Ultraproducts

<u>Def.</u>: • $\mathcal{D} \subseteq \wp(I)$ is a <u>filter</u> on a set I if

(i) $D_1, D_2 \in \mathcal{D} \Rightarrow D_1 \cap D_2 \in \mathcal{D}$

(ii) $\emptyset \notin \mathcal{D}$

(iii) $D \in \mathcal{D}$, $D \subseteq E \subseteq I \Rightarrow E \in \mathcal{D}$

• $\mathcal{D}$ <u>ultrafilter</u> if also

(iv) for all $D \subseteq I$: $D \in \mathcal{D}$ or $I \setminus D \in \mathcal{D}$

• $\mathcal{D}$ <u>principal</u> ultrafilter if $\mathcal{D} = \{D \subseteq I \mid x \in D\}$ for some fixed $x \in I$ (o/w <u>non-principal</u>)

<u>Example</u> ① $\mathcal{D} = \{D \subseteq \mathbb{N} \mid D \text{ cofinite}\}$ is a filter on $\mathbb{N}$ (not an ultrafilter)

② X a topological space, $\mathcal{D}$ = the filter of nbds. of pt. x

<u>Note</u> Every filter is contained in some ultrafilter (Zorn)

<u>Def.</u>: Let $\mathcal{D}$ be an ultrafilter on I, $(\mathcal{A}_i = \langle A_i; \ldots \rangle)_{i \in I}$ a family of L-structures. Then the <u>ultraproduct</u> $\mathcal{A} = \prod_{i \in I} \mathcal{A}_i / \mathcal{D}$ is the L-structure with domain $A = \{[(a_i)_{i \in I}] \mid a_i \in A_i\}$ where $[\ ]$ denotes equ. cl. w.r.t. $(a_i) \sim (b_i) \Leftrightarrow \{i \in I \mid a_i = b_i\} \in \mathcal{D}$

$$P^{\mathcal{A}}([(\bar{a}_i)_{i \in I}]) \Leftrightarrow \{i \in I \mid P^{\mathcal{A}_i}(\bar{a}_i)\} \in \mathcal{D}$$

$$f^{\mathcal{A}}([\bar{a}_i]) = [(b_i)] \Leftrightarrow \{i \in I \mid f^{\mathcal{A}_i}(\bar{a}_i) = b_i\} \in \mathcal{D}$$

$$c^{\mathcal{A}} = [(c^{\mathcal{A}_i})_{i \in I}]$$

well-defined, as $\mathcal{D}$ is <u>ultrafilter</u>.

<u>Theorem of Łoś</u> $\varphi = \varphi(v_1, \dots, v_n)$ L-formula, $\bar{a} \in A^n$

$$\mathcal{A} \models \varphi[\bar{a}] \iff \{i \in I \mid \mathcal{A}_i \models \varphi(\bar{a}_i)\} \in \mathcal{D} \qquad \overset{\shortparallel}{\bar{a}} = [(\bar{a}_i)]$$

<u>Pf.:</u> for atomic φ by def.

$\varphi = \psi \wedge \chi$: by (i)

$\varphi = \neg \psi$: by (iv)

$\varphi = \exists x \; \psi(x, \bar{a})$:

$\mathcal{A} \models \varphi$ iff there are $b_i \in \mathcal{A}_i$ s.t. $\mathcal{A} \models \psi([(b_i)], \bar{a})$

IH iff $\{i \in I \mid \mathcal{A}_i \models \psi(b_i, \bar{a}_i)\} \in \mathcal{D}$

$$\cap\!\!|$$

$\{i \in I \mid \mathcal{A}_i \models \exists x \, \psi(x, \bar{a}_i)\} \in \mathcal{D}$ ∎

<u>Example 1</u> $\mathcal{D} = \{D \subseteq I \mid i \in D\}$ for some $i \in I$

$$\Rightarrow \prod_{i \in I} \mathcal{A}_i / \mathcal{D} \equiv \mathcal{A}_i$$

<u>Example 2</u>

p prime, $\mathbb{F}_p$ the field with p elements

$\mathcal{D}$ ultrafilter on $\mathbb{P} := \{p \mid p \text{ prime}\}$

containing the filter of cofinite subsets of $\mathbb{P}$

$L = \{+, \cdot; 0, 1\}$

$\Sigma = \{\sigma \in \mathrm{Sent}(L) \mid \mathbb{F}_p \models \sigma \text{ for almost all } p\}$

Then $K = \prod_{p \in \mathbb{P}} \mathbb{F}_p / \mathcal{D} \models \Sigma$ with char $K = 0$

"pseudofinite fields"

<u>Obs.:</u> $\{p \in \mathbb{P} \mid p \equiv 1 \bmod 4\} \in \mathcal{D} \Rightarrow \sqrt{-1} \in K$

$\{p \in \mathbb{P} \mid p \equiv 3 \bmod 4\} \in \mathcal{D} \Rightarrow \sqrt{-1} \notin K$

Ax' Theorem (model-theoretic pf. of a purely algebraic fact)

Let $f: \mathbb{C}^n \to \mathbb{C}^n$ be a function given by polynomials $f_1, \dots, f_n \in \mathbb{C}[X_1, \dots, X_n]$.

Then: f 1-1 $\Rightarrow$ f onto $(*)$

Pf.: As the f_i are of bounded degree, there is an $L = \{+, \cdot\, ; 0, 1\}$-sentence φ for $(*)$

$\varphi \models \overline{\mathbb{F}}_p$ for each $p \in \mathbb{P}$:

$\overline{\mathbb{F}}_p = \bigcup_n \mathbb{F}_{p^{n!}}$: choose n_0 s.t. f defined over $\mathbb{F}_{p^{n_0!}}$

$\Rightarrow \forall n \geq n_0$: $f \upharpoonright \mathbb{F}_{p^n}$ is 1-1, hence onto

given $y \in \overline{\mathbb{F}}_p$ choose $n \geq n_0$ with $y \in \mathbb{F}_{p^{n!}}$

$\Rightarrow \exists \bar{x} \in \mathbb{F}_{p^{n!}}^{\,n}$ with $f(\bar{x}) = y$

$\Rightarrow \varphi \models \prod_{p \in \mathbb{P}} \overline{\mathbb{F}}_p / \mathcal{D} \models ACF_0$ hence $\equiv \mathbb{C}$ ∎

($\mathcal{D}$ non-pr.)

New proof of Compactness Theorem

$\Sigma \subseteq Sent(L)$ f.s., $I = \{\emptyset \neq \Sigma_0 \subseteq \Sigma \mid \Sigma_0 \text{ finite}\}$

let $\mathcal{D}_0 = \{D \subseteq \Sigma \mid D = \{\Sigma_0^* \subseteq \Sigma \mid \Sigma_0^* \text{ finite and } \Sigma_1 \subseteq \Sigma_0\}$ for some $\Sigma_1 \in I\}$

For each $i = \Sigma_0 \in I$ ~~then~~ let $\mathcal{A}_i \models \Sigma_0$

$\mathcal{D}_0$ closed under $\cap$, under oversets and $\emptyset \notin \mathcal{D}_0$

$\Rightarrow$ there is an ultrafilter $\mathcal{D} \supseteq \mathcal{D}_0$ on I

$\Rightarrow \prod_{i \in I} \mathcal{A}_i / \mathcal{D} \models \Sigma$:

let $\varphi \in \Sigma \Rightarrow \underbrace{\{i = \Sigma_0 \in I \mid \varphi \in \Sigma_0\}}_{\in \mathcal{D}} \subseteq \{i = \Sigma_0 \in I \mid \mathcal{A}_i \models \varphi\} \in \mathcal{D}$ ∎