Exercise sheet 1. Chapters 1-4.

Part A

Question 1.1. (1) Describe the Zariski topology of k.

(2) Show that the Zariski topology of k^2 is not the product topology of $k \times k = k^2$.

Question 1.2. Let $V \subseteq k^n$ be an algebraic set. Show that V is the disjoint union of two non empty algebraic sets in k^n iff there are two non-zero finitely generated reduced k-algebras T_1 and T_2 and an isomorphism of k-algebras $T_1 \oplus T_2 \simeq \mathcal{C}(V)$.

Part B

Question 1.3. Let $V \subseteq k^3$ be the set

$$V := \{(t, t^2, t^3) \,|\, t \in k\}.$$

Show that V is an algebraic set and that it is isomorphic to k as an algebraic set. Provide generators for $\mathcal{I}(V)$.

Question 1.4. (1) Let $V \subseteq k^2$ be the set of solutions of the equation $y = x^2$. Show that V is isomorphic to k as an algebraic set.

(2) Let $V \subseteq k^2$ be the set of solutions of the equation $xy = 1$. Show that V is not isomorphic to k as an algebraic set.

(3) [difficult] (optional) Let $P(x, y) \in k[x, y]$ be an irreducible quadratic polynomial and let $V \subseteq k^2$ be the set of zeroes of $P(x, y)$. Show that V is isomorphic to one of the algebraic sets defined in (1) and (2).

Question 1.5. Let $V \subseteq k^n$ and $W \subseteq k^t$ be two algebraic sets. Let $\psi : V \to W$ be a regular map.

(1) Show that $\psi(V)$ is dense in W iff the map of rings $\psi^* : \mathcal{C}(W) \to \mathcal{C}(V)$ is injective.

(2) Show that ψ^* is surjective iff $\psi(V)$ is closed and the induced map $V \to \psi(V)$ is an isomorphism of algebraic sets.

Question 1.6. Let $V \subseteq k^3$ be the algebraic set described by the ideal $(x^2 - yz, xz - x)$. Show that V has three irreducible components. Find generators for the radical (actually prime) ideals associated with these components.

Question 1.7. Let $V \subseteq k^n$ and $W \subseteq k^t$ be algebraic subsets. Let $V_0 \subseteq V$ and $W_0 \subseteq W$ be open subsets. View V_0 and W_0 as open subvarieties of V and W respectively. For $i \in \{1, \ldots, t\}$ let $\pi_i : k^t \to k$ be the projection on the i-coordinate. Let $\psi : V_0 \to W_0$ be a map. Show that ψ is a morphism of varieties iff $\pi_i \circ \psi$ is a regular function on V_0 for all $i \in \{1, \ldots, t\}$.

Part C

Question 1.8. Show that the open subvariety $k^2 \backslash \{0\}$ of k^2 is not affine.