Exercise sheet 4. All lectures.

Part A.

Question 4.1. Let $f : X \to Y$ be a regular map between varieties. Suppose that X is quasi-projective. Let $\sigma : Y \to X$ be a regular map such that $f \circ \sigma = \mathrm{Id}_Y$ (such a map is called a *section* of f). Show that $\sigma(Y)$ is closed in X.

Part B.

Question 4.2. Suppose in this exercise that $\mathrm{char}(k) = 0$. Find the singularities of the following curves C in k^2. For each singular point $P \in C$ compute the dimension of $\mathfrak{m}_P/\mathfrak{m}_P^2$ as a k-vector space. Here $\mathfrak{m}_P$ is the maximal ideal of $\mathcal{O}_{C,P}$.

(1) $\mathrm{Z}(x^6 + y^6 - xy)$

(2) $\mathrm{Z}(y^2 + x^4 + y^4 - x^3)$

You may assume that the polynomials $x^6 + y^6 - xy$ and $y^2 + x^4 + y^4 - x^3$ are irreducible.

Question 4.3. Let C be the plane curve considered in (1) of question 4.2. Consider the blow-up B of C at each of its singular points in turn. How many irreducible components does the exceptional divisor of B have? Is B nonsingular?

Question 4.4. Let $V \subseteq k^2$ be the algebraic set defined by the equation $x_1x_2 = 0$. Show that $\mathrm{Bl}(V, 0)$ has two disjoint irreducible components and that each of these components is isomorphic to k.

Question 4.5. Let $C \subseteq k^2$ be defined by the equation $P(x_1, x_2) = 0$, where $P(x_1, x_2)$ is an irreducible polynomial. Suppose that C goes through the origin 0 of k^2 and is non singular there. Show that the natural morphism $\mathrm{Bl}(C, 0) \to C$ is an isomorphism. *[Hint: construct an inverse map directly, without looking at coordinate charts]*

Part C.

Question 4.6. (1) Let $f : X \to Y$ be a dominant morphism of varieties. Suppose that Y is irreducible. Show that $\dim(X) \geqslant \dim(Y)$.

(2) Let $f : X \to Y$ be a dominant morphism of irreducible varieties. Suppose that the field extension $\kappa(X)|\kappa(Y)$ is algebraic. Show that there are affine open subvarieties $U \subseteq X$ and $W \subseteq Y$ such that $f(U) = W$ and such that the map of rings $\mathcal{O}_X(U) \to \mathcal{O}_Y(V)$ is injective and finite.

(3) Let $f : X \to Y$ be a dominant morphism of irreducible quasi-projective varieties. Show that there is a $y \in Y$ such that we have $\dim(f^{-1}(\{y\})) \geqslant \dim(X) - \dim(Y)$. [*Hint. Reduce to the situation where Y is affine and apply Noether's normalisation lemma to show that you may assume wlog that $Y = k^n$ for some n. Now use the existence of transcendence bases and (2) to show that there is an open subvariety $U \subseteq X$ and an open subvariety W of $k^{\dim(X)-\dim(Y)} \times k^n$ such that $f|_U$ factors as a finite and surjective morphism $U \to W$, followed by the projection to k^n. Now deduce the result from (1) and a computation of the dimension of the fibres of the projection $k^{\dim(X)-\dim(Y)} \times k^n \to k^n$.*]

(4) Deduce that in the situation of (3), the set of $y \in Y$ such that we have $\dim(f^{-1}(\{y\})) \geqslant \dim(X) - \dim(Y)$ is dense in Y.

Question 4.7. (1) Show that all the morphisms from $\mathbb{P}^2(k)$ to $\mathbb{P}^1(k)$ are constant. *[Hint: Use question 4.6 and the projective dimension theorem.]*

(2) Using (1) or using another method, show that the morphisms from $\mathbb{P}^n(k)$ to $\mathbb{P}^1(k)$ are constant.