C3.5 Lie Groups

Sheet 1 — HT25

Section A contains an introductory question. Section B contains material to test understanding of the course. Section C contains a more advanced question which is optional. Only answers to Section B should be submitted for marking.

Section A

1. Let G be the group of Möbius transformations which map the upper half-plane

$$\{z = x + iy \in \mathbb{C} : y > 0\}$$

to itself. These are of the form

$$z \mapsto \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. Show that G is a 3-dimensional non-compact connected Lie group.

Solution: The coefficients in a Möbius transformation are only defined up to a scalar multiple, so we cover G with two charts.

Since $ad - bc > 0$, a and b are not simultaneously zero, so define U as the subset on which $a \neq 0$ and take coordinates $x_1 = c/a, x_2 = b/a, x_3 = d/a$ in the open subset of $\mathbb{R}^3$ defined by $x_3 - x_1x_2 > 0$, which is equivalent to $ad - bc > 0$. This is one chart.

For another take V to be the open subset where $b \neq 0$ and set $\tilde{x}_1 = c/b, \tilde{x}_2 = a/b, \tilde{x}_3 = d/b$ so that $\tilde{x}_3 - \tilde{x}_1\tilde{x}_2 > 0$. Then on $U \cap V$, where $y = b/a \neq 0$,we have

$$\tilde{x}_1 = x_1/x_2, \quad \tilde{x}_2 = 1/x_2, \quad \tilde{x}_3 = x_3/x_2$$

which is smooth and invertible.

This makes G into a 3-dimensional manifold with a countable basis of open sets. Composition of Möbius transformations follows from multiplication of the 2×2 matrices

$$\begin{pmatrix} a & b \\ a' & b' \end{pmatrix} \begin{pmatrix} c & d \\ c' & d' \end{pmatrix},$$

which is polynomial and hence smooth in the coordinates x_i, $\tilde{x}_i$ for $i = 1, 2, 3$. Inversion is

$$z \mapsto \frac{dz - b}{-cz + a}$$

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which is smooth.

We need to prove that G is Hausdorff; it is sufficient to prove that any $g \in G$ and e, the identity, can be separated by open sets. The identity is given by $a = d$ and $b = c = 0$, or $(x_1, x_2, x_3) = (0, 0, 1)$. Since the topology of an open set in $\mathbb{R}^3$ is Hausdorff it is separated from anything in U. So if $g \in V$ is not in U then $a = 0$ so $\tilde{x}_2 = 0$. A neighbourhood of this point has $\tilde{x}_2$ small and hence in $U \cap V$ where $\tilde{x}_2 = 1/x_2$ we must have $|x_2|$ large. But then a neighbourhood of $y = 0$ will not intersect this.

The subset U is homeomorphic to the open subset of $\mathbb{R}^3$ defined by $x_3 - x_1 x_2 > 0$, which is connected (think of the half-planes $x_3 > m x_1$ in the (x_1, x_3)-plane as m varies) – and likewise V. Since $U \cap V$ is non-empty, G is connected.

The group G is non-compact, for consider the well-defined function $a^2/(ad - bc)$. Restrict to $b = c = 0, a = \lambda \in \mathbb{R}^+, d = 1$ and it is the unbounded function λ.

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Section B

2. (a) Suppose G_1, G_2 are Lie groups.
   - (i) Show that $G_1 \times G_2$ is a Lie group in a natural way. (You may assume that the product of two manifolds is naturally a manifold).
   - (ii) Show that $T^n = S^1 \times \cdots \times S^1$ is a Lie group.

   (b) (i) Find a map $\pi : \mathbb{R}^n \to T^n$ that allows you to identify T^n with the quotient group $\mathbb{R}^n/\mathbb{Z}^n$.
   - (ii) Which vector fields on $\mathbb{R}^n$ project under the map induced by π to vector fields on T^n? Do all vector fields on T^n arise in this way?
   - (iii) Which vector fields X on T^n are left-invariant?

3. (a) Show that
$$\mathrm{U}(n) = \{A \in M_n(\mathbb{C}) \, : \, \overline{A^{\mathrm{T}}} A = I\}$$
   is a Lie group and compute its dimension.
   [*Hint: Use the Regular Value Theorem.*]

   (b) Find the tangent space $T_I \, \mathrm{U}(n)$.

   (c) Show that $\mathrm{U}(n)$ is compact.

4. (a) Let G be a Lie group with identity e.
   - (i) Show that the *tangent bundle* $TG = \bigsqcup_{g \in G} T_g G$ of a Lie group G is canonically identifiable with $G \times T_e G$.
     [*Hint: Consider left-translation.*]
   - (ii) Deduce that any Lie group of dimension n has n non-vanishing vector fields which are linearly independent at each point of G.

   (b) (i) Show that the 3-dimensional sphere S^3 is a Lie group by identifying it with
$$\mathrm{SU}(2) = \{A \in M_2(\mathbb{C}) \, : \, \overline{A^{\mathrm{T}}} A = I, \, \det A = 1\}.$$
   - (ii) Show that the 2-dimensional sphere S^2 *cannot* be a Lie group.
     [*Hint: apply the "Hairy Ball Theorem".*]

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5. (a) Let $\varphi : M \to N$ be a diffeomorphism of manifolds. For a vector field X on M define the *push-forward* vector field $Z = \varphi_* X$ on N by

$$Z_y = d\varphi_x(X_x)$$

where $x = \varphi^{-1}(y)$.

(i) Show that for any smooth function $f : N \to \mathbb{R}$,

$$(\varphi_* X) \cdot f = (X \cdot (f \circ \varphi)) \circ \varphi^{-1}.$$

(ii) Deduce that $[\varphi_* X, \varphi_* Y] \cdot f = \varphi_*[X, Y] \cdot f$, and hence that

$$[\varphi_* X, \varphi_* Y] = \varphi_*[X, Y].$$

(b) Let G be a Lie group with identity e and let $\operatorname{Lie} G$ be the set of left-invariant vector fields on G.

(i) Show that

$$(L_g)_* X = X \text{ for all } g \in G \quad \Leftrightarrow \quad d(L_g)_e(X_e) = X_g \text{ for all } g \in G$$

(ii) Show that if $X, Y \in \operatorname{Lie} G$, then also $[X, Y] \in \operatorname{Lie} G$.

6. Let G be a Lie group, and let G_0 denote the connected path component of G containing the identity (we call G_0 the *identity component* of G).

(a) Show that G_0 is a normal subgroup of G.

(b) If $G = \mathrm{O}(n)$ what is G_0? Is it true in this example that $G \cong G_0 \times (G/G_0)$ as groups?

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Section C

7. (a) By considering the action of a matrix of the form

$$\begin{pmatrix} A_{11} & A_{12} & a_1 \\ A_{21} & A_{22} & a_2 \\ 0 & 0 & 1 \end{pmatrix}$$

on the plane $x_3 = 1$ in $\mathbb{R}^3$, find the condition on A_{ij} for this to define an isometry of $\mathbb{R}^2$, and then show that the set of such matrices is a 3-dimensional Lie group G.

(b) Is G connected?

(c) Show that G is diffeomorphic to $\mathbb{R}^2 \times \mathrm{O}(2)$ as a manifold.

(d) Show that G has a subgroup isomorphic as a group to the additive group $\mathbb{R}^2$, and another isomorphic to $\mathrm{O}(2)$, but G is not a product of these two groups.

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