Hand out Notes for Infinite Groups, 2018

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These notes expand on certain notions introduced in the course, as further reading for students who wish to have a better understanding of the latter. The material in these notes is not examinable.

0.1. Graphs

An *unoriented graph* Γ consists of the following data:

- a set V called the *set of vertices* of the graph;
- a set E called the *set of edges* of the graph;
- a map ι called *incidence map* defined on E and taking values in the set of subsets of V of cardinality one or two.

We will use the notation $V = V(\Gamma)$ and $E = E(\Gamma)$ for the vertex and respectively the edge set of the graph Γ. When $\{u, v\} = \iota(e)$ for some edge e, the two vertices u, v are called the *endpoints* of the edge e; we say that u and v are *adjacent vertices*.

Note that in the definition of a graph we allow for *monogons* (i.e. edges connecting a vertex to itself)[1] and *bigons*[2] (pairs of distinct edges with the same endpoints). A graph is *simplicial* if the corresponding cell complex is a simplicial complex. In other words, a graph is simplicial if and only if it contains no monogons or bigons[3].

The incidence map ι defining a graph Γ is set-valued; converting ι into a map with values in $V \times V$, equivalently into a pair of maps $E \to V$ is the choice of an *orientation* of Γ: An orientation of Γ is a choice of two maps

$$o : E \to V, \quad t : E \to V$$

such that $\iota(e) = \{o(e), t(e)\}$ for every $e \in E$. In view of the Axiom of Choice, every graph can be oriented.

Definition 0.1. An *oriented* or *directed* graph is a graph Γ equipped with an orientation. The maps o and t are called the *head (or origin) map* and the *tail map* respectively.

We will in general denote an oriented graph by $\overline{\Gamma}$, its edge-set by $\overline{E}$, and oriented edges by $\bar{e}$.

Convention 0.2. Unless we state otherwise, all graphs are assumed to be unoriented.

The *valency (or valence, or degree) of a vertex* v of a graph Γ is the number of edges having v as an endpoint, where every monogon with both endpoints equal to v is counted twice. The *valency* of Γ is the supremum of valencies of its vertices.

Examples of graphs. Below we describe several examples of well-known graphs.

Example 0.3 (n-rose). This graph, denoted R_n, has one vertex and n edges connecting this vertex to itself.

Example 0.4. [i-star or i-pod] This graph, denoted T_i, has $i + 1$ vertices, $v_0, v_1, \ldots, v_i$. Two vertices are connected by a unique edge if and only if one of these vertices is v_0 and the other one is different from v_0. The vertex v_0 is the *center* of the star and the edges are called its *legs*.

[1] Not to be confused with *unigons*, which are hybrids of unicorns and dragons.
[2] Also known as *digons*.
[3] and, naturally, no unigons, because those do not exist anyway.

Example 0.5 (n-circle). This graph, denoted C_n, has n vertices which are identified with the n–th roots of unity:

$$v_k = e^{2\pi ik/n}.$$

Two vertices u, v are connected by a unique edge if and only if they are adjacent to each other on the unit circle:

$$uv^{-1} = e^{\pm 2\pi i/n}.$$

Example 0.6 (n-interval). This graph, denoted I_n, has the vertex set equal to $[1, n+1] \cap \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. Two vertices n, m of this graph are connected by a unique edge if and only if

$$|n - m| = 1.$$

Thus, I_n has n edges.

Example 0.7 (Half-line). This graph, denoted H, has the vertex set equal to $\mathbb{N}$ (the set of natural numbers). Two vertices n, m are connected by a unique edge if and only if

$$|n - m| = 1.$$

The subset $[n, \infty) \cap \mathbb{N} \subset V(H)$ is the vertex set of a subgraph of H also isomorphic to the half-line H. We will use the notation $[n, \infty)$ for this subgraph.

Example 0.8 (Line). This graph, denoted L, has the vertex set equal to $\mathbb{Z}$, the set of integers. Two vertices n, m of this graph are connected by a unique edge if and only if

$$|n - m| = 1.$$

A *morphism* of graphs $f : \Gamma \to \Gamma'$ is a pair of maps $f_V : V(\Gamma) \to V(\Gamma')$, $f_E : E(\Gamma) \to E(\Gamma')$ such that

$$\iota' \circ f_E = f_V \circ \iota$$

where ι and ι' are the incidence maps of the graphs Γ and Γ' respectively.

subgraph in a graph Γ' is defined by subsets $V \subset V(\Gamma'), E \subset E(\Gamma')$ such that

$$\iota'(e) \subset V$$

for every $e \in E$. A subgraph Γ' of Γ is called *full* if every $e = [v, w] \in E(\Gamma)$ connecting vertices of Γ', is an edge of Γ'.

A morphism $f : \Gamma \to \Gamma'$ of graphs which is invertible (as a morphism) is called an *isomorphism* of graphs: More precisely, we require that the maps f_V, f_E are invertible and the inverse maps define a morphism $\Gamma' \to \Gamma$.

We use the notation $\mathrm{Aut}(\Gamma)$ for the group of automorphisms of a graph Γ.

An edge connecting two vertices u, v of a graph Γ will sometimes be denoted by $[u, v]$: This is unambiguous if Γ is simplicial. A finite ordered set of edges of the form $[v_1, v_2], [v_2, v_3], \ldots, [v_n, v_{n+1}]$ is called an *edge-path* in Γ. The number n is called the *combinatorial length* of the edge-path. An edge-path in Γ is a *cycle* if $v_{n+1} = v_1$. A *simple* cycle (or a *circuit*) is a cycle with all vertices $v_i, i = 1, \ldots, n$, pairwise distinct. In other words, a simple cycle is a subgraph isomorphic to the n-circle for some n. A graph Γ is *connected* if any two vertices of Γ are connected by an edge-path.

A subgraph $\Gamma' \subset \Gamma$ is called a *connected component* of Γ if Γ' is a maximal (with respect to the inclusion) connected subgraph of Γ.

A *simplicial tree* is a connected graph without circuits.

Maps of graphs. Sometimes, it is convenient to consider maps of graphs which are not morphisms. A *map of graphs* $f : \Gamma \to \Gamma'$ consists of a pair of maps (g, h):

1. A map $g : V(\Gamma) \to V(\Gamma')$ sending adjacent vertices to adjacent or equal vertices;

2. A *partially defined* map of the edge-sets:

$$h : E_o \to E(\Gamma'),$$

where E_o consists only of edges e of Γ whose endpoints $v, w \in V(\Gamma)$ have distinct images by g:

$$g(v) \neq g(w).$$

For each $e \in E_o$, we require the edge $e' = h(e)$ to connect the vertices $g(o(e)), g(t(e))$. In other words, f amounts to a morphism of graphs $\Gamma_o \to \Gamma'$, where the vertex set of Γ_o is $V(\Gamma)$ and the edge-set of Γ_o is E_o.

Collapsing a subgraph. Given a graph Γ and a (non-empty) subgraph Λ of it, we define a new graph, $\Gamma' = \Gamma/\Lambda$, by "collapsing" the subgraph Λ to a vertex. Here is the precise definition. Define the partition $V(\Gamma) = W \sqcup W^c$,

$$W = V(\Lambda), \quad W^c = V(\Gamma) \setminus V(\Lambda).$$

The vertex set of Γ' equals

$$W^c \sqcup \{v_o\}.$$

Thus, we have a natural surjective map $V(\Gamma) \to V(\Gamma')$ sending each $v \in W^c$ to itself and each $v \in W$ to the vertex v_o. The edge-set of Γ' is in bijective correspondence to the set of edges in Γ which *do not* connect vertices of Λ to each other. Each edge $e \in E(\Gamma)$ connecting $v \in W^c$ to $w \in W$ projects to an edge, also called e, connecting v to v_0. If an edge e connects two vertices in W^c, it is also retained and connects the same vertices in Γ'.

The map $V(\Gamma) \to V(\Gamma')$ extends to a *collapsing* map of graphs $\kappa : \Gamma \to \Gamma'$.

EXERCISE 0.9. If Γ is a tree and Λ is a subtree, then Γ' is again a tree.

0.2. Connected graphs as metric spaces

We introduce a metric dist on a graph Γ as follows. We declare every edge of Γ to be isometric to the unit interval in $\mathbb{R}$. The distance between any vertices of Γ is the length of the shortest edge-path connecting these vertices. Of course, points on the edges of Γ that are not vertices are not connected by any edge-paths. Thus, we consider *fractional* edge-paths, where in addition to the edges of Γ we allow intervals contained in the edges. The length of such a fractional path is the sum of lengths of the intervals in the path. Then, for $x, y \in \Gamma$,

$$\mathrm{dist}(x, y) = \inf_{\mathfrak{p}} \left(\mathrm{length}(\mathfrak{p})\right),$$

where the infimum is taken over all fractional edge-paths $\mathfrak{p}$ in Γ connecting x to y. The metric dist is called the *standard* metric on the graph Γ.