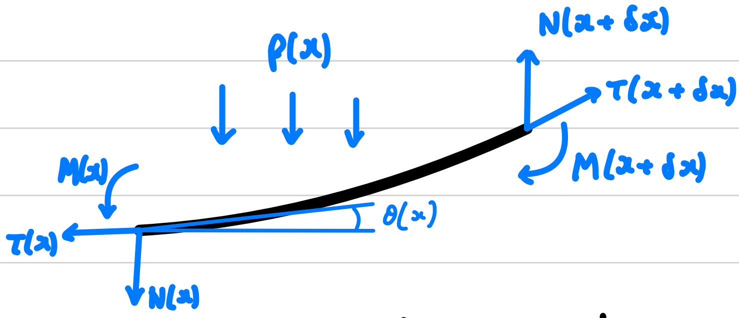
From sketch, approximation for θ_N invalid for $aR \in [(N-\frac{1}{2})\pi, N\pi)$ in which case ⊛ $\Rightarrow$ $\sqrt{a^2R^2-\theta_N^2} = -\frac{\varepsilon\,\theta_N}{\tan\theta_N}$ $\Rightarrow$ $\theta_N \sim aR$ $\Rightarrow$ $c \sim c_1\sqrt{1 + \frac{a^2}{h^2}}$

N3

$\boxed{9}$

CS.2/2023/Q2

(a) Tension T, transverse shear force N and moment M about y-axis act on a segment $[x, x+\delta x]$ in an open region of contact as shown:



Force balance $\Rightarrow \left[\begin{pmatrix} T\cos\theta \\ T\sin\theta + N \end{pmatrix}\right]_x^{x+\delta x} + \int_x^{x+\delta x} \begin{pmatrix} 0 \\ -p(\tilde{x}) \end{pmatrix} d\tilde{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \int_x^{x+\delta x} \frac{d}{dx}\begin{pmatrix} T\cos\theta \\ T\sin\theta + N \end{pmatrix} + \begin{pmatrix} 0 \\ -p(\tilde{x}) \end{pmatrix} d\tilde{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

True $\forall_{x, x+\delta x}$ and integrand cts (or taking limit $\delta x \to 0$)

$$\Rightarrow \frac{d}{dx}\left(T\cos\theta\right) = 0, \quad \frac{d}{dx}\left(T\sin\theta + N\right) - p = 0$$

Small displacement $\Rightarrow |\theta| \ll 1 \Rightarrow \cos\theta \sim 1, \; \sin\theta \sim \theta \sim \tan\theta = \frac{dw}{dx}$

So $\boxed{\frac{dT}{dx} = 0, \; T\frac{d^2w}{dx^2} + \frac{dN}{dx} = p}$

Moment balance $\Rightarrow M(x) + N(x+\delta x)\delta x - M(x+\delta x) = 0$

So proceeding as above or taking limit $\delta x \to 0 \Rightarrow \boxed{N = \frac{dM}{dx}}$

Assume constitutive relation $\boxed{M = -B\frac{d^2w}{dx^2}}$

Combine equations in boxes $\Rightarrow \boxed{B\frac{d^4w}{dx^4} - T\frac{d^2w}{dx^2} + p = 0}$

B5

In an open region of contact, an additional upward reaction force $R(x)$ is exerted by the obstacle on the beam, so that $\boxed{B\frac{d^4w}{dx^4} - T\frac{d^2w}{dx^2} + p = R}$

For a physically relevant solution we impose the constraint $R \geq 0$ because the obstacle can push but not pull on the beam. B2

$\boxed{7}$

(b) The jump conditions follow from continuity of the beam (i.e. it cannot break) and a force and moment balance at the contact point.

For a physically relevant solution we impose the constraint $R_i \geq 0$ because the obstacle can push but not pull on the beam (as in part (a)) S2

$\boxed{3}$

(c)(i) No contact for $\alpha = 0$, $w = 0$ and hence for α sufficiently small.

$$\Rightarrow w'''' = 0 \text{ for } |x| < L \text{ with } w(\pm L) = 0,\ w'(\pm L) = \pm\alpha$$

$$\Rightarrow w = \frac{\alpha}{2L}(x^2 - L^2) \text{ for } \alpha \text{ s.t. } w(x) \geq -H \text{ for } |x| \leq L$$

Since w minimum at $x = 0$ where $w(0) = -\frac{\alpha L}{2}$, there is no contact for $0 \leq \alpha < \frac{2H}{L}$ and contact is first made at $x = 0$ when $\alpha = \frac{2H}{L}$ B2

(c)(ii) Since $w''(0) = \frac{\alpha}{L} = \frac{2H}{L^2} > 0$ when $\alpha = \frac{2H}{L}$, contact is initially made at an isolated point at $(x, z) = (0, -H)$.

As α increases further, this configuration persists until the curvature of the beam matches that of the obstacle, i.e. until $w''(0) = 0$.

So look for a solution with this configuration and exploit symmetry about $x = 0$ to obtain the BVP

$w'''' = 0$ for $0 < x < L$ with $w(0) = -H$, $w'(0) = 0$, $w''(0) \geq 0$, $w(L) = 0$, $w'(L) = \alpha$

BCs at $x = 0 \Rightarrow w = -H + a\left(\frac{x}{L}\right)^2 + b\left(\frac{x}{L}\right)^3$, where $a, b \in \mathbb{R}$ TBD

BCs at $x = L \Rightarrow a + b = H$, $2a + 3b = \alpha L$

$\Rightarrow a = 3H - \alpha L$, $b = \alpha L - 2H$

Since $w''(0) = \frac{2A}{L^2} = \frac{2}{L^2}(3H - \alpha L) \geq 0$ for $\alpha \leq \frac{3H}{L}$, the beam makes contact with the obstacle at an isolated contact point for $\frac{2H}{L} \leq \alpha \leq \frac{3H}{L}$.

S/N5

(c)(iii) For $\alpha > \frac{3H}{L}$, introduce an open contact set in $|x| < s < L$ in which $w = 0$.

By symmetry, just focus on $s < x < L$.

BVP : $w'''' = 0$ for $s < x < L$ with $w(s) = -H$, $w'(s) = 0$, $w''(s) = 0$, $w(L) = 0$ and $w'(L) = \alpha$.

BCs at $x = s \Rightarrow w = -H + c(x - s)^3$ where $c \in \mathbb{R}$ TBD

BCs at $x = L \Rightarrow c = \frac{H}{(L-s)^3}$, $3c(L-s)^2 = \alpha$

Hence, contact set is $|x| \leq s$ where $\boxed{s = L - \frac{3H}{\alpha} \quad \text{for} \quad \alpha > \frac{3H}{L}}$

S/N4

(c)(iv) By part (b) the net upward force F exerted by the obstacle on the beam is given by

$$F = \begin{cases} 0 & \text{for } 0 \leq \alpha < \frac{2H}{L} \\ B(w'''(0+) - w'''(0-)) = 2B \cdot \frac{6b}{L^3} = \frac{12B}{L^3}(\alpha L - 2H) & \text{for } \frac{2H}{L} \leq \alpha \leq \frac{3H}{L} \\ 2B(w'''(s+) - w'''(s-)) = 2B \cdot 6c = \frac{12BH}{(L-s)^3} = \frac{4B\alpha^3}{9H^2} & \text{for } \alpha > \frac{3H}{L} \end{cases}$$

N3

$\boxed{15}$

C5.2/2023/Q3

(a) While the material remains elastic, the displacement satisfies the Navier equations with given constitutive relations, so

$$\frac{d\tau_{rr}}{dr} + \frac{2}{r}\left(\tau_{rr} - \tau_{\theta\theta}\right) = 0$$

$$\Rightarrow \frac{d}{dr}\left((\lambda+2\mu)\frac{du}{dr} + 2\lambda\frac{u}{r}\right) + \overbrace{\frac{2}{r}\left(2\mu\frac{du}{dr} - 2\mu\frac{u}{r}\right)}^{=\frac{d}{dr}\left(4\mu\frac{u}{r}\right)} = 0$$

$$\Rightarrow \frac{d}{dr}\left((\lambda+2\mu)\left(\frac{du}{dr} + 2\frac{u}{r}\right)\right) = 0 \qquad \text{B3}$$

$$\Rightarrow \frac{d}{dr}\left((3\lambda+2\mu)\left(\frac{du}{dr} + 2\frac{u}{r}\right)\right) = 0$$

$$\Rightarrow \frac{d}{dr}\left(\tau_{rr} + \tau_{\theta\theta} + \tau_{\phi\phi}\right) = 0 \qquad \text{B2}$$

$\boxed{5}$

(b) Pre-yield part (a) $\Rightarrow \tau_{rr} + 2\tau_{\theta\theta} = 3A$ say $(A \in \mathbb{R})$

So radial Navier equation $\Rightarrow \dfrac{d\tau_{rr}}{dr} + \dfrac{3}{r}\tau_{rr} = \dfrac{3A}{r}$

$$\Rightarrow \frac{d}{dr}\left(r^3\tau_{rr}\right) = 3Ar^2$$

$$\Rightarrow \tau_{rr} = A - \frac{2B}{r^3},\quad \tau_{\theta\theta} = \tau_{\phi\phi} = A + \frac{B}{r^3} \quad (A, B \in \mathbb{R})$$

BCs $\tau_{rr}(a) = -P$, $\tau_{rr}(\infty) = 0 \Rightarrow A = 0$, $2B = Pa^3$

So $\boxed{\tau_{rr} = -\dfrac{Pa^3}{r^3},\quad \tau_{\theta\theta} = \tau_{\phi\phi} = \dfrac{Pa^3}{2r^3} \text{ for } r > a}$ B4

Since $P \geq 0$, the Tresca yield function $f = \frac{1}{2}(\tau_{\theta\theta} - \tau_{rr}) = \dfrac{3Pa^3}{4r^2}$ has its maximum at $r = a$, so as P increases gradually from zero yield first occurs at $r = a$ when $f|_{r=a} = \tau_Y$, i.e. $\boxed{P = P_c = \frac{4}{3}\tau_Y.}$ B2

$\boxed{6}$

(c) For $P > P_c$, material must yield in a neighbourhood of $r = a$ say $a < r < s$.

In $r > s$, still have elastic solution $\tau_{rr} = -\frac{2B}{r^3}$, $\tau_{\theta\theta} = \tau_{\phi\phi} = \frac{B}{r^3}$ by part (b) with $A = 0$ because $\tau_{rr}(\infty) = 0$.

Yield condition at $r = s$ gives $\frac{1}{2}(\tau_{\theta\theta} - \tau_{rr}) = \frac{3B}{2s^3} = \tau_Y$, with the sign determined by how yield condition was satisfied initially.

So $B = \frac{2}{3}\tau_Y s^3 \Rightarrow$ $\boxed{\tau_{rr} = -\frac{4\tau_Y s^3}{3r^3},\ \tau_{\theta\theta} = \tau_{\phi\phi} = \frac{2\tau_Y s^3}{3r^3} \text{ for } r > s}$ S2

In $r < s$ apply the Yield condition $\tau_{\theta\theta} - \tau_{rr} = 2\tau_Y$

So radial Navier equation $\Rightarrow \frac{d\tau_{rr}}{dr} = \frac{4\tau_Y}{r}$, with $\tau_{rr}(a) = -P$

So $\boxed{\tau_{rr} = -P + 4\tau_Y \log\left(\frac{r}{a}\right),\ \tau_{\theta\theta} = \tau_{\phi\phi} = 2\tau_Y - P + 4\tau_Y \log\left(\frac{r}{a}\right) \text{ for } a < r < s}$

Stress balance at $r = s \Rightarrow \tau_{rr}(s-) = \tau_{rr}(s+)$

$$\Rightarrow -P + 4\tau_Y \log\left(\frac{s}{a}\right) = -\frac{4}{3}\tau_Y$$

$$\Rightarrow \boxed{s = a\exp\left(\frac{P - P_c}{4\tau_Y}\right)}$$

S4

7

(d) Now superimpose a purely elastic stress on the plastic stress obtained at the maximum pressure $P_m > P_c$, viz.

$$\boxed{\begin{aligned} \tau_{rr} &= \frac{2\tilde{B}}{r^3} + \begin{cases} -P_m + 4\tau_Y \log\left(\frac{r}{a}\right) & \text{for } a < r < s_m \\ -\frac{4\tau_Y s^3}{3r^3} & \text{for } r > s_m \end{cases} \\ \tau_{\theta\theta} &= \tau_{\phi\phi} = -\frac{\tilde{B}}{r^3} + \begin{cases} 2\tau_Y - P_m + 4\tau_Y \log\left(\frac{r}{a}\right) & \text{for } a < r < s_m \\ \frac{2\tau_Y s^3}{3r^3} & \text{for } r > s_m \end{cases} \end{aligned}}$$

where $\boxed{s_M = a\exp\left(\frac{P_M - P_c}{4\tau_Y}\right)}$

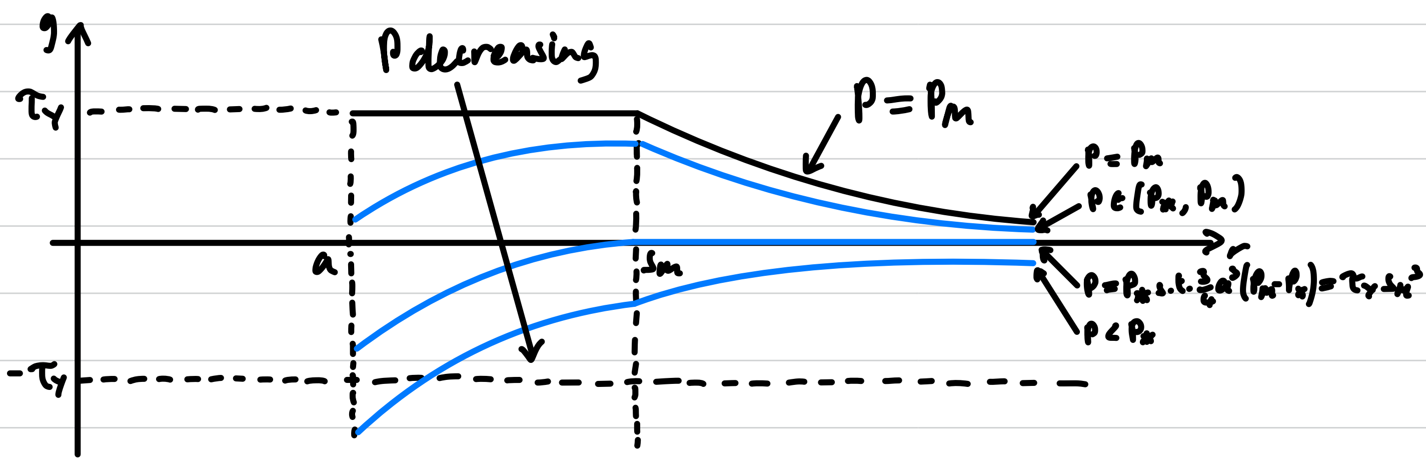
BC $\tau_{rr}(a) = -P \Rightarrow -P = \frac{2\tilde{B}}{a^3} - P_M$

$$\Rightarrow \boxed{2\tilde{B} = a^3(P_M - P) \text{ for } P < P_M}$$

S/N3

So Tresca yield function $f = |g|$, where

$$g(r) = \frac{1}{2}(\tau_{\theta\theta} - \tau_{rr}) = -\frac{3a^3(P_M - P)}{4r^3} + \begin{cases} \tau_Y & \text{for } a \le r \le s_M \\ \frac{\tau_Y s_M^3}{r^3} & \text{for } r > s_M \end{cases}$$



As P decreases from P_M, the maximum of g is always less than τ_Y, while the minimum of g is $\min\{g(a), 0\}$, as illustrated.

So the material yields again if $g(a)$ reaches $-\tau_Y$, which occurs when $-\frac{3}{4}(P_M - P) + \tau_Y = -\tau_Y$, i.e. $P = P_M - \frac{8}{3}\tau_Y$

Hence material yields again while being unloaded iff $P_M - \frac{8}{3}\tau_Y > 0$, i.e. $P_M > 2P_c$.

N4

$\boxed{7}$