1 Axioms

The point of this is to stay flexible. Definitions aren't true or false, they are convenient or inconvenient, depending on what one wants to study. If you encounter a problem where a variation of the standard definition of topology makes more sense, then feel free to change to that variation. Carefully check which 'standard' results are still true, though.

1.1 Open Sets

1. $X = \{0, 1, 2\}$, $\tau = \{\emptyset, X, \{0, 1\}, \{0, 2\}\}$ satisfies Unions, Non-triviality but not Intersections.

2. $X = \{0, 1, 2\}$, $\tau = \{\emptyset, X, \{0\}, \{1\}, \{2\}\}$ satisfies Non-triviality, Intersections but not Unions.

3. Of course the discrete topology ($\tau = \mathcal{P}(X)$) and the indiscrete topology ($\tau = \{\emptyset, X\}$) satisfy 'arbitrary Intersections'.

   More interesting is the following (a version of which appeared on a Part A Topology example sheet): Let X be an uncountable set and add a new point $\star$ to X. The topology on $Y = X \cup \{\star\}$ is given by

   $$\mathcal{P}(X) \cup \{Y \setminus C \colon C \text{ is countable }\}.$$

   One can easily check that this is a topology satisfying 'countable Intersections'. The critical observation is that a countable intersection of co-countable sets (co-countable = countable complement) is co-countable since a countable union of countable sets is countable.

   In a similar spirit, the co-countable topology on any set X (i.e. $\tau = \{X \setminus C \colon C \text{ is countable}\}$) will have 'countable Intersections'.

4. Let X be infinite [uncountable] and τ be the collection of all finite [countable] subsets of X together with X.

1.2 Closure

I find thinking in terms of closure much more intuitive than in terms of open sets. Think of x is in the closure of A as x can be approximated to arbitrary precision from within A.

5. Suppose that $x \in \overline{A}$ (according to the first definition), that U is open and contains x and that $U \cap A = \emptyset$. As $X \setminus (X \setminus U) = U \in \tau$ and $A \subseteq X \setminus U$ we get a contradiction. Conversely, if $x \notin \overline{A}$ then there is a closed subset C containing A and not x and then $U = X \setminus C$ witnesses that x does not belong to the second definition fo $\overline{A}$.

6. Non-triviality is trivial, Increasing follows straight from the first definition. They correspond (intuitively) to: you can't approximate anything from nothing and everything approximates itself.

   Next we note that $\overline{A}$ is closed (by duality, intersections of closed sets are closed) and that by definition the closure operator is monotone, i.e. preserves $\subseteq$.

   For Idempotency, by the fact that $\overline{A}$ is closed, $\overline{\overline{A}} \subseteq \overline{A}$ (by minimality) but also $\overline{A} \subseteq \overline{\overline{A}}$ by Increasing giving equality. Idempotency is a 'diagonality' property (or the triangle law): if you have a very good approximation and a very good approximation to the approximation, then this is still a reasonably good approximation to the original.

   For distributity over finite (binary) unions we have: the RHS is closed (finite unions of closed sets are closed) so $\subseteq$ follows from Increasing. Conversely $A \subseteq A \cup B$ so by monotonicity $\overline{A} \subseteq \overline{A \cup B}$ and by symmetry $\overline{B} \subseteq \overline{A \cup B}$ giving $\supseteq$. Finite Union Distributivity is hard(er) to justify intuitively.

7. $X = \mathbb{R}$ with its usual topology, $A_n = [2^{-(n+1)}, 2^{-n}] = \overline{A_n}$ gives $\bigcup_{n\in\omega} \overline{A_n} = \bigcup_{n\in\omega} A_n = (0, 1]$ but $\overline{\bigcup_{n\in\omega} A_n} = \overline{(0, 1]} = [0, 1]$, so closures need not distribute over infinite unions. Similarly $X = \mathbb{R}$, $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$ gives $\overline{A} = \mathbb{R} = \overline{B}$ but $\overline{A \cap B} = \overline{\emptyset} = \emptyset$.

8. X any non-empty set, and $\overline{A} = X$ fails Non-triviality, but satisfies (trivially) everything else.

   Similarly, X any non-empty set and $\overline{A} = \emptyset$ fails Increasing but satisfies (again trivially) everything else.

   For failing Idempotency, either the artificial $X = \{0, 1, 2\}$ with $\overline{\emptyset} = \emptyset$, $\overline{\{0\}} = \{0, 1\}$, $\overline{\{1\}} = \{0, 1\}$, $\overline{\{2\}} = \{2\}$, and $\overline{A} = X$ for any other A will do. Alternatively, consider $X = \{(0, 0)\} \cup \{(0, 2^{-n}) : n \in \omega\} \cup \{(2^{-m}, 2^{-n}) : n, m \in \omega\}$ (draw it!) with closure as follows: $(2^{-m}, 2^{-n}) \in \overline{A}$ if and only if $(2^{-m}, 2^{-n}) \in A$; $(0, 2^{-n}) \in \overline{A}$ if and only if A contains infinitely many $(2^{-m}, 2^{-n})$ and $(0, 0) \in \overline{A}$ if and only if A contains infinitely many $(0, 2^{-n})$. This shows what I meant by 'diagonality' property.

   To fail Finite Union Distributivity, again an artificial example with $X = \{0, 1, 2\}$ can be easily found. Or consider $\mathbb{R}$ with the following 'closure' operator: $x \in \overline{A}$ if and only if there are $B, C \subseteq A$ such that $x = \inf B = \sup C$ (i.e. x can be approximated from above **and** below from within A).

9. As suggested, we start with monotonicity: suppose $A \subseteq B$. Then

$$c(A) \subseteq c(A) \cup c(B) = c(A \cup B) = c(B).$$

   Next, we show that the set of fixed points of c is closed under intersections: so, suppose that $c(C_i) = C_i$ for $C_i \subseteq X, i \in I$. Then $\bigcap_{j\in I} C_j \subseteq C_i$ for each $i \in I$ so $c\left(\bigcap_{j\in I} C_j\right) \subseteq c(C_i) = C_i$ for each $i \in I$. Hence

$$c\left(\bigcap_{j\in I} C_j\right) \subseteq \bigcap_{i\in I} C_i \subseteq c\left(\bigcap_{i\in I} C_i\right)$$

(from Increasing) giving the result.

That the set of fixed points of c is closed under finite unions follows from Finite Union Distributivity. Finally, $\emptyset$ (Non-triviality) and X (Increasing) are fixed points of c. By duality τ_c is a topology on X.

10. We need to show $c(A) = \overline{A}$ (where the RHS is the closure according to τ_c). By definition of τ_c and Idempotency, $c(A)$ is τ_c-closed and contains A (Increasing), so certainly $\overline{A} \subseteq c(A)$. Conversely, if $A \subseteq B$ and B is τ_c-closed, then $c(B) = B$ so that by monotonicity of c we have $c(A) \subseteq c(B) = B$. But then by definition of $\overline{A}$ we get $c(A) \subseteq \overline{A}$ as required.

    Finally for uniqueness, suppose that τ' is any topology with closure operator c. Then if A is τ'-closed we have $A = \overline{A}^{\tau'} = c(A)$ so that A is τ_c-closed. On the other hand if A is not τ'-closed then $A \neq \overline{A}^{\tau'} = c(A)$ so that A is not τ_c-closed.

11. This is (more or less - more later) the formal version of the A-level result that a function is continuous if you can draw its graph without lifting the pen.

    Suppose that f is continuous and $x \in \overline{A}$. Let $V \ni f(x)$ be open. Observe that$f^{-1}(V) \ni x$ is open and hence meets A in some a. Then $f(a) \in V \cap f(A)$ so V meets $f(A)$ as required (by the second definition of closure).

    Now suppose f has the given property and that $C \subseteq Y$ is closed. Let $x \in \overline{f^{-1}(C)}$. Then $f(x) \in \overline{f(f^{-1}(C))} \subseteq \overline{C} = C$ so that $x \in f^{-1}(C)$. Hence $f^{-1}(C)$ is closed, showing that f is continuous.

2 Metrics

1. The given set clearly contains $\emptyset$ and X. For binary intersections, take the minimum of witnessing ϵs. For unions, the result is immediate. If $x \in \overline{A}$ then for each $n \in \omega$ choose $a_n \in A \cap B_{2^{-n}}(x)$ (which is non-empty by the second definition of closure) to get a sequence in A converging to x. Conversely, if (a_n) is a sequence in A converging to x and $U \ni x$ is open, then find $\epsilon > 0$ with $B_\epsilon(x) \subseteq U$ and some $N \in \omega$ with $n \geq N \implies a_n \in B_\epsilon(x)$ to see that $a_N \in B_\epsilon(x) \cap A \subseteq U \cap A$ as required. The result about continuity follows from the characterization of continuity in terms of closures (which we have just shown to be characterized by converging sequences).

2. For a point x in a metric space, it is straightforward to verify that $\{B_{2^{-n}}(x) : n \in \omega\}$ is a countable neighbourhood base (don't forget to remark that open balls are indeed open by the triangle law).

3. The next bit requires some messiness: if $y \in X$ and $U \ni y$ is open, first find $n \in \omega$ with $B_{2^{-n}}(y) \subseteq U$ and then use density of D to find $x \in D$ with $d(x, y) < 2^{-(n+1)}$. Finally check that by the triangle law $y \in B_{2^{-(n+1)}}(x) \subseteq B_{2^{-n}}(y) \subseteq U$. We note that for any $N \in \omega$ we can add the condition $n \geq N$ to the definition of $\mathcal{B}$.

4. The results in this section were, I think, all in the Part A Topology course and are mostly messy and not too difficult. Skip them, but note the results, if you are short on time.

   Symmetry and positive definiteness is clear. For the triangle law an easy case distinction does the trick. The topologies are the same by the basis given above (with $X = D$) (where we require $n \geq 1$) which coincides for d and d'.

5. First note that as each d_n is bounded by 1, d_H and d_{sup} are well-defined. Again, symmetry and positive definiteness for d_H and d_{sup} are clear. For the triangle law we observe for d_H that all terms in the sum are non-negative, so applying the triangle law to each d_n and collecting the terms (by non-negativity, this doesn't change the resulting infinite sum) gives the result. For d_{sup} the triangle law follows from $\sup A + B \leq \sup A + \sup B$ proven in first year Analysis.

6. Now suppose that all but the first N many X_n are trivial (so that the sup is in fact a max). Then check that $2^{-N} d_{\text{sup}}(x, y) \leq d_H(x, y) \leq 2d_{\text{sup}}(x, y)$ and observe that this implies that the identity map is continuous (in fact Lipschitz) in both directions, so a homeomorphism.

7. Equipping $X \times X$ with d_{sup} gives $d_{\text{sup}}((x, y), (x', y')) < \epsilon/2$ so that $d(x, x'), d(y, y') < \epsilon/2$ and hence

   $$d(x, y) - d(x', y') \leq d(x, x') + d(x', y') + d(y', y) - d(x', y') < \epsilon.$$

   Symmetry gives continuity in the classic metric sense.

8. For continuity of d_A observe that $d_A(x) \leq d(x, a) \leq d(x, y) + d(y, a)$ for each $a \in A$. Taking the infimum over all $a \in A$ gives $d_A(x) \leq d(x, y) + d_A(y)$ i.e. $d_A(x) - d_A(y) \leq d(x, y)$ and hence continuity ($\delta = \epsilon$ works).

3 Normality and compactness

1. We note that a collection $\mathcal{D}$ of (closed) sets of X has empty intersection if and only if the collection $\{X \setminus D \colon D \in \mathcal{D}\}$ is a (open) cover of X. Now apply this to get the if and only if result immediately.

2. $\Rightarrow$: Suppose $\mathcal{U}$ is an X-open cover of Y. Then $\{U \cap Y : U \in \mathcal{U}\}$ is a Y-open cover of Y so has a finite subcover $\mathcal{V}$. For each $V \in \mathcal{V}$ choose $U_V \in \mathcal{U}$ with $U_V \cap Y = V$. Then note that $\{U_V : V \in \mathcal{V}\}$ is the required finite subcover.

   $\Leftarrow$: Suppose $\mathcal{U}$ is a Y-open cover of Y. For $U \in \mathcal{U}$ choose X-open V_U with $U = V_U \cap Y$. Note that $\{V_U : U \in \mathcal{U}\}$ is then an X-open cover of Y so has a finite subcover $\mathcal{V}$. For each $V \in \mathcal{V}$ choose $U_V \in \mathcal{U}$ with $V = V_{U_V}$ and finally observe that $\{U_V : V \in \mathcal{V}\}$ is the required finite subcover.

3. That compactness is closed hereditary follows from the dual version of compactness (which we just proved) and the fact that closed subsets of closed subsets are globally closed.

4. The proof that compact Hausdorff implies normal is 'prototypical' and we will encounter variations of it later on. We first show that X is regular: if $x \in X$, $C \subseteq X$ closed and $x \notin C$ then by Hausdorffness for each $c \in C$ we can find disjoint open U_c, V_c with $x \in U_c$, $c \in V_c$. Then V_c is an open cover of the compact set C so has a finite subcover $\{V_{c_0}, \dots V_{c_n}\}$. Set $U = \bigcap_{i \le n} U_{c_i}$ (open because of the finiteness of the intersection) and $V = \bigcup_{i \le n} V_{c_i}$ (open as a union of open sets). Clearly $x \in U$, $C \subseteq V$ and if U and V are not disjoint then some U_{c_i} must meet V so in particular must meet V_{c_i} a contradiction.

   Next repeat this proces but replace x by a closed set D and apply regularity instead of Hausdorffness.