C4.1 Further Functional Analysis

Sheet 0 — MT 2024

Initial Sheet

This problem sheet is not for handing in. It is intended for revision and consolidation (during Week 0 and the beginning of Week 1 of MT) of some important concepts in Functional Analysis.

1. Let X be a normed vector space. Prove that X is a Banach space if and only if every absolutely convergent series with terms in X converges to a limit in X.

2. Given an example of Banach spaces X, Y and a bounded linear operator $T : X \to Y$ such that $\operatorname{Ran} T$ is not closed in Y.

3. Let X_n, $n \geq 1$, be normed vector spaces. Consider the vector space X of sequences $(x_n)_{n=1}^{\infty}$ such that $x_n \in X_n$, $n \geq 1$, and $\sum_{n=1}^{\infty} \|x_n\| < \infty$, endowed with the norm

$$\|x\| = \sum_{n=1}^{\infty} \|x_n\|, \quad x = (x_n)_{n=1}^{\infty} \in X.$$

   (a) Prove that if X_n is complete for each $n \geq 1$ then so is X.

   (b) Let X_n^* denote the dual space of X_n, $n \geq 1$. Show that the dual space X^* of X is isometrically isomorphic to the vector space Y of all sequences $(f_n)_{n=1}^{\infty}$ such that $f_n \in X_n^*$, $n \geq 1$, and $\sup_{n\geq 1} \|f_n\| < \infty$, endowed with the norm given by $\|f\| = \sup_{n\geq 1} \|f_n\|$, $f = (f_n)_{n=1}^{\infty} \in Y$.

   [Think about the proof that the dual space of ℓ^1 is isometrically isomorphic to ℓ^∞. If you've not seen this result in your earlier courses, this problem will probably be hard, and I'd encourage you instead to spend time considering finding out about dual spaces of ℓ^p for $1 \leq p < \infty$ and for c_0.]

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4. Let X be a Banach space.

   (a) What does it mean to say that an operator $T \in \mathcal{B}(X)$ is *invertible*?

   (b) Suppose that $T \in \mathcal{B}(X)$ and that $\|T\| < 1$. Show that $I - T$ is invertible.

   (c) Let $S, T \in \mathcal{B}(X)$ and suppose that T is invertible and that $\|S\| < \|T^{-1}\|^{-1}$. Prove that $S + T$ is invertible and that

   $$(S+T)^{-1} = \sum_{n=1}^{\infty} (-1)^n (T^{-1}S)^n T^{-1},$$

   where the series converges in the norm of $\mathcal{B}(X)$.

   (d) Deduce that the set of invertible operators is an open subset of $\mathcal{B}(X)$ and that the spectrum

   $$\sigma(T) = \{\lambda \in \mathbb{F} : \lambda - T \text{ is not invertible}\}$$

   of any operator $T \in \mathcal{B}(X)$ is a compact subset of the field $\mathbb{F}$.

   (e) Given a non-empty compact subset K of $\mathbb{F}$, show that there exist a Banach space X and $T \in \mathcal{B}(X)$ such that $\sigma(T) = K$. What can you say if K is empty? [Does it make a difference whether $\mathbb{F}$ is $\mathbb{C}$ or $\mathbb{R}$?]

5. Let X be a Banach space, Y a normed vector space and let $T \in \mathcal{B}(X, Y)$.

   (a) Suppose there exist $\varepsilon \in (0, 1)$ and $M > 0$ such that $\operatorname{dist}(y, T(B_X^\circ(M))) < \varepsilon$ for all $y \in B_Y^\circ$. Prove that $B_Y^\circ \subseteq T(B_X^\circ(M(1-\varepsilon)^{-1}))$. [Take $y_1 = y \in B_Y^\circ$ and take $x_1 \in B_X^\circ(M)$ with $\|Tx_1 - y_1\| < \epsilon$. Now take $y_2 = Tx_1 - y_1$. How well can you approximate y_2 by something in the range of T?]

   (b) Deduce that if $T(B_X^\circ(M))$ contains a dense subset of B_Y° then $B_Y^\circ \subseteq T(B_X^\circ(M))$.

   [In this course the notation $B_X^\circ(r)$ is used for the open unit ball in X of radius r. This result is the successive approximation lemma - if you get stuck, you might have a look at the proof of the open mapping theorem from B.4.2.]

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