Question 1

We will often be using the following easy Lemma:

Lemma 0.1. *Suppose $\mathcal{B}$ is a family of open subsets of a topological space X.*
$\mathcal{B}$ is a basis for X if and only if for every open $U \subseteq X$ and $x \in U$ there is $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Proof. **$\Rightarrow$:** $U = \bigcup \mathcal{B}'$ for some $\mathcal{B}' \subseteq \mathcal{B}$. Thus $x \in B$ for some $B \in \mathcal{B}'$ and then $x \in B \subseteq U$.

$\Leftarrow$: If U is open, for each $x \in U$ choose $B_x \in \mathcal{B}$ with $x \in B_x \subseteq U$ and observe that $U = \bigcup_{x \in U} B_x$. $\square$

Countable Products

Suppose (X_n, d_n) are metric spaces. From Sheet 0, we may assume all d_n are bounded by 1 (otherwise replace with $\min\{d_n, 1\}$ which induces the same topology) so that d_H from Sheet 0 is a metric on $\prod_n X_n$.

We need to check that the topology induced by d_H coincides with the Tychonoff topology.

Note that $\pi_m \colon (X, d_H) \to (X_m, d_m)$ is continuous since $d_m(x_m, y_m) \leq 2^m d(x, y)$ (for $x, y \in \prod_n X_n$). Since the Tychonoff product topology is the smallest topology that makes all π_m continuous, the every Tychonoff-open set must be d_H-open.

For the converse it is enough to check that for $x \in \prod_n X_n$ and $\epsilon > 0$ there is Tychonoff-open $U = U_{x,\epsilon}$ with $x \in U \subseteq B_\epsilon(x)$. Then if V is d_H-open for $x \in V$ we can choose $\epsilon_x > 0$ such that $B_{\epsilon_x}(x) \subseteq V$ and thus $V = \bigcup_{x \in V} U_{x,\epsilon_x}$ is Tychonoff open.

So let $x \in \prod_n X_n$ and $\epsilon > 0$ and choose N such that $\sum_{n \geq N} 2^{-n} < \epsilon/2$. Now let $\delta = \frac{\epsilon}{2(N+1)}$ and observe that

$$x \in B_\delta(x_0) \times B_\delta(x_1) \times \cdots \times B_\delta(x_{N-1}) \times \prod_{n \geq N} X_n = \bigcap_{n<N} \pi_n^{-1}(B_\delta(x_n)) \subseteq B_\epsilon(x).$$

Remarks

If you live in the category of metric spaces, what should the 'morphisms', i.e. the 'structure-preserving maps' be?

We know about the isomorphisms (namely $f \colon (X, d_X) \to (Y, d_Y)$ is an isomorphism if and only if f is a bijection such that for all $x, x' \in X$ we have $d_Y(f(x), f(x')) = d_X(x, x')$.

But we also want that if $f \colon (X, d_X) \to (Y, d_Y), g \colon (X, d_X) \to (Y, d_Y)$ are morphisms with $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$ then f and g are isomorphisms. If we simply use continuous maps as morphisms, this fails (e.g. $X = Y = \mathbb{R}$, $f(x) = 2x$, $g(x) = x/2$). So the 'right' choice of morphism are (can be) the 'non-expansions', i.e. maps such that $d_Y(f(x), f(x')) \leq d_X(x, x')$.

Note however that π_m ($m > 1$) is not a non-expansion from $(\prod_n X_n, d_H)$ to (X_m, d_m), so d_H is not the 'right' metric on $\prod_n X_n$. In some sense this is reassuring since d_H is sensitive to the order of the metric spaces which 'should not' happen for a product-metric.

In fact, you can check that the sup-metric $d_\infty(x, y) = \sup_n d_n(x_n, y_n)$ is the 'right' product-metric, but this (for non-trivial X_n) does not induce the Tychonoff topology.

Uncountable Products

Let I be uncountable and for $i \in I$ let $X_i = \{0, 1\}$ with the discrete topology.

We will show that $\{0\}$ is not a countable intersection of open sets in $\prod_i X_i$. But in every metric space $\{x\} = \bigcap_{n\in\omega} B_{2^{-n}}(x)$ is a countable intersection of open sets.

So, let $U_n, n \in \omega$ be open sets containing 0. We shrink the U_n to basic open sets, i.e. choose finite $F_n \subseteq I$ such that $0 \in B_n = \bigcap_{i\in F_n} \pi_i^{-1}(0) \subseteq U_n$. Then $\bigcup_n F_n$ is a countable union of finite sets so countable and hence we can choose $i_0 \in I \setminus \bigcup_n F_n$. Let $y \in \prod_i X_i$ be given by $y_i = \begin{cases} 1; \text{ if } i = i_0 \\ 0; \text{ otherwise} \end{cases}$. Then $0 \neq y \in \bigcap_n B_n \subseteq \bigcap_n U_n$ as required.

Remarks

First note that the proof above works for any non-trivial X_i.

Next observe that being first countable implies every singleton being a countable intersection of open sets.

However, the above argument does not show that the set of **continuous** real-valued functions on $[0, 1]$, $\mathcal{C}([0, 1])$, as a subspace of $\mathbb{R}^{[0,1]}$ (this gives it the topology of pointwise convergence) is non-metrizable: continuous functions on $[0, 1]$ are determined by their values on $\mathbb{Q} \cap [0, 1]$, so f is the only continuous function in

$$\bigcap_{n\in\omega, q\in\mathbb{Q}\cap[0,1]} \pi_q^{-1}((f(q) - 2^{-n}, f(q) + 2^{-n})).$$

However, the space is still not first countable: for suppose that $U_n, n \in \omega$ is a collection of open sets containing f.

Assume first that for each n, there is finite $F_n \subseteq [0, 1]$ and $\epsilon > 0$ such that $U_n = \bigcap_{x\in F_n} \pi_x^{-1}(B_{\epsilon_n}(f(x)))$. Now let $y \in [0, 1] \setminus \bigcup_n F_n$ (this exists as $\bigcup_n F_n$ is countable and $[0, 1]$ is uncountable) and let $V = \pi_y^{-1}((f(x) - 1, f(x) + 1))$ which is open and contains f. For each n, we can construct a continuous function $g_n\colon [0, 1] \to \mathbb{R}$ (in fact a polynomial) through all the $(x, f(x)), x \in F_n$ and $(y, f(x) + 2)$. Then $g_n \in U_n \setminus V$ so that $U_n \not\subseteq V$.

If the U_n are not of the form above, we can shrink them to this form and again $U_n \not\subseteq V$.

Thus we have shown: For every countable family of open sets $U_n \ni f$, $n \in \omega$ there is open $V \ni f$ such that for every $n \in \omega$ $U_n \not\subseteq V$.

Hence $\mathcal{C}([0,1])$ with the topology of pointwise convergence is not first countable and hence not metrizable.

Question 2

The 'slick' way of doing this is by using the d_C from Sheet 0: if C, D are disjoint closed then $d_C, d_D \colon X \to \mathbb{R}$ are continuous and by disjointness and $C = d_C^{-1}(0), D = d_D^{-1}(0)$ we have that $d_C + d_D$ is non-zero on X. Thus

$$f = \frac{d_C}{d_C + d_D} : X \to [0,1]$$

(it is into $[0,1]$ since $0 \leq d_C \leq d_C + d_D$) is continuous and still $C = f^{-1}(0), D = f^{-1}(1)$ so that $f^{-1}([0,1/3)), f^{-1}((2/3,1])$ are disjoint open sets containing C and D respectively.

Remarks

If you try this 'manually' you need to be careful: the two components of the graph of $1/x^2$ are closed subsets of $\mathbb{R}^2$ but have 'distance' 0. So you need to do things pointwise, i.e. for $c \in C$ choose $\epsilon_c > 0$ such that $B_{\epsilon_c}(c) \cap D = \emptyset$ and similarly for $d \in D$. Then you need to halve the ϵs and union up i.e. set

$$U = \bigcup_{c \in C} B_{\epsilon_c/2}(c); \quad V = \bigcup_{d \in D} B_{\epsilon_d/2}(d)$$

and note that $C \subseteq U$, $D \subseteq V$ and if $x \in U \cap V$ then there is $c \in C$ and $d \in D$ such that $d(c,d) \leq d(c,x) + d(x,d) < \epsilon_c/2 + \epsilon_d/2 \leq \epsilon_c, \epsilon_d$ so that either $d \in B_{\epsilon_c}(c) \subseteq X \setminus D$ (impossible) or $c \in B_{\epsilon_d}(d) \subseteq X \setminus C$ (also impossible).

Question 3

second countable implies separable

Suppose $\mathcal{B}$ is a countable basis. For $B \in \mathcal{B}$ choose $d_B \in B$ and set $D = \{d_B \colon B \in \mathcal{B}\}$.

Clearly D is countable.

To see that it is dense, let U be a non-empty open subset of X, choose $x \in U$ then $B \in \mathcal{B}$ with $x \in B \subseteq U$. Then $d_B \in B \cap U$ as required.

second countable implies Lindelöf

Let $\mathcal{B}$ be a countable basis.

If $\mathcal{U}$ is an open cover of X, for each $B \in \mathcal{B}$ choose a $V_B \in \mathcal{U}$ such that $B \subseteq V_B$ if some such V_B exists.

Then

$$\mathcal{V} = \{V_B \colon B \in \mathcal{B} \text{ such that } V_B \text{ is defined}\}$$

is a countable subcollection of $\mathcal{U}$. We'll show it covers X: if $x \in X$ find $U \in \mathcal{U}$ such that $x \in U$ and then $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Then V_B is defined (since $B \subseteq U \in \mathcal{U}$) and $x \in B \subseteq V_B \in \mathcal{V}$ as required.

Remarks

This is a typical way to see that usually (but not always) for compactness-like properties, we may assume that the open cover is in fact a basic open cover: shrink the open sets to basic open sets and for each such basic open set fix an element of the cover containing it.

metric+separable implies second countable

Let D be a countable dense subset and let

$$\mathcal{B} = \{B_{2^{-n}}(d) \colon d \in D, n \in \omega\}.$$

Clearly $\mathcal{B}$ is a countable collection of open sets. We'll verify it is a basis: let $x \in U$ open $\subseteq X$ and choose N such that $x \in B_{2^{-N}}(x) \subseteq U$. Find $d \in D \cap B_{2^{-(N+1)}}(x)$ (by density of D) and note that by symmetry and the triangle law

$$x \in B_{2^{-(N+1)}}(d) \subseteq B_{2^{-N}}(x) \subseteq U$$

as required.

metric + Lindelöf implies second countable

For each $n \in \omega$, note that $\mathcal{U}_n = \{B_{2^{-n}}(x) \colon x \in X\}$ is an open cover of X so has a countable subcover $\mathcal{B}_n$.

Then $\mathcal{B} = \bigcup_n \mathcal{B}_n$ is a countable union of countable sets so countable and consists of open sets. We verify it is a basis: let $x \in U$ open $\subseteq X$ and choose N such thath $x \in B_{2^{-N}}(x) \subseteq U$. Choose $B = B_{2^{-(N+1)}}(y) \in \mathcal{B}_n \subseteq \mathcal{B}$ (for some $y \in X$) such that $x \in B$ and as before

$$x \in B \subseteq B_{2^{-N}}(x) \subseteq U.$$

Remarks

You could be tempted to directly go from metric+separable to Lindelöf: a typical attempt may go as follows: let D be countable dense and $\mathcal{U}$ be an open cover. For $d \in D$ choose $U_d \in \mathcal{U}$ such that $d \in U_d$. Now we hope that $\{U_d \colon d \in D\}$ covers X, but that fails unless you are clever about choosing the U_d (e.g. $X = \mathbb{R}, D = \mathbb{Q} \setminus \{0\}, \mathcal{U} = \{(-\infty, 0), (0, \infty), \mathbb{R}\}$, $U_q = (-\infty, 0)$ or $(0, \infty)$).

An interesting question is under which conditions (other than metric) separable implies Lindelöf and conversely. We will see one of these later in the course.

Also, from the 1960s onwards, a lot of research went into the question of whether there exists a hereditarily separable space that is not Lindelöf (called an S-space) and whether there eixsts a hereditarily Lindelöf space that is not separable (called an L-space). For more information see the bottom half of page 2 of `http://web.mat.bham.ac.uk/C.Good/research/pdfs/ency-lind.pdf`.

Question 4

metric spaces have halving operators

Define $H(x, B_\epsilon(x)) = B_{\epsilon/2}(x)$ and extend as follows: if $x \in U$ open $\subseteq X$ choose $\epsilon_x > 0$ such that $x \in B_{\epsilon_x}(x) \subseteq U$ and set $H(x, U) = H(x, B_{\epsilon_x}(x))$.

If $H(x, U)$ meets $H(y, V)$ in z say then wlog $\epsilon_x \leq \epsilon_y$ and then

$$d(x, y) \leq d(x, z) + d(z, y) < \epsilon_x/2 + \epsilon_y/2 \leq \epsilon_y$$

so that $x \in B_{\epsilon_y}(y) \subseteq V$ as required.

having halving operators is hereditary

Next if H is a halving operator for X and $Y \subseteq X$ we define a halving operator fo Y as follows: for each Y-open $U \subseteq Y$, choose X-open $W_U \subseteq X$ such that $U = W_U \cap Y$ and if $x \in V$ then set $H^Y(x, U) = H(x, W_U) \cap Y$.

If then $z \in H^Y(x, U) \cap H^Y(y, V) = H(x, W_U) \cap H(y, W_V) \cap Y$ then $x \in W_V$ or $y \in W_U$. But $x, y \in Y$ so $x \in W_V \cap Y = V$ or $y \in W_U \cap Y = U$ as required.

Hausdorff+halving operator implies normal

We follow the 'manual' proof of metric implies normal: Suppose C, D are disjoint closed. For $c \in C$ note that $c \in X \setminus D$ which is open and similarly for $d \in D$ we have $d \in X \setminus C$ which is open (from disjointness). Thus we can set

$$U = \bigcup_{c \in C} H(c, X \setminus D); \quad V = \bigcup_{d \in D} H(d, X \setminus C)$$

and observe that these are open and $C \subseteq U$, $D \subseteq V$.

If U and V were to meet, then some $H(c, X \setminus D)$ meets some $H(d, X \setminus C)$ giving $c \in X \setminus C$ or $d \in X \setminus D$ a contradiction. thus U, V are disjoint as required.

Remark

When considering normality, we can view it as a function $N : (C, D) \mapsto (U_{C,D}, V_{C,D})$ from the set of pairs of disjoint closed subsets to the set of pairs of disjoint open subsets. We would expect this function to be 'monotone' in both C and D, i.e. if $C \subseteq C'$ (and D is disjoint from C) then $U_{C,D} \subseteq U_{C',D}$ and $V_{C',D} \subseteq V_{C,D}$ (and similarly if $D \subseteq D'$). If such a monotone function exists, the space is called monotonically normal. It turns out (see Q7) the in regular spaces, monotone normality is equivalent to having halving operators! It also turns out that monotone normality is a very important property and much research has been carried out into it.

Question 5

First note that just like for compactness,

Lemma 0.2. *If X is Lindelöf and C is a closed subset of X then every X-open cover of C has a countable subset covering C.*

Proof. Let $\mathcal{U}$ be an X-open cover of C. Then $\mathcal{U} \cup \{X \setminus C\}$ is an open cover of X so has a countable subcover $\mathcal{U}'$. Then $\mathcal{U} \cap \mathcal{U}' = \mathcal{U}' \setminus \{X \setminus C\}$ is a countable subset of $\mathcal{U}$ that covers C (since the removed set $X \setminus C$ does not contribute to covering C). □

So, let C, D be disjoint closed subsets of X. By regularity, for $c \in C$ and $d \in D$ choose open $U_c \ni c$, $V_d \ni d$ respectively such that

$$c \in U_c \subseteq \overline{U_c} \subseteq X \setminus D$$

and similarly

$$d \in V_d \subseteq \overline{V_d} \subseteq X \setminus C.$$

Then $\{U_c \colon c \in C\}$ and $\{V_d \colon d \in D\}$ are open covers of C and D respectively, so have countable subcovers $\mathcal{U} = \{U_n \colon n \in \omega\}$ and $\mathcal{V} = \{V_n \colon n \in \omega\}$ respectively. Now let

$$\hat{U}_n = \bigcup_{k \leq n} U_k \setminus \bigcup_{k \leq n} \overline{V_k}; \quad \hat{V}_n = \bigcup_{k \leq n} V_k \setminus \bigcup_{k \leq n} \overline{U_k}.$$

These are open sets (finite union of closed sets are closed) and $U_n \cap C \subseteq \hat{U}_n$ since the removed stuff, $\bigcup_{k \leq n} \overline{V_k}$, does not meet C and similarly $V_n \cap D \subseteq \hat{V}_n$.

Therefore

$$U = \bigcup_n \hat{U}_n \supseteq \bigcup_n (U_n \cap C) = C \cap \bigcup_n U_n = C$$

and similarly $V = \bigcup_n \hat{V}_n \supseteq D$.

Finally, if U and V would meet, then some $\hat{U}_n$ meets some $\hat{V}_m$. Wlog $n \leq m$ but by the definition of $\hat{V}_m$, we have $\hat{V}_m \subseteq X \setminus \bigcup_{k \leq m} U_k \subseteq X \setminus \hat{U}_n$, a contradiction.

Thus U, V are as required showing that C, D are separated by open sets.

Remarks

This is a typical use of countability. We construct our open sets in stages and because each stage is finite, we can use unions over closed sets in our construction as well.

Question 6

⇒: Suppose A, B are separated subsets of X, i.e. $\overline{A} \cap B = \emptyset = A \cap \overline{B}$.

Let $Y = X \setminus (\overline{A} \cap \overline{B})$. Note that $A, B \subseteq Y$ by construction and that Y is open in X. Now observe that

$$\overline{A}^Y = \overline{A}^X \cap Y; \quad \overline{B}^Y = \overline{B}^X \cap Y$$

and hence by choice of Y we have that $\overline{A}^Y$ and $\overline{B}^Y$ are disjoint (and of course Y-closed).

Since Y is normal by assumption there are disjoint Y-open $U \supseteq \overline{A}^Y \supseteq A$ and $V \supseteq \overline{B}^Y \supseteq B$. But since Y is open in X, U and V are in fact X-open as required.

⇐: Suppose $Y \subseteq X$ and A, B are disjoint Y-closed subsets of Y. Then

$$\overline{A}^X \cap B = \overline{A}^X \cap (B \cap Y) = (\overline{A}^X \cap Y) \cap B = \overline{A}^Y \cap B = A \cap B = \emptyset$$

and similarly

$$\overline{B}^X \cap A = \emptyset$$

so that by assumption there are disjoint X-open $U \supseteq A$ and $V \supseteq B$.

Then $U \cap Y$ and $V \cap Y$ witness separation of A, B in Y by open sets and hence normality of Y (at A, B).

Remark

This is nice because it gives a straightforward internal definition of hereditary normality. Also, the property that (some) separated sets are separated by open sets is useful in many instances.

Question 7

Monotonically Normal implies existence of halving operator: For closed $C \subseteq U$ open write $W(C, U)$ for the open set containing C and whose closure is contained in U and assume W is monotonically increasing in C and monotonically decreasing in U (by setting $D = X \setminus U$).

We first make this 'symmetric' by replacing $W(C, U)$ by $W(C, U) \setminus \overline{W(U, C)}$ so that $W(C, U) \cap W(X \setminus U, X \setminus C) = \emptyset$ (and it is still monotone in the right ways).

Then we define $H(x, U) = N(\{x\}, U)$ and show this works: if $x \notin V \ni y$ and $y \notin U \ni x$ then

$$H(x, U) \cap N(X \setminus U, X \setminus \{x\}) = \emptyset$$
$$H(y, V) = N(\{y\}, V) \subseteq N(X \setminus U, X \setminus \{x\})$$

where the first line is the symmetry and the second the monotonicity ($y \notin U$ means $\{y\} \subseteq X \setminus U$ and $x \notin V$ means $V \subseteq X \setminus \{x\}$). But this gives $H(x, U) \cap H(y, V) = \emptyset$ as required.

Halving operator implies monotonically normal: Assume $H(x,U)$ is the halving operator.

We first make it monotone in U by replacing $H(x,U)$ with $\bigcup\{H(x,V)\colon x \in V \text{ open } \subseteq U\}$ so that wlog $x \in U \subseteq U' \Rightarrow H(x,U) \subseteq H(x,U')$.

If C, D are disjoint closed then set

$$U = \bigcup_{c\in C} H(c, X \setminus D)$$
$$V = \bigcup_{d\in D} H(d, X \setminus C)$$

and note that these are open sets with the right monotony properties (because H is monotonic) and if $U \cap V \neq \emptyset$ then there is $c \in C$, $d \in D$ such that $H(c, X \setminus D) \cap H(d, X \setminus C) \neq \emptyset$ so that $c \in \setminus C$ (contradiction) or $d \in X \setminus D$ (contradiction).

Totally ordered spaces are monotonically normal: We do this for a dense total order without endpoints since we can embed every total order in a dense total order without endpoints and subspaces of monotonically normal spaces are monotonically normal.

Well order X by $\hat{\leq}$. For a non-empty open interval (a,b) we let $m_{(a,b)}$ be the $\hat{\leq}$-minimal element of (a,b).

We then set $H(x,(\alpha,\beta)) = (m_{(\alpha,x)}, m_{(x,\beta)})$ for $\alpha < x < \beta$ and claim that this works (note that by density this is well-defined).

The relevant cases for pairs of triples $\alpha < x < \beta$ and $\alpha' < x' < \beta'$ are

- $x \leq \alpha'$ and $\beta \leq x'$
- $\beta' \leq x$ and $x' \leq \alpha$

(otherwise $x \in (\alpha', \beta')$ or $x' \in (\alpha, \beta)$).

So consider the first of them and assume that $H(x,(\alpha,\beta))$ meets $H(x',(\alpha',\beta'))$ in some z. Then $x \leq \alpha' < m_{(\alpha',x')} < z < m_{(x,\beta)} < \beta \leq x'$ and so $m_{(\alpha',x')} \in (x,\beta)$ and $m_{(x,\beta)} \in (\alpha', x')$ which contradicts their $\hat{\leq}$-minimality as they are different.

If we now want to define $H(x,U)$ for $x \in U$ open we simply choose a, b with $x \in (a,b) \subseteq U$ and define $H(x,U) = H(x,(a,b))$.