Problem Sheet 0

Rolf Suabedissen

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For the benefit of students whose first class is in week 3, we have been asked to provide an extra problem sheet. This should allow you to get an idea of the flavour of the course. It contains mainly results which could have appeared in the Part A Topology course (or indeed have, even if not in this generality) and which will be used in future sheets. It is thus longer than the typical example sheet.

1 Axioms

1. **Open Sets**

   A topology τ on a set X is a collection of subsets of X (i.e. $\tau \subseteq \mathcal{P}(X)$) such that

   - τ is closed under arbitrary unions [Unions];
   - τ is closed under finite intersections [Intersections];
   - τ contains $\emptyset$ and X [Non-triviality].

   Give examples of subsets of X that

   (a) satisfy Unions and Non-triviality, but not Intersections (these are called generalized or weak topologies and apparently have been studied by psychologists under the name of 'Knowledge Spaces');

   (b) satisfy Intersections and Non-triviality, but not Unions;

   (c) satisfy Unions, Non-triviality and 'countable Intersections'(i.e. τ is closed under countable intersections);

   (d) satisfy Intersections, Non-triviality and 'finite [countable] Unions'but not Unions.

2. **Closure**

   Now assume that some topology τ on X is fixed. For a subset A of X, define
   $$\overline{A} = \bigcap \{C \subseteq X : A \subseteq C, X \setminus C \in \tau\}$$
   .

(a) Verify that

$$\overline{A} = \{x \in X : \forall U \in \tau \ (x \in U \implies U \cap A \neq \emptyset)\}\,.$$

(b) Show that $\overline{\cdot} \colon \mathcal{P}(X) \to \mathcal{P}(X)$ is an operator satisfying (for all $A, B \subseteq X$)

- $\emptyset = \overline{\emptyset}$ [Non-triviality];
- $A \subseteq \overline{A}$ [Increasing];
- $\overline{\overline{A}} = \overline{A}$ [Idempotent];
- $\overline{A \cup B} = \overline{A} \cup \overline{B}$ [Finite Union Distributivity];

An operator like this is called a 'Kuratowski closure operatore'.

(c) Give examples of topologies showing that $\overline{\cdot}$ does not need to distribute over countably infinite unions and does not need to distribute over finite intersections.

(d) Give examples of operators satisfying any 3 but not the fourth of the closure axioms.

(e) Show that if c is a Kuratowski closure operator, then

$$\tau_c = \{U \subseteq X : c(X \setminus U) = X \setminus U\}$$

is a topology on X. **Hint:** You may want to start with showing monotonicity of c, i.e. $A \subseteq B \implies c(A) \subseteq c(B)$.

(f) Verify that c is the closure operator of τ_c and that τ_c is the unique topology on X that has c as its closure operator.

(g) Show that a map $f \colon X \to Y$ between topological spaces is continuous if and only if for all $A \subseteq X$ we have $f(\overline{A}) \subseteq \overline{f(A)}$.

3. **Metrics**

(a) Given a metric d on a set X, verify that

$$\{U \subseteq X : \forall x \in U \ \exists \epsilon > 0 \ B_\epsilon(x) \subseteq U\}$$

is a topology on X with closure operator given by $x \in \overline{A}$ if and only if there is a sequence (a_n) in A which converges to x. Deduce that a map between metric spaces is continuous if and only if it preserves sequential limits.

If X is a topological space, a neighbourhood base at a point $x \in X$ is a family $\mathcal{N}$ of open subsets of X such that

- $\forall B \in \mathcal{N} \ x \in B$;
- if $x \in U$ open $\subseteq X$ then there is $B \in \mathcal{N}$ with $x \in B \subseteq N$;

(b) Verify that in a metric space every point has a countable neighbourhood base. A topological space in which every point has a countable neighbourhood base is called *first countable*.

(c) Show that if D is a subset of a metric space X with $\overline{D} = X$ (i.e. D is dense in X) then

$$\mathcal{B} = \{B_{2^{-n}}(x) : n \in \omega, x \in D\}$$

is a basis for X, i.e. that every (non-empty) open set is a union of elements of $\mathcal{B}$.

(d) Verify that if d is a metric on X then so is $d' = \min\{d, 1\}$ given by $d'(x, y) = \min\{d(x, y), 1\}$. Show that the topologies induced by d and d' are the same.

(e) Show that if d_n are metrics on X_n, $n \in \omega$ bounded by 1, then both

$$d_H(x, y) = \sum_{n \in \omega} d_n(x_n, y_n)/2^n$$

and

$$d_{\sup}(x, y) = \sup\{d_n(x_n, y_n) \colon n \in \omega\}$$

are metrics on $\prod_{n \in \omega} X_n$.

(f) Show that if all but finitely many X_n are trivial (i.e. contain only one point) then d_H and $d_{\sup}$ induce the same topology.

(g) If d is a metric on X, show that $d \colon X \times X \to \mathbb{R}$ is continuous. Also show that for a (non-empty) subset A of X the function $d_A \colon X \to \mathbb{R}$ given by

$$d_A(x) = \inf\{d(x, a) \colon a \in A\}$$

is continuous.

4. **Normality and Compactness**

A topological space is compact if and only if every open cover has a finite subcover.

(a) Show that a toplogical space is compact if and only if every collection of closed sets with the finite intersection property (i.e. every finite subcollection has non-empty intersection) has non-empty intersection.

(b) Verify that compactness is an absolute property (is independent of the surrounding space) i.e. if Y is a subset of X then every Y-open cover of Y has a finite subcover of Y if and only if every X-open cover of Y has a finite subcover of Y.

(c) Show that closed subsets of compact topological spaces are compact.

(d) Show that a compact Hausdorff space is normal.