C4.1 Further Functional Analysis

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Michaelmas Term 2024

Thank you to Stuart White for providing these materials.

Functional analysis prerequisites from B4.1 and B4.2

A lot of the material found in undergraduate functional analysis courses will be very similar (such as the items in the first two bullet points below). Depending on your courses you may not have seen every item in the last 3 bullet points below. I don't think this is a problem, provided you've seen some of this material, and are prepared to familiarise yourself with the relevant statements.

- Normed spaces and Banach spaces. Definitions and fundamental examples. Familiarity with Hilbert spaces, and their fundamental properties.[1] We set out some of this background, and in particular our notation, in Section 1 of the notes.
- Operators between normed spaces, Continuity and boundedness, Completeness of $\mathcal{B}(X,Y)$ when Y is complete. Again we'll review some of these and set out our notation in Section 1 of the notes.
- The Baire category theorem.[2] We will briefly describe Baire's category theorem in the Appendix.
- The consequences of Baire's category theorem for maps between Banach spaces. Open mapping theorem, closed graph theorem, and inverse mapping theorem.[3]. These are discussed a bit in appendix **??**, where we deduce them from Baire's category theorem (in a very similar fashion to as done in B.4.2). We will see the equivalence of these three classical theorems in sheet 2 (and they're also equivalent to the uniform boundeness principle).

[1] Most of this course will focus on the structure of Banach spaces and the operators between them, but it'll be useful to contrast the behaviour with known results for Hilbert spaces, such as the projection theorem: there is an orthogonal projection onto a closed subspace of Hilbert space.

[2] If you've not seen this before, this shouldn't be a problem. We will state it, and use it once in Section 2 in order to show that Hamel bases on Banach spaces are necessarily uncountable.

[3] Ideally you've seen these before, but for our course the statements will suffice, though we will use part of the proof of the open mapping theorem the so called 'successive approximation lemma' which is developed in the notes and exercises

- We will reprove the Hahn-Banach extension theorem in the course (so as to avoid any separability assumptions), but as the one step extension lemma is done in B4.1 I will omit this proof from the lectures. The proof is given in the notes and discussed further in the exercises.

- Definitions and basic properties of the spectrum of a bounded operator on a complex Banach space. (This will come up right at the end of the course - though we recap some features on problem sheet 0 - so if your course didn't cover any spectral theory, there's plenty of time to take a look at this).

Other Prerequisites

- The fundamentals of linear algebra (bases etc) are likely to be familar to anyone considering taking this course. If you've not seen quotient vector spaces for a while (or at all), it's worth checking these out from the beginning of the A0 course.

- Basics of metric spaces, particularly aspects relating to completeness. For the first part of the course, all our topologies will come from metrics.

- Basics of topological spaces, closures and interiors, and in particular compactness. You'll also want to be familiar with constructing topological spaces by specifiying a basis of open sets. This becomes most relevant in the middle of the course when we start to work with topologies on normed spaces which do not come from the norm, and are not a-priori metrisable. In terms of the part A topology course, this means we will essentially use the first half of it, but we won't need any of the aspects of the classification of surfaces. If you've not studied the first half of part A topology or its equivalent, I really recommend the topological spaces part of Wilson Sutherland's little book 'An Introduction to metric and topological spaces and please do try and get up to speed with this before about week 4.'[4]

[4] We will use Tychonoff's theorem that the product of compact spaces is compact as a black box in the course. This is equivalent to the axiom of choice, and not in my list of basic facts about topological spaces.